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Strong Converse Inequalities for Averages in Weighted L^p Spaces on $[-1, 1]$

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Abstract. Averages in weighted spaces $L^p_\phi[-1,1]$ defined by additions on $[-1,1]$ will be shown to satisfy strong converse inequalities of type A and B with appropriate K-functionals. Results for higher levels of smoothness are achieved by combinations of averages. This yields, in particular, strong converse inequalities of type D between K-functionals and suitable difference operators.

1 Introduction

In this section we introduce the concept of averages and give the main definitions needed for the paper.

We will investigate the averaging operators

$$(A_t f)(x) := \frac{1}{\phi(t)} \int_0^t f(x \oplus u)\, d\phi(u), \quad x \in [-1,1], \tag{1.1}$$

for $t > 0$ where $\phi \in AC[-1,1]$ is an odd function with $\phi'(x) > 0$ almost everywhere and $\oplus$ an inner addition on $[-1,1]$ suitably defined by means of the function ϕ (see below). We denote by $L^p_\phi[-1,1]$, $p \in [1,\infty)$, the set of all measurable functions $f\colon [-1,1] \to R$ for which the weighted norm

$$\|f\|_{L^p_\phi} := \left(\int_{-1}^1 |f(x)|^p \, d\phi(x) \right)^{\frac{1}{p}}$$

is finite.

The smoothness of functions in $L^p_\phi[-1,1]$ is described by the K-functional

$$K^r(f, t^r) := \inf\{\|f-g\|_{L^p_\phi} + t^r \|D^r g\|_{L^p_\phi} \mid D^r g \in L^p_\phi[-1,1]\}, \quad t \geq 0, \tag{1.2}$$

in which the differential operator is given by

$$(Df)(x) := \frac{f'(x)}{\phi'(x)} \quad x \in [-1,1] \text{ a.e.} \tag{1.3}$$

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and D^r is defined by composition of D r times.

In this paper we will relate the approximation of f by $A_t f$ (or linear combinations of A_u) with the K-functional (1.2). We will investigate strong equivalence relations of type A, B and D in the classification introduced and discussed in [Di-Iv].

For first order $r = 1$, $K(f,t) := K^1(f,t)$, we will prove that the relation

$$K\big(f,\phi(t)\big) \sim \|A_t f - f\|_{L^p_\phi} \tag{1.4}$$

holds for $p > \frac{\ln 2}{\ln(\sqrt{13}-1)-\ln 2} = 2.62\ldots$ where $a(t) \sim b(t)$ means that there exists a constant $c > 0$ with $ca(t) \le b(t) \le c^{-1}a(t)$. (1.4) is a strong converse inequality of type A in the sense of [Di-Iv]. In addition, we will show the strong converse inequality of type B

$$K\big(f,\phi(t)\big) \sim \|A_t f - f\|_{L^p_\phi} + \|A_{\rho t} f - f\|_{L^p_\phi}$$

for $1 \le p \le \frac{\ln 2}{\ln(\sqrt{13}-1)-\ln 2}$, where $\rho \in (0,1]$ is a constant independent of t and f.

Equivalence relations with K-functionals of higher orders are achieved if the operators $A_{r,t}$ are defined as linear combinations of A_u with different variables u. For that we need to define a scalar multiplication $n \odot t$, $n \in N_0$, which matches the addition $\oplus$. The multiplication is given by

$$n \odot t := \underbrace{t \oplus \cdots \oplus t}_{n \text{ times}}, \quad t \in [-1,1], \tag{1.5}$$

for $n \in N$ and $n \odot t := 0$ for $n = 0$. For average operators $A_{r,t}$ of order $r \in N$, defined by

$$A_{r,t} := \sum_{k=1}^{r} \binom{r}{k} (-1)^{k+1} A_{k \odot t} \tag{1.6}$$

we will show, in Section 5, that there exists a $\rho \in (0,1]$ such that

$$K^r\big(f,\phi(t)^r\big) \sim \|A_{r,t} f - f\|_{L^p_\phi} + \|A_{r,\rho t} f - f\|_{L^p_\phi}. \tag{1.7}$$

From this strong converse inequality of type B we will show, in Section 6, a strong converse relation of type D, namely

$$K^r\big(f,\phi(t)^r\big) \sim \sup_{0<u\le t} \|\Delta_u^r f\|_{L^p_\phi} \tag{1.8}$$

where $(\Delta_u^r f)(x) := \sum_{k=0}^{r} \binom{r}{k} (-1)^{k+1} f\big(x \oplus (k \odot u)\big)$ is a difference operator of order r, that is to say, the right hand side of (1.8) is a modulus of smoothness which is equivalent to the K-functional in question.

The addition $\oplus$ in (1.1) is defined by means of the function ϕ as follows. From the assumptions that $\phi \in AC[-1,1]$ is odd and $\phi' > 0$ a.e., it follows that ϕ maps the unit interval $[-1,1]$ bijectively onto $[-l,l]$, $l = \phi(1)$ and $\phi^{-1}\colon [-l,l] \to [-1,1]$ is also an absolutely continuous function. Then $\oplus\colon [-1,1]^2 \to [-1,1]$ is defined as

$$a \oplus b := \phi^{-1}\big(\phi(a) + \phi(b)\big) \quad \text{for } a,b \in [-1,1] \tag{1.9}$$

where $\phi^{-1}\colon R \to [-1,1]$ is interpreted as a $4l$-periodic function (*i.e.*, $\phi^{-1}(t+4l) = \phi^{-1}(t)$, $t \in R$) which satisfies

$$\phi^{-1}(t) = \phi^{-1}(2l - t) \quad \text{for } t \in (l, 3l]. \tag{1.10}$$

The extension (1.10) of ϕ from $[-l,l]$ to a $4l$-periodic function on R makes (1.9) well-defined because $\phi(a) + \phi(b)$ can lie outside of the interval $[-l,l]$. Because of (1.10) the inverse ϕ^{-1} satisfies the relations $\phi^{-1}(l+x) = \phi^{-1}(l-x)$ and $\phi^{-1}(-l-x) = \phi^{-1}(-l+x)$ for all $x \in [0,l]$.

The properties of the addition (1.9) have been discussed in [Fe1]. It should be mentioned that the addition does not yield a group in $[-1,1]$ because the associative law is not fulfilled on the whole interval (see [Fe1]). However, $\oplus$ has the following group properties: $a \oplus b = b \oplus a$, $a \oplus 0 = a$ and $a \ominus a = 0$ if we define $a \ominus b$ by $a \oplus (-b)$. Further discussions can be found in [Fe1].

Below are some special cases of the measures $d\phi$. Obviously, in all cases $d\phi$ is a positive measure on the unit interval $[-1,1]$ and we can write $d\phi(x) = w(x)dx$ where $w\colon [-1,1] \to R$ is the weight $w = \phi'$. If $\phi(x) = x$, the weight w is equal to 1 and the averages (1.1), namely

$$(A_t f)(x) = \frac{1}{t}\int_0^t f(x \oplus u)\,du, \quad x \in [-1,1],$$

are Steklov functions and $x \oplus u$ is given by

$$x \oplus u = \begin{cases} x+u, & x+u \in [-1,1] \\ 2-(x+u), & x+u \in [1,2] \end{cases}$$

for $u \in [0,t]$, which means that $x \oplus u$ is the ordinary addition $x+u$, if $x+u$ lies within $[-1,1]$, otherwise $x+u$ is mirrored back to $[-1,1]$ at 1. The differential operator (1.3) is equal to the ordinary derivative, *i.e.*, $Df = f'$ and it follows that $\|f - A_{r,t}f\|_p + \|f - A_{r,\rho t}f\|_p$ for some $\rho \in (0,1]$ is equivalent to the K-functional

$$K^r(f,t^r) = \inf\{\|f-g\|_p + t^r\|g^{(r)}\|_p \mid g^{(r)} \in L^p[-1,1]\}$$

and to the modulus $\sup_{0<u\le t}\|\Delta_u^r f\|_{L^p_\phi}$ which is in essence the ordinary modulus of smoothness. This shows that the unweighted case $w=1$ gives results which are comparable to those of the classical case.

If $d\phi$ is the arcsin measure, then $w(x) = 1/\sqrt{1-x^2}$, $(Df)(x) = \sqrt{1-x^2}\,f'(x)$,

$$(A_t f)(x) = \frac{1}{\arcsin t}\int_0^t f(x \oplus u)\frac{du}{\sqrt{1-u^2}} \quad x \in [-1,1],$$

and $x \oplus u = x \cdot \sqrt{1-u^2} + u \cdot \sqrt{1-x^2}$. This specific addition is connected with best approximation by algebraic polynomials on $[-1,1]$ and has been considered in [Fe2] and [Fe3] in relation with moduli of smoothness. This translation can be considered in a more general frame (see [Fe1]) if we take as $d\phi$ the measure $d\arcsin g(x)$ where $g\colon [-1,1] \to [-1,1]$ is an odd and absolutely continuous function with $g' > 0$ a.e. Then (1.9) can be represented by

$$a \oplus b = g^{-1}\left(g(a) \cdot \sqrt{1 - \big(g(b)\big)^2} + g(b) \cdot \sqrt{1 - \big(g(a)\big)^2}\right)$$

for each $a, b \in [-1, 1]$. For $g(x) = x^\alpha$, $\alpha = 1, 3, 5, \dots$, in particular, we obtain

$$a \oplus b = \left(a^\alpha \cdot \sqrt{1 - b^{2\alpha}} + b^\alpha \cdot \sqrt{1 - a^{2\alpha}}\right)^{\frac{1}{\alpha}}.$$

It should be noted that the investigation of the averages (1.1) and (1.6) was mainly influenced by the ideas of Ditzian and Runovskii in [Di-Ru]. Many of their techniques could be transferred to the case of weighted spaces which will be considered here.

2 Preliminary Considerations

First we must obtain some elementary relations of derivatives of averages and properties of the operator O_t, given in (2.4), which will be used frequently in the following sections. Moreover, some properties and techniques of the addition (1.9) will be discussed.

We begin with

Theorem 2.1 *Let $f \in L^1_\phi[-1, 1]$ and $t \in (0, 1]$. Then $A_t f$, given in (1.1), is absolutely continuous on $[-1, 1]$ and fulfills*

$$(DA_t f)(x) = \frac{f(x \oplus t) - f(x)}{\phi(t)}, \quad x \in [-1, 1] \text{ a.e.}, \tag{2.1}$$

$$D^2 A_t f = A_t D^2 f, \quad \text{if } D^2 f \text{ exists}, \tag{2.2}$$

$$A_t f - f = O_t(DA_u f), \tag{2.3}$$

whereby the operator O_t is defined by

$$O_t(g_u)(x) := \frac{1}{\phi(t)} \int_0^t \phi(u) g_u \, d\phi(u). \tag{2.4}$$

Proof Equation (2.1) has been proved in [Fe1, Theorem 4].

Let $t \in (0, 1]$ arbitrary but fixed. To prove (2.2) we introduce the translation

$$\tau_u \colon [-1, 1] \to [-1, 1], \quad \tau_u(x) := x \oplus u.$$

For the sake of brevity we omit x in the notation of $(A_t f)(x)$ in (1.1) and write

$$A_t f = \frac{1}{\phi(t)} \int_0^t f \circ \tau_u \, d\phi(u). \tag{2.5}$$

The derivative of $\tau_u(x)$ with respect to x is given by (see [Fe1, Theorem 2])

$$\tau_u'(x) = \begin{cases} \dfrac{\phi'(x)}{\phi'(x \oplus u)}, & x \in [-1, 1 \ominus u] \\ -\dfrac{\phi'(x)}{\phi'(x \oplus u)}, & x \in (1 \ominus u, 1] \end{cases} \quad \text{a.e.}$$

or rewritten

$$\tau_u'(x) = \mu_u(x) \cdot \frac{\phi'(x)}{\phi'(x \oplus u)} \quad \text{a.e.} \tag{2.6}$$

if we define the sign function by $\mu_u(x) := \begin{cases} +1, & x \in [-1, 1 \ominus u] \\ -1, & x \in (1 \ominus u, 1] \end{cases}$. Then (2.6) and definition (1.3) of D give

$$D(f \circ \tau_u) = \mu_u \cdot (Df) \circ \tau_u \tag{2.7}$$

and consequently

$$D^2(f \circ \tau_u) = (\mu_u)^2 \cdot (D^2 f) \circ \tau_u = (D^2 f) \circ \tau_u. \tag{2.8}$$

Now, from (2.5) and (2.8) it follows that

$$D^2(A_t f) = \frac{1}{\phi(t)} \int_0^t (D^2 f) \circ \tau_u \, d\phi(u) = A_t(D^2 f),$$

proving (2.2). Now, we use $A_t 1 = 1$ and (2.1) to obtain

$$\begin{aligned} (A_t f)(x) - f(x) &= \frac{1}{\phi(t)} \int_0^t \big(f(x \oplus u) - f(x)\big) \, d\phi(u) \\ &= \frac{1}{\phi(t)} \int_0^t \phi(u)(DA_u f)(x) \, d\phi(u) \\ &= O_t(DA_u f) \end{aligned}$$

which leads to (2.3) and concludes the proof of our theorem. ■

The results of Theorem 2.1 will be crucial for direct and converse estimates. In particular, equation (2.3) will be the starting point for the investigation of properties of the averages in Section 3. A similar connection between averages, derivatives and O_t of Theorem 2.1 can be found in [Di-Ru] and [Di-Fe].

The notation of the operator O_t in (2.4) has shown itself to be useful (see [Di-Ru], [Di-Fe]) because we do not have to write so many integration signs if we consider iterations $O_t\big(O_u(1)\big)$, $O_t\Big(O_u\big(O_\eta(1)\big)\Big)$, *etc.* Furthermore, for the sake of brevity, we will use the notation $O_{t_1} \cdots O_{t_n}(1)$ instead of $O_{t_1}\Big(\cdots\big(O_{t_n}(1)\big)\cdots\Big)$.

It should be noted that the function $O_t(g_u)(x)$ does not depend on the variable u because u is just an inner variable for integration (see definition (2.4)). Moreover, if g_u is independent of x then so is $O_t(g_u)(x)$. In particular, the function $O_t(1)$ is realvalued and depends only on t. Hence, $O_{t_1} \cdots O_{t_n}(1)$ is a function which depends only on t_1 but not on $t_2, \ldots, t_n$. The following lemma gives an explicit representation of these functions.

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Lemma 2.1 *For $t_1 \in (0,1]$ and $k \in N$ we have*

$$O_{t_1} O_{t_2} \cdots O_{t_k}(1) = \frac{1}{(k+1)!}\phi(t_1)^k. \tag{2.9}$$

Proof We will prove the statement by induction with respect to k. From

$$\begin{aligned} O_{t_1}(1) &= \frac{1}{\phi(t_1)} \int_0^{t_1} \phi(u)\phi'(u)\,du \\ &= \frac{1}{\phi(t_1)} \frac{1}{2}\big(\phi(t_1)^2 - \phi(0)^2\big) = \frac{1}{2}\phi(t_1)^2 \end{aligned}$$

we obtain (2.9) for $k = 1$. Now let the statement hold true for $k \geq 1$. Then

$$\begin{aligned} O_{t_1}\big(O_{t_2} \cdots O_{t_{k+1}}(1)\big) &= \frac{1}{(k+1)!} O_{t_1}\big(\phi(t_2)^k\big) \\ &= \frac{1}{(k+1)!} \frac{1}{\phi(t_1)} \int_0^{t_1} \big(\phi(t_2)\big)^{k+1} \phi'(t_2)\,dt_2 \\ &= \frac{1}{(k+2)!}\phi(t_1)^{k+1}. \end{aligned}$$

■

Lemma 2.2 *Let $n \in N_0$ and suppose $D^n f$ exists in $L^p_\phi[-1,1]$. Then*

$$\|O_{t_1} O_{t_2} \cdots O_{t_k}(D^n A_t f)\|_{L^p_\phi} \leq \frac{2^{1/p}}{(k+1)!}\phi(t_1)^k \, \|D^n f\|_{L^p_\phi}$$

for $k \in N$ and $t_1 \in (0,1]$.

Proof For each $t \in (0,1]$ let $g_t \colon [-1,1] \to R$ be a function $g_t \in L^p_\phi[-1,1]$. Taking into account that

$$\|O_{t_1}(g_t)\|_{L^p_\phi} \leq \frac{1}{\phi(t_1)} \int_0^{t_1} \phi(t)\, \|g_t\|_{L^p_\phi} \, d\phi(t)$$

we obtain

$$\|O_{t_1}(g_t)\|_{L^p_\phi} \leq O_{t_1}(\|g_t\|_{L^p_\phi}) \leq O_{t_1}(1) \sup_{0<t\leq t_1} \|g_t\|_{L^p_\phi}$$

and

$$\begin{aligned} \|O_{t_1} O_{t_2} \cdots O_{t_k}(g_t)\|_{L^p_\phi} &\leq O_{t_1} O_{t_2} \cdots O_{t_k}(\|g_t\|_{L^p_\phi}) \\ &\leq O_{t_1} O_{t_2} \cdots O_{t_k}(1) \cdot \sup_{0<t\leq t_1} \|g_t\|_{L^p_\phi} \\ &= \frac{1}{(k+1)!}\phi(t_1)^k \sup_{0<t\leq t_1} \|g_t\|_{L^p_\phi}, \end{aligned} \tag{2.10}$$

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where we used Lemma 2.1 in the last step.

We need the inequality

$$\|f(\bullet \oplus u)\|_{L^p_\phi} \leq 2^{1/p} \|f\|_{L^p_\phi}, \quad u \in [-1, 1], \tag{2.11}$$

which was proved in [Fe1, eq. (19)]. From (2.7) and (2.11) it follows that

$$\begin{aligned} \|D^n A_t f\|_{L^p_\phi} &= \left\| \frac{1}{\phi(t)} \int_0^t D^n\{f(\bullet \oplus u)\}\, d\phi(u) \right\|_{L^p_\phi} \\ &= \left\| \frac{1}{\phi(t)} \int_0^t \mu_u^n \cdot (D^n f)(\bullet \oplus u)\, d\phi(u) \right\|_{L^p_\phi} \\ &\leq 2^{1/p} \|D^n f\|_{L^p_\phi} \end{aligned}$$

which, with (2.10) for $g_t = D^n A_t f$, concludes the proof of our lemma. ∎

We have already mentioned in the introductory section that the associative law of the addition $\oplus$ is not valid on $[-1, 1]$, which means that the order of the parentheses of the sum $\big(\cdots(x_1 \oplus x_2) \oplus \cdots\big) \oplus x_n$ cannot in general be omitted. However, we find that the associative law is satisfied on certain subintervals $[-\delta_n, \delta_n]$ of $[-1, 1]$ which depend on the number n of summands. The following lemma makes this clear.

Lemma 2.3 *Let $n \in N$ and $\delta_n := \phi^{-1}\big(\frac{\phi(1)}{n}\big)$. Then the associative law of n summands in $[-\delta_n, \delta_n]$ with respect to $\oplus$ holds true, i.e., whenever $x_1, \ldots, x_n \in [-\delta_n, \delta_n]$ the parentheses of $\big(\cdots(x_1 \oplus x_2) \oplus \cdots\big) \oplus x_n$ may be omitted without ambiguity.*

Proof Obviously, $[-\delta_n, \delta_n] \subset [-1, 1]$. If $x_j \in [-\delta_n, \delta_n]$, $j = 1, \ldots, n$, then $n \cdot \phi(x_j) \in [-\phi(1), \phi(1)] = [-l, l]$. Since ϕ^{-1} maps $[-l, l]$ onto $[-1, 1]$ bijectively it follows, as can be easily seen from (1.9), that

$$\big(\cdots(x_1 \oplus x_2) \oplus \cdots\big) \oplus x_n = \phi^{-1}\big(\phi(x_1) + \cdots + \phi(x_n)\big). \tag{2.12}$$

The order of the summands of the right hand side of (2.12) can be changed, which concludes the proof of the lemma. ∎

The next lemma shows a relationship between the $\odot$, given in (1.5), and the ordinary multiplication.

Lemma 2.4 *Let $n \in N$. We then obtain*

$$\phi(k \odot t) = k \cdot \phi(t)$$

for $k = 0, \ldots, n$ and $t \in [0, \delta_n]$ with $\delta_n := \phi^{-1}\big(\frac{\phi(1)}{n}\big)$.

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Proof We will prove the lemma by induction with respect to n. Obviously, for $n = 1$ the equation $\phi(k \odot t) = k \cdot \phi(t)$ holds for $k \in \{0, 1\}$ and $t \in [0, \delta_1] = [0, 1]$. Let the assertion hold true for n. We must now show that $\phi(k \odot t) = k \cdot \phi(t)$ is fulfilled for $k = 0, \ldots, n+1$ and $t \in [0, \delta_{n+1}]$. Because of $[0, \delta_{n+1}] \subset [0, \delta_n]$ we must now show that $\phi\big((n+1) \odot t\big) = (n+1) \cdot \phi(t)$, $t \in [0, \delta_{n+1}]$. Indeed,

$$\phi\big((n+1) \odot t\big) = \phi\big((n \odot t) \oplus t\big) = \phi\Big(\phi^{-1}\big(\phi(n \odot t) + \phi(t)\big)\Big)$$
$$= \phi\Big(\phi^{-1}\big((n+1)\phi(t)\big)\Big).$$

In view of $(n+1)\phi(t) \in [0, \phi(1)] = [0, l]$ for $t \in [0, \delta_{n+1}]$ and the fact that ϕ maps $[0, l]$ bijectively onto $[0, 1]$ we obtain

$$\phi\big((n+1) \odot t\big) = (n+1)\phi(t). \qquad \blacksquare$$

Lemma 2.5 *Let $n \in N$. Then*

$$\int_0^{k \odot t} f(u)\, d\phi(u) = k \int_0^t f(k \odot u)\, d\phi(u)$$

for $k = 0, \ldots, n$ and $t \in [0, \delta_n]$ with $\delta_n := \phi^{-1}\big(\frac{\phi(1)}{n}\big)$.

Proof Firstly, for $n = 2, 3, \ldots$ we will prove the following identity:

$$\frac{d}{dx}(x \oplus t) = \frac{\omega(x)}{\omega(x \oplus t)} \quad \text{for } x, t \in [0, \delta_n] \tag{2.13}$$

in which w denotes the weight function ϕ'. With respect to (2.6) we must show that $x \in [-1, 1 \ominus t]$ or $x \le 1 \ominus t$ respectively. This is done if we show that $\delta_n \le 1 \ominus \delta_n$ because $x \le \delta_n$ and $t \le \delta_n$. Indeed, the inequality

$$\delta_n = \phi^{-1}\left(\frac{\phi(1)}{n}\right) \le 1 \ominus \phi^{-1}\left(\frac{\phi(1)}{n}\right) = \phi^{-1}\left(\left(1 - \frac{1}{n}\right)\phi(1)\right)$$

is satisfied for $n = 2, 3, \ldots$ due to the monotonicity of ϕ^{-1}. Hence, equation (2.13) is established.

Let us now prove the assertion of our lemma. There is nothing to prove for $n = 1$. Let $n \ge 2$ and let us define the function

$$g\colon [0, \delta_n]^k \to [-1, 1], \quad g(u_1, \ldots, u_k) := u_1 \oplus \cdots \oplus u_k$$

whereby, in view of Lemma 2.3, the parentheses in the sum can be omitted, *i.e.*, $u_1 \oplus \cdots \oplus u_k$ is well-defined. Moreover, we can rearrange the order of the summands since $\oplus$ is commutative. Then, from (2.13) it follows that

$$\frac{dg}{du_j}(u_1, \ldots, u_k) = \frac{\omega(u_k)}{\omega(u_1 \oplus \cdots \oplus u_k)}, \quad j = 1, \ldots, k,$$

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and consequently for $u \in [0, \delta_n]$

$$\frac{d}{du}(k \odot u) = \sum_{j=1}^{k} \frac{dg}{du_j}(u, \dots, u) = \sum_{j=1}^{k} \frac{\omega(u)}{\omega(u \oplus \cdots \oplus u)} = k \frac{\omega(u)}{\omega(k \odot u)}.$$

Finally, this yields

$$\begin{aligned} \int_0^{k\odot t} f(u)\, d\phi(u) &= \int_0^{k\odot t} f(u)\omega(u)\, du \\ &= \int_0^t f(k \odot u) \left\{ \frac{d}{du}(k \odot u) \right\} \omega(k \odot u)\, du \\ &= k \int_0^t f(k \odot u) \omega(u)\, du. \end{aligned}$$

■

A similar argument as in the last step of the previous proof shows that

$$\int_x^{x\oplus t} f(u)\, d\phi(u) = \int_0^t f(x \oplus u)\, d\phi(u) \tag{2.14}$$

is satisfied.

3 Properties of Averages

The relation (2.3) between the error of $f - A_t f$ and the operator O_t given in (2.4) is the motivation for the present paper and in particular this section. Here we will use (2.3) to obtain Theorem 3.1 which will yield several other important results needed for strong converse relations later on. The technique used in this section stems from the paper [Di-Ru]. Similar ideas have also been used in [Di-Fe].

We begin with the following Taylor-type formula.

Theorem 3.1 *Suppose $D^n f$ exists. Then, for $t \in (0, 1]$ we have*

$$A_t f - f - \sum_{j=1}^{n} \frac{D^j f}{(j+1)!} \phi(t)^j = O_t O_{t_1} \cdots O_{t_n} D^{n+1} A_{t_{n+1}} f$$

for $n = 1, 2, \dots$.

Proof For $n = 1$ it follows from (2.3) that

$$A_t f = O_t(D A_{t_1} f) + f$$

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and iteration of this formula gives

$$\begin{aligned} A_t f &= O_t D O_{t_1}(D A_{t_2} f) + O_t(Df) + f \\ &= O_t O_{t_1}(D^2 A_{t_2} f) + O_t(1)\cdot Df + f \\ &\ \vdots \\ &= O_t O_{t_1}\cdots O_{t_n} D^{n+1} A_{t_{n+1}} f + \sum_{j=1}^{n} O_t O_{t_1}\cdots O_{t_{j-1}}(1)\cdot D^j f. \end{aligned}$$

Lemma 2.1 concludes the proof of our theorem. ■

To achieve results for higher levels of smoothness, combinations of averages are considered. We define the average operator of order $n \in N$ by

$$A_{n,t} := \sum_{k=1}^{n} \binom{n}{k} (-1)^{k+1} A_{k\odot t}, \quad t \in (0,1]. \tag{3.1}$$

In particular,

$$\begin{aligned} A_{1,t} &= A_t, \\ A_{2,t} &= 2A_t - A_{2\odot t}, \\ A_{3,t} &= 3A_t - 3A_{2\odot t} + A_{3\odot t}. \end{aligned}$$

As a consequence of Theorem 3.1 we obtain the following corollary for averages of higher orders.

Corollary 3.1 *Suppose $D^{n-1}f$ exists. Then,*

$$A_{n,t} f - f = \sum_{k=1}^{n} \binom{n}{k} (-1)^{k+1} O_{k\odot t} O_{t_1} \cdots O_{t_{n-1}} D^n A_{t_n} f$$

for $n = 2, 3, \ldots$ and $t \in (0, \delta_n]$ with $\delta_n := \phi^{-1}\big(\frac{\phi(1)}{n}\big)$.

Proof With the aid of Theorem 3.1 and Lemma 2.4 we obtain

$$A_{k\odot t} f - f - \sum_{j=1}^{n-1} \frac{D^j f}{(j+1)!} k^j \phi(t)^j = O_{k\odot t} O_{t_1} \cdots O_{t_{n-1}} D^n A_{t_n} f$$

for $k = 1, 2, \ldots, n$ and $n \geq 2$. Multiplication of both sides by $\binom{n}{k}(-1)^{k+1}$ and summation from $k = 1$ up to n gives (using (3.1) and $\sum_{k=1}^{n} \binom{n}{k}(-1)^{k+1} = 1$)

$$\begin{aligned} A_{n,t} t - f - \sum_{j=1}^{n-1} \frac{D^j f}{(j+1)!} \phi(t)^j \sum_{k=1}^{n} \binom{n}{k} (-1)^{k+1} k^j \\ = \sum_{k=1}^{n} \binom{n}{k} (-1)^{k+1} O_{k\odot t} O_{t_1} \cdots O_{t_{n-1}} D^n A_{t_n} f. \end{aligned}$$

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From $\sum_{k=1}^{n}\binom{n}{k}(-1)^{k+1}k^j = 0$ for $j = 1,\dots,n-1$ we obtain the assertion. ■

Corollary 3.2 *Suppose $D^n f$ exists. Then,*

$$A_{n,t}f - f - \frac{D^n f}{n+1}\phi(t)^n = \sum_{k=1}^{n}\binom{n}{k}(-1)^{k+1}O_{k\odot t}O_{t_1}\cdots O_{t_n}D^{n+1}A_{t_{n+1}}f$$

for $n = 1,2,\dots$ and $t \in (0,\delta_n]$ with $\delta_n := \phi^{-1}\big(\frac{\phi(1)}{n}\big)$.

Proof From Theorem 3.1 and Lemma 2.4 we see that

$$A_{k\odot t}f - f - \sum_{j=1}^{n}\frac{D^j f}{(j+1)!}k^j\phi(t)^j = O_{k\odot t}O_{t_1}\cdots O_{t_n}D^{n+1}A_{t_{n+1}}f.$$

A similar argument to the one used in the proof of the previous Corollary 3.1 yields

$$A_{n,t}t - f - \sum_{j=1}^{n}\frac{D^j f}{(j+1)!}\phi(t)^j\sum_{k=1}^{n}\binom{n}{k}(-1)^{k+1}k^j$$
$$= \sum_{k=1}^{n}\binom{n}{k}(-1)^{k+1}O_{k\odot t}O_{t_1}\cdots O_{t_n}D^{n+1}A_{t_{n+1}}f.$$

Since $\sum_{k=1}^{n}\binom{n}{k}(-1)^{k+1}k^n = n!$ we obtain the corollary. ■

4 The Direct Result

Direct estimates for averages are obtained from the results in the previous sections, in particular from Corollary 3.1.

Theorem 4.1 *Let $f \in L^p_\phi[-1,1]$. Then, for averages (1.1) of first order, we have*

$$\|A_t f - f\|_{L^p_\phi} \le (1+2^{1/p})K\big(f,\phi(t)\big), \quad t \in (0,1].$$

Let $r \in N$ and $r \ge 2$. Then, for averages $A_{r,t}f$, given in (1.6), we have

$$\|A_{r,t}f - f\|_{L^p_\phi} \le c_r 2^{1/p}K^r\big(f,\phi(t)^r\big), \quad t \in (0,\delta_r],$$

where $\delta_r = \phi^{-1}\big(\frac{\phi(1)}{r}\big)$ and $c_r = \max\{1+2^{1/p}(2^r-1), \frac{1}{(r+1)!}\sum_{k=1}^{r}\binom{r}{k}k^r\}$.

Proof We begin by proving direct estimates for smooth functions, namely

$$\|A_{r,t}g - g\|_{L^p_\phi} \le \tilde{c}_r 2^{1/p}\phi(t)^r\,\|D^r g\|_{L^p_\phi}, \quad t \in (0,\delta_r], \tag{4.1}$$

if $D^r g$ exists, $r \in N$ and $\tilde{c}_r = \frac{1}{(r+1)!}\sum_{k=1}^{r}\binom{r}{k}k^r$.

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If Dg exists, then equation (2.3) in Theorem 2.1 yields the direct estimate for the error $A_t g - g$ as follows: In conjunction with Lemma 2.2 for $n = k = 1$ we have

$$\|A_t g - g\|_{L^p_\phi} \le \frac{1}{2}\phi(t)2^{1/p}\,\|Dg\|_{L^p_\phi}, \quad 0 < t \le 1. \tag{4.2}$$

Because of $A_{1,t} = A_t$, inequality (4.2) implies (4.1) for $r = 1$.

For $r \ge 2$ we use Corollary 3.1, Lemma 2.2 and Lemma 2.4 to obtain

$$\begin{aligned}\|A_{r,t} g - g\|_{L^p_\phi} &\le \sum_{k=1}^{r}\binom{r}{k}\left\|O_{k\odot t}O_{t_1}\cdots O_{t_{r-1}}D^r A_{t_r} g\right\|_{L^p_\phi}\\ &\le \sum_{k=1}^{r}\binom{r}{k}\frac{2^{1/p}}{(r+1)!}k^r\phi(t)^r\,\|D^r g\|_{L^p_\phi}\\ &= \tilde{c}_r 2^{1/p}\phi(t)^r\,\|D^r g\|_{L^p_\phi}\end{aligned}$$

which establishes the validity of (4.1).

To prove the direct result for functions $f \in L^p_\phi[-1,1]$ we estimate the norm of the averages $A_{r,t} f$. From definition (1.1) and inequality (2.11) we obtain

$$\|A_t f\|_{L^p_\phi} \le 2^{1/p}\,\|f\|_{L^p_\phi}$$

and, analogously, from (1.6)

$$\|A_{r,t} f\|_{L^p_\phi} \le 2^{1/p}(2^r - 1)\,\|f\|_{L^p_\phi}.$$

Finally, by making use of (4.1), we arrive at

$$\begin{aligned}\|A_{r,t} f - f\|_{L^p_\phi} &\le \|(A_{r,t} - I)(f-g) + (A_{n,t} - I)g\|_{L^p_\phi}\\ &\le \left(2^{1/p}(2^r-1)+1\right)\|f-g\|_{L^p_\phi} + \tilde{c}_r 2^{1/p}\phi(t)^r\,\|D^r g\|_{L^p_\phi}\\ &\le c_r 2^{1/p}\left(\|f-g\|_{L^p_\phi} + \phi(t)^r\,\|D^r g\|_{L^p_\phi}\right).\end{aligned}$$

Taking the infimum on both sides over all smooth functions g with $D^r g \in L^p_\phi[-1,1]$ we obtain the result of our theorem for $r = 1$ and $r \ge 2$. ■

The following Voronovskaja-type estimate will be of importance when proving strong converse inequalities of type A and B in Section 5.

Theorem 4.2 *Let $r \in N$ and suppose that $D^{r+1}g$ exists. Then*

$$\left\|A_{r,t} g - g - \frac{\phi(t)^r}{r+1}D^r g\right\|_{L^p_\phi} \le d_r 2^{1/p}\phi(t)^{r+1}\left\|D^{r+1} g\right\|_{L^p_\phi}$$

for all t with $0 < t \le \delta_r = \phi^{-1}\left(\frac{\phi(1)}{r}\right)$ where $d_r = \frac{1}{(r+2)!}\sum_{k=1}^{r}\binom{r}{k}k^{r+1}$.

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Proof With the aid of Corollary 3.2, Lemma 2.2 and Lemma 2.4 we have

$$\begin{aligned}\left\|A_{r,t}g - g - \frac{\phi(t)^r}{r+1}D^r g\right\|_{L^p_\phi} &\le \sum_{k=1}^{r}\binom{r}{k}\left\|O_{k\odot t}O_{t_1}\cdots O_{t_r}D^{r+1}A_{t_{r+1}}g\right\|_{L^p_\phi}\\ &\le \sum_{k=1}^{r}\binom{r}{k}\frac{2^{1/p}}{(r+2)!}k^{r+1}\phi(t)^{r+1}\left\|D^{r+1}g\right\|_{L^p_\phi}\\ &= d_r 2^{1/p}\phi(t)^{r+1}\left\|D^{r+1}g\right\|_{L^p_\phi}.\end{aligned}$$

■

From Theorem 3.1 we also have the estimate

$$\left\|A_t g - g - \sum_{k=1}^{r}\frac{D^k g}{(k+1)!}\phi(t)^k\right\|_{L^p_\phi} \le \frac{2^{1/p}}{(r+2)!}\phi(t)^{r+1}\left\|D^{r+2}g\right\|_{L^p_\phi},$$

if $D^{r+2}g$ exists.

5 Strong Converse Inequalities of Types A and B

In this section we will investigate strong converse inequalities of types A and B in the sense of [Di-Iv] for the approximation process $A_{r,t}f - f$. First we will establish type B, *i.e.*, we will relate the K-functional (1.2) by two terms, $\|A_{r,t}f - f\|_{L^p_\phi}$ and $\|A_{r,\rho t}f - f\|_{L^p_\phi}$ with some ρ. Later, we will establish type A (*i.e.*, one term is sufficient) for averages $A_t f$ of first order if p is not too small, that is $p > 2.62\ldots$.

The strong converse inequality of type B is given in

Theorem 5.1 *Suppose that $f \in L^p_\phi[-1,1]$. Suppose also that the weight $\omega = \phi'$ is equivalent to a positive constant in a neighborhood of 0, i.e., $0 < c \le \omega(t) \le d$ for $t \in (-t_0, t_0)$. Then there exist constants $C > 0$ and $\rho \in (0,1]$ being independent of f and t such that*

$$K^r\big(f,\phi(t)^r\big) \le C(\|A_{r,t}f - f\|_{L^p_\phi} + \|A_{r,\rho t}f - f\|_{L^p_\phi}) \tag{5.1}$$

for all t with $0 < t \le t_0$.

Proof The proof follows a method in Ditzian and Ivanov [Di-Iv] which was developed for proving strong converse inequalities of type B and A.

Obviously,

$$K^r\big(f,\phi(t)^r\big) \le \left\|A^{r+1}_{r,t}f - f\right\|_{L^p_\phi} + \phi(t)^r\left\|D^r A^{r+1}_{r,t}f\right\|_{L^p_\phi}. \tag{5.2}$$

For the first summand in (5.2) we have

$$\left\|A^{r+1}_{r,t}f - f\right\|_{L^p_\phi} = \left\|\sum_{k=0}^{r}A^k_{r,t}(f - A_{r,t}f)\right\|_{L^p_\phi} \le \tilde{C}_1\left\|f - A_{r,t}f\right\|_{L^p_\phi} \tag{5.3}$$

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where $\tilde{C}_1$ is independent of f and t. Thus, the first summand in (5.2) is dominated by a term as claimed in (5.1).

The rest of the proof is concerned with the second summand in (5.2). For an estimation of the second summand we will use the following Voronovskaja inequality of Theorem 4.2 with $A_{r,t}^{r+1} f$ taking the place of g:

$$\left\| A_{r,\rho t} A_{r,t}^{r+1} f - A_{r,t}^{r+1} f - \frac{\phi(\rho t)^r}{r+1} D^r A_{r,t}^{r+1} f \right\|_{L_\phi^p} \le d_r 2^{1/p} \phi(\rho t)^{r+1} \left\| D^{r+1} A_{r,t}^{r+1} f \right\|_{L_\phi^p}$$

for $t \in (0, \delta_r]$ and $\rho \in (0, 1]$. Let $j \in N$. Using (1.6), equation (2.1), with $f(x \oplus t) - f(x) =: (\Delta_t f)(x)$, and Lemma 2.4 we have

$$\begin{aligned} \left\| D^j A_{r,t}^j f \right\|_{L_\phi^p} &= \left\| D^{j-1} D A_{r,t} A_{r,t}^{j-1} f \right\|_{L_\phi^p} \\ &= \left\| D^{j-1} \sum_{k=1}^{r} \binom{r}{k} (-1)^{k+1} \frac{1}{\phi(k \odot t)} \Delta_{k \odot t} A_{r,t}^{j-1} f \right\|_{L_\phi^p} \\ &\le \frac{1}{\phi(t)} \sum_{k=1}^{r} \binom{r}{k} \frac{1}{k} \left\| D^{j-1} \Delta_{k \odot t} A_{r,t}^{j-1} f \right\|_{L_\phi^p}. \end{aligned} \tag{5.4}$$

We note that, by a simple calculation (using (2.11) and (2.7)), $\left\| D^{j-1} \Delta_h g \right\|_{L_\phi^p}$ can be estimated by $(1 + 2^{1/p}) \left\| D^{j-1} g \right\|_{L_\phi^p}$. Therefore, with

$$e_r := \sum_{k=1}^{r} \binom{r}{k} \frac{1}{k}, \tag{5.5}$$

we observe from (5.4) that

$$\left\| D^j A_{r,t}^j f \right\|_{L_\phi^p} \le (1 + 2^{1/p}) e_r \frac{1}{\phi(t)} \left\| D^{j-1} A_{r,t}^{j-1} f \right\|_{L_\phi^p}. \tag{5.6}$$

Repeating inequality (5.6) yields

$$\left\| D^j A_{r,t}^j f \right\|_{L_\phi^p} \le (1 + 2^{1/p})^j e_r^j \frac{1}{\phi(t)^j} \| f \|_{L_\phi^p}$$

for $j = 1, 2, \ldots$. Moreover,

$$\begin{aligned} \left\| D^{r+1} A_{r,t}^{r+1} f \right\|_{L_\phi^p} &\le \frac{1 + 2^{1/p}}{\phi(t)} e_r \left\| D^r A_{r,t}^r f \right\|_{L_\phi^p} \\ &\le \frac{1 + 2^{1/p}}{\phi(t)} e_r \left(\left\| D^r A_{r,t}^{r+1} f \right\|_{L_\phi^p} + \left\| D^r A_{r,t}^r (A_{r,t} f - f) \right\|_{L_\phi^p} \right) \\ &\le \frac{1 + 2^{1/p}}{\phi(t)} e_r \left(\left\| D^r A_{r,t}^{r+1} f \right\|_{L_\phi^p} + \frac{(1 + 2^{1/p})^r}{\phi(t)^r} e_r^r \| A_{r,t} f - f \|_{L_\phi^p} \right). \end{aligned}$$

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Hence, for $\rho \in (0,1]$

$$\left\| A_{r,\rho t} A_{r,t}^{r+1} f - A_{r,t}^{r+1} f - \frac{\phi(\rho t)^r}{r+1} D^r A_{r,t}^{r+1} f \right\|_{L^p_\phi}$$
$$\leq 2^{1/p}(1+2^{1/p}) d_r e_r \phi(\rho t)^r \frac{\phi(\rho t)}{\phi(t)} \left\| D^r A_{r,t}^{r+1} f \right\|_{L^p_\phi} + C_r \left\| A_{r,t} f - f \right\|_{L^p_\phi}.$$

Now, by triangle inequality and (5.3)

$$\left\{ \frac{1}{r+1} - 2^{1/p}(1+2^{1/p}) d_r e_r \frac{\phi(\rho t)}{\phi(t)} \right\} \phi(\rho t)^r \left\| D^r A_{r,t}^{r+1} f \right\|_{L^p_\phi}$$
$$\leq C_r \left\| A_{r,t} f - f \right\|_{L^p_\phi} + \left\| A_{r,\rho t} A_{r,t}^{r+1} f - A_{r,t}^{r+1} f \right\|_{L^p_\phi}$$
$$\leq (C_r + \tilde{C}_1) \left\| A_{r,t} f - f \right\|_{L^p_\phi} + \left\| A_{r,\rho t} A_{r,t}^{r+1} f - A_{r,\rho t} f \right\|_{L^p_\phi} + \left\| A_{r,\rho t} f - f \right\|_{L^p_\phi}$$
$$\leq (C_r + \tilde{C}_1 + \tilde{C}_2) \left\| A_{r,t} f - f \right\|_{L^p_\phi} + \left\| A_{r,\rho t} f - f \right\|_{L^p_\phi}.$$

We may choose $\rho \in (0,1]$ such that

$$2^{1/p}(1+2^{1/p}) d_r e_r \frac{\phi(\rho t)}{\phi(t)} < \frac{1}{r+1}, \quad t \in (0, t_0]. \tag{5.7}$$

This is possible because

$$\frac{\phi(\rho t)}{\phi(t)} = \frac{\phi(\rho t) - \phi(0)}{\phi(t) - \phi(0)} = \rho \cdot \frac{\phi'(\xi_1)}{\phi'(\xi_2)}$$

with $0 < \xi_1 < \rho t$ and $0 < \xi_2 < t$. Using the assumption $0 < c \leq \omega(t) \leq d$ for $t \in [0, t_0)$ we can find a $\rho \in (0,1]$, such that

$$c_2 \rho \leq \frac{\phi(\rho t)}{\phi(t)} \leq c_3 \rho. \tag{5.8}$$

Then, (5.7) implies

$$\left\| D^r A_{r,t}^{r+1} f \right\|_{L^p_\phi} \leq c_4 \frac{1}{\phi(\rho t)^r} \Big((C_r + \tilde{C}_1 + \tilde{C}_2) \left\| A_{r,t} f - f \right\|_{L^p_\phi} + \left\| A_{r,\rho t} f - f \right\|_{L^p_\phi} \Big),$$

showing, with (5.8), (5.3) and (5.2), that there is a $C > 0$, such that

$$K^r\big(f, \phi(t)^r\big) \leq C(\| A_{r,t} f - f \|_{L^p_\phi} + \| A_{r,\rho t} f - f \|_{L^p_\phi}). \qquad \blacksquare$$

In a special case we can improve the result of Theorem 5.1 in the sense that we can choose $\rho = 1$ in (5.1). We will prove that for first order $r = 1$ the second term $\| A_{r,\rho t} f - f \|_{L^p_\phi}$ in (5.1) can be dropped if p is sufficiently large. This is a strong converse inequality of type A.

Theorem 5.2 *Let* $f \in L^p_\phi[-1,1]$ *and* $p > \frac{\ln 2}{\ln(\sqrt{13}-1)-\ln 2} = 2.62\ldots$. *Then there exists a constant* $C > 0$, *such that*

$$K\big(f,\phi(t)\big) \le C\,\|A_t f - f\|_{L^p_\phi}$$

for all $t \in (0,1]$.

Proof Let $r = 1$ and $\rho = 1$ and follow the proof of Theorem 5.1. Since $d_1 = 1/6$ and $e_1 = 1$ (see Theorem 4.2 and (5.5)) the inequality (5.7) reads

$$2^{1/p}(1+2^{1/p})\frac{1}{6} < \frac{1}{2} \tag{5.9}$$

and it can be easily seen that (5.9) is satisfied for $p > \frac{\ln 2}{\ln(\sqrt{13}-1)-\ln 2}$. It is clear that we can choose $t \in (0,1]$. ■

Finally, we combine the results of Theorems 4.1, 5.1 and 5.2 and close this section with

Corollary 5.1 *Let the assumptions of Theorem 5.1 hold. Then there exists a* $\rho \in (0,1]$, *such that*

$$K^r\big(f,\phi(t)^r\big) \sim \|A_{r,t}f - f\|_{L^p_\phi} + \|A_{r,\rho t}f - f\|_{L^p_\phi}$$

for all $t \in (0,t_0]$. *In particular, if* $r = 1$ *and* $p > \frac{\ln 2}{\ln(\sqrt{13}-1)-\ln 2}$ *then*

$$K\big(f,\phi(t)\big) \sim \|A_t f - f\|_{L^p_\phi}$$

for all $t \in (0,1]$.

6 Strong Converse Inequalities of Type D

From the strong converse inequality of type B for $A_{r,t}f - f$ we will derive a strong converse inequality of type D for the differences

$$(\Delta^r_u f)(x) := \sum_{k=0}^{r}\binom{r}{k}(-1)^{k+1} f\big(x \oplus (k \odot u)\big), \quad x,u \in [-1,1], \tag{6.1}$$

which is an estimate of the K-functional by $\sup_{0<u\le t}\|\Delta^r_u f\|_{L^p_\phi}$.

The differences (6.1) are closely connected with the definition (1.6) of the averages $A_{r,t}f$. Using (1.1), Lemma 2.4 and Lemma 2.5 we can write

$$\begin{aligned}(A_{r,t}f - f)(x) &= \sum_{k=1}^{r}\binom{r}{k}(-1)^{k+1}(A_{k\odot t}f - f)(x)\\ &= \sum_{k=1}^{r}\binom{r}{k}(-1)^{k+1}\frac{1}{k\phi(t)}\int_0^{k\odot t}\big(f(x\oplus u) - f(x)\big)\,d\phi(u)\end{aligned}$$

$$= \sum_{k=1}^{r} \binom{r}{k} (-1)^{k+1} \frac{1}{\phi(t)} \int_0^t \Big(f\big(x \oplus (k \odot u)\big) - f(x) \Big)\, d\phi(u)$$

$$= \frac{1}{\phi(t)} \int_0^t (\Delta_u^r f)(x)\, d\phi(u)$$

for $t \in (0, \delta_r]$. Hence, we can represent $A_{r,t} f - f$ by means of an integral over the difference operator. This immediately yields the estimate

$$\|A_{r,t} f - f\|_{L^p_\phi} \le \sup_{0<u\le t} \|\Delta_u^r f\|_{L^p_\phi}. \tag{6.2}$$

The right hand side of (6.2) is a modulus of smoothness. We now can relate the left hand side to the K-functional $K^r\big(f, \phi(t)^r\big)$ via the strong converse inequality of type B, *i.e.*, from Theorem 5.1 we deduce that

$$K^r\big(f, \phi(t)^r\big) \le C \sup_{0<u\le t} \|\Delta_u^r f\|_{L^p_\phi}. \tag{6.3}$$

The following Corollary shows that also a lower estimate of the K-functional is possible. Hence, K-functional $K^r\big(f, \phi(t)^r\big)$ and modulus $\sup_{0<u\le t} \|\Delta_u^r f\|_{L^p_\phi}$ are equivalent, which is a strong converse inequality of type D.

Theorem 6.1 *Suppose that $f \in L^p_\phi[-1,1]$ and suppose that the weight $\omega = \phi'$ is equivalent to a positive constant in a neighborhood of 0, i.e., $0 < c \le \omega(t) \le d$ for $t \in (-t_0, t_0)$. Then*

$$K^r\big(f, \phi(t)^r\big) \sim \sup_{0<u\le t} \|\Delta_u^r f\|_{L^p_\phi} \tag{6.4}$$

for $t \in [0, t_0]$.

Proof It remains to show that a lower estimate of (6.3) is satisfied, *i.e.*,

$$\sup_{0<u\le t} \|\Delta_u^r f\|_{L^p_\phi} \le C K^r\big(f, \phi(t)^r\big). \tag{6.5}$$

To establish the lower estimate (6.5), let us define the measure $d(u_1, \dots, u_r)$ by $d\phi(u_1)\cdots d\phi(u_r)$. Clearly, (6.5) holds true for $t = 0$. Therefore let $t \in (0, t_0]$ arbitrary but fixed. Below, x is always in $[-1,1]$. If $D^r g$ exists, we will prove that

$$\int_0^t \cdots \int_0^t (D^r g)\big(x \oplus (u_1 \oplus \cdots \oplus u_r)\big)\, d(u_1, \dots, u_r) = (\Delta_t^r g)(x) \tag{6.6}$$

holds for $r = 0, 2, 4, \dots$ and

$$\int_0^t \cdots \int_0^t \mu_{u_1 \oplus \cdots \oplus u_r}(x) (D^r g)\big(x \oplus (u_1 \oplus \cdots \oplus u_r)\big)\, d(u_1, \dots, u_r) = (\Delta_t^r g)(x) \tag{6.7}$$

holds for $r = 1, 3, 5, \ldots$ (for definition of μ see (2.6)).

It is important to note that $\Delta_t^r \neq (\Delta_t)^r$, $r \geq 2$. However, by defining an appropriate functional we can represent Δ_t^r in terms of a power. If we let the functional $\bar{\Delta}_t f := f(t) - f(0)$ we obtain

$$\bar{\Delta}_t f(x \oplus \bullet) = (\Delta_t f)(x)$$

and

$$\begin{aligned} \bar{\Delta}_t \bar{\Delta}_t f\big(x \oplus (\bullet_1 \oplus \bullet_2)\big) &= \bar{\Delta}_t f\big(x \oplus (t \oplus \bullet_1)\big) - \bar{\Delta}_t f(x \oplus \bullet_1) \\ &= f\big(x \oplus (t \oplus t)\big) - 2f(x \oplus t) + f(x) \end{aligned} \tag{6.8}$$

which yields

$$\bar{\Delta}_t \bar{\Delta}_t f\big(x \oplus (\bullet_1 \oplus \bullet_2)\big) = (\Delta_t^2 f)(x). \tag{6.9}$$

Without fear of confusion we write $\bar{\Delta}_t^r f\big(x \oplus (\bullet_1 \oplus \cdots \oplus \bullet_r)\big)$ instead of the composition $\bar{\Delta}_t \cdots \bar{\Delta}_t f\big(x \oplus (\bullet_1 \oplus \cdots \bullet_r)\big)$. From (6.9) it follows, by iteration, that

$$\bar{\Delta}_t^r f\big(x \oplus (\bullet_1 \oplus \cdots \oplus \bullet_r)\big) = (\Delta_t^r f)(x) \tag{6.10}$$

which we need for the proof of (6.6) and (6.7).

In view of (1.1) and (2.1) we can write

$$D\left(\frac{1}{\phi(t)} \int_0^t g(x \oplus u)\, d\phi(u)\right) = \frac{1}{\phi(t)}\big(g(x \oplus t) - g(x)\big), \tag{6.11}$$

i.e., (using (2.7))

$$\int_0^t \mu_u(x)(Dg)(x \oplus u)\, d\phi(u) = g(x \oplus t) - g(x) \tag{6.12}$$

if Dg exists. Furthermore, if $D^2 g$ exists then

$$\int_0^t (D^2 g)(x \oplus u)\, d\phi(u) = \mu_t(x)(Dg)(x \oplus t) - (Dg)(x). \tag{6.13}$$

First let us consider the case where r is an even integer bigger than 1. Let $u_1, \ldots, u_{r-1}$ be in $[0, \delta_r]$ and let us bear in mind that the associative law of r summands in $[0, \delta_r]$ holds, that is to say, we can write, for example, $t \oplus u_1 \oplus \cdots \oplus u_{r-1}$ without ambiguity (see Lemma 2.3). Making use of (2.14) and (6.13) it follows that

$$\begin{aligned} &\int_0^t (D^r g)\big(x \oplus (u_1 \oplus \cdots \oplus u_r)\big)\, d\phi(u_r) \\ &\qquad = \int_{u_1 \oplus \cdots \oplus u_{r-1}}^{t \oplus u_1 \oplus \cdots \oplus u_{r-1}} (D^r g)(x \oplus u_r)\, d\phi(u_r) \end{aligned}$$

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$$= \int_0^{t\oplus u_1\oplus\cdots\oplus u_{r-1}} (D^r g)(x \oplus u_r)\, d\phi(u_r)$$

$$- \int_0^{u_1\oplus\cdots\oplus u_{r-1}} (D^r g)(x \oplus u_r)\, d\phi(u_r)$$

$$= \mu_{t\oplus u_1\oplus\cdots\oplus u_{r-1}}(x)(D^{r-1}g)\big(x \oplus (t \oplus u_1 \oplus \cdots \oplus u_{r-1})\big)$$

$$- \mu_{u_1\oplus\cdots\oplus u_{r-1}}(x)(D^{r-1}g)\big(x \oplus (u_1 \oplus \cdots \oplus u_{r-1})\big)$$

and (using (6.12))

$$\int_0^t \int_0^t (D^r g)\big(x \oplus (u_1 \oplus \cdots \oplus u_r)\big)\, d(u_{r-1}, u_r)$$

$$= \int_0^t \mu_{t\oplus u_1\oplus\cdots\oplus u_{r-1}}(x)(D^{r-1}g)\big(x \oplus (t \oplus u_1 \oplus \cdots \oplus u_{r-1})\big)\, d\phi(u_{r-1})$$

$$- \int_0^t \mu_{u_1\oplus\cdots\oplus u_{r-1}}(x)(D^{r-1}g)\big(x \oplus (u_1 \oplus \cdots \oplus u_{r-1})\big)\, d(u_{r-1})$$

$$= \int_{t\oplus u_1\oplus\cdots\oplus u_{r-2}}^{t\oplus t\oplus u_1\oplus\cdots\oplus u_{r-2}} \mu_{u_{r-1}}(x)(D^{r-1}g)(x \oplus u_{r-1})\, d\phi(u_{r-1})$$

$$- \int_{u_1\oplus\cdots\oplus u_{r-2}}^{t\oplus u_1\oplus\cdots\oplus u_{r-2}} \mu_{u_{r-1}}(x)(D^{r-1}g)(x \oplus u_{r-1})\, d\phi(u_{r-1})$$

$$= (D^{r-2}g)\big(x \oplus (t \oplus t \oplus u_1 \oplus \cdots \oplus u_{r-2})\big)$$

$$- 2(D^{r-2}g)\big(x \oplus (t \oplus u_1 \oplus \cdots \oplus u_{r-2})\big)$$

$$+ (D^{r-2}g)\big(x \oplus (u_1 \oplus \cdots \oplus u_{r-2})\big).$$

This can be written (using (6.8)) as

$$\int_0^t \int_0^t (D^r g)\big(x\oplus(u_1\oplus\cdots\oplus u_r)\big)\, d(u_{r-1}, u_r) = \bar{\Delta}_t^2\{(D^{r-2}g)\big(x\oplus(u_1\oplus\cdots\oplus u_{r-2}\bullet_2\oplus\bullet_1)\big)\}$$

and so iteration of this equation gives

$$\begin{aligned}
&\int_0^t \cdots \int_0^t (D^r g)\big(x \oplus (u_1 \oplus \cdots \oplus u_r)\big)\, d(u_1, \ldots, u_r) \\
&= \bar{\Delta}_t^2 \Big\{ \int_0^t \cdots \int_0^t (D^{r-2}g)\big(x \oplus (u_1 \oplus \cdots \oplus u_{r-2} \bullet_2 \oplus \bullet_1)\big) d(u_1, \ldots, u_{r-2}) \Big\} \\
&\;\;\vdots \\
&= \bar{\Delta}_t^2 \cdots \bar{\Delta}_t^2 \{ g\big(x \oplus (\bullet_r \oplus \cdots \oplus \bullet_1)\big)\} \\
&= (\Delta_t^r g)(x)
\end{aligned} \tag{6.14}$$

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where in the last step we used (6.10). Therefore (6.6) is proved.

Now, let r be an odd positive integer. As in (6.14) we obtain by using (2.7), (6.10) and (6.11), such that

$$\begin{aligned}
&\int_0^t \cdots \int_0^t \mu_{u_1\oplus\cdots\oplus u_r}(x)(D^r g)\big(x\oplus(u_1\oplus\cdots\oplus u_r)\big)\,d(u_1,\ldots,u_r)\\
&\quad= D\int_0^t\left(\int_0^t\cdots\int_0^t (D^{r-1}g)\big(x\oplus(u_1\oplus\cdots\oplus u_r)\big)\,d(u_1,\ldots,du_{r-1})\right)d\phi(u_r)\\
&\quad= D\int_0^t \bar{\Delta}_t^{r-1}\{g\big(x\oplus(\bullet_{r-1}\oplus\cdots\oplus\bullet_1\oplus u_r)\big)\}\,d\phi(u_r)\\
&\quad= \bar{\Delta}_t^{r-1}\{g\big(x\oplus(\bullet_{r-1}\oplus\cdots\oplus\bullet_1\oplus t)\big)\} - \bar{\Delta}_t^{r-1}\{g\big(x\oplus(\bullet_{r-1}\oplus\cdots\oplus\bullet_1)\big)\}\\
&\quad= \bar{\Delta}_t^{r}\{g\big(x\oplus(\bullet_r\oplus\cdots\oplus\bullet_1)\big)\}\\
&\quad= (\Delta_t^r g)(x),
\end{aligned}$$

proving (6.7). After having established equations (6.6) and (6.7), the difference operator can be estimated as

$$\begin{aligned}
\|\Delta_t^r g\|_{L_\phi^p} &\le \int_0^t\cdots\int_0^t 2^{1/p}\,\|D^r g\|_{L_\phi^p}\ d(u_1,\ldots,u_r)\\
&\le 2^{1/p}\phi(t)^r\,\|D^r g\|_{L_\phi^p}
\end{aligned}$$

which leads to

$$\begin{aligned}
\|\Delta_t^r f\|_{L_\phi^p} &\le \|\Delta_t^r(f-g)\|_{L_\phi^p} + \|\Delta_t^r g\|_{L_\phi^p}\\
&\le 2^{1/p}(2^r-1)\,\|f-g\|_{L_\phi^p} + 2^{1/p}\phi(t)^r\,\|D^r g\|_{L_\phi^p}\,.
\end{aligned}$$

Taking the infimum over all g with $D^r g\in L_\phi^p[-1,1]$ we obtain the lower inequality (6.5) which concludes the proof of our theorem. ■

References

[Di-Fe] Z. Ditzian and M. Felten, *Averages using translation induced by Laguerre and Jacobi expansions.* Constr. Approx., to appear.

[Di-Iv] Z. Ditzian and K. Ivanov, *Strong converse inequalities.* J. Anal. Math. **61**(1993), 61–111.

[Di-Ru] Z. Ditzian and K. Runovskii, *Averages and K-functionals related to the Laplacian.* J. Approx. Theory, to appear.

[Fe1] M. Felten, *A Smoothness Concept for Weighted L^p Spaces on* $[-1,1]$. Acta Sci. Math. **62**(1996), 217–230.

[Fe2] ______, *A modulus of smoothness based on an algebraic addition.* Aequationes Math. **54**(1997), 56–73.

[Fe3] ______, *Characterization of best algebraic approximation by an algebraic modulus of smoothness.* J. Approx. Theory **89**(1997), 1–25.

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