Dear Editor,

Comments on a paper by A. J. Branford

Branford [1] states and proves that the overflow process from an N-state birth–death process is a renewal process with inter-event times characterized by a hyperexponential distribution with N components. The same result and a similar proof were published earlier in [3]. The proof in [3] is in fact somewhat shorter since use was made of Chihara's important result ([2], Theorem I.9.1 and Corollary) that a sequence of polynomials defined by a recurrence relation of the form

$$P_{n+1}(x) = (x - \alpha_n)P_n(x) - \beta_n P_{n-1}(x), \qquad n = 1, 2, \cdots$$
$$P_0(x) = 1, \quad P_1(x) = x - \alpha_0$$

is orthogonal on $[0, \infty)$ (that is, the zeros of $P_n(x)$ are all positive) if and only if positive numbers λ_n and μ_{n+1}, $n \geqq 0$, exist such that $\alpha_0 = \lambda_0$, $\alpha_{n+1} = \lambda_{n+1} + \mu_{n+1}$ and $\beta_{n+1} = \lambda_n\mu_{n+1}$, $n \geqq 0$. The same theorem can be employed to shorten considerably the proof of the second result in [1] which says that any hyperexponential distribution with N components can be viewed as the interoverflow time distribution of an N-state birth–death process. Namely, Wendroff's theorem in [4] guarantees the existence of a sequence of monic polynomials $\{q_n(x)\}$ which is not only orthogonal, that is,

$$q_{n+1}(x) = (x - c_n)q_n(x) - d_n q_{n-1}(x),$$

but also orthogonal on $(-\infty, 0]$. Writing $r_n(x) = (-1)^n q_n(-x)$, it follows that

$$r_{n+1}(x) = (x + c_n)r_n(x) - d_n r_{n-1}(x),$$

while $\{r_n(x)\}$ is orthogonal on $[0, \infty)$. Application of Chihara's theorem to $\{r_n(x)\}$ then yields that positive numbers λ_n and μ_{n+1}, $n \geqq 0$, exist such that (3.4) and (3.5) of [1] are satisfied.

References

[1] BRANFORD, A. J. (1986) On a property of finite-state birth and death processes. *J. Appl. Prob.* **23**, 859–866.

[2] CHIHARA, T. S. (1978) *An Introduction to Orthogonal Poynomials*. Gordon and Breach, New York.

[3] VAN DOORN, E. A. (1984) On the overflow process from a finite Markovian queue. *Performance Evaluation* **4**, 233–240.

[4] WENDROFF, B. (1961) On orthogonal polynomials. *Proc. Amer. Math. Soc.* **12**, 554–555.

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Yours sincerely,
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Dear Editor,

I write this in reply to a letter from Erik A. van Doorn concerning my recent paper Branford (1986). The material in this paper appeared in unpublished form in Branford (1980).

In his letter, Dr van Doorn points out that the result stated as Theorem 1 of Branford (1986) appeared with a similar proof in van Doorn (1984). This result, however, was never claimed to be new, and several earlier references were given in Branford (1986), in particular

> '... have been demonstrated in, or can easily be derived from, existing results (Karlin and McGregor (1959), ...'.

An application of the result alluded to in Karlin and McGregor (1959) gives the formula for the Laplace–Stieltjes transform of the probability distribution function for the time between successive overflows, that is, (2.6) of Branford (1986). This formula of Karlin and McGregor (1959) forms the cornerstone of the proof offered in van Doorn (1984), and so the proofs in Branford (1986) and van Doorn (1984) overlap only in the application of standard orthogonal polynomial results to provide an inversion of the Laplace-Stieltjes transform.

As stated, the reasons for offering the proof of Theorem 1 in Branford (1986) were to give a derivation of (2.6) *directly* from first principles rather than by appealing to it as a corollary to the result of Karlin and McGregor (1959) which is itself a corollary, this direct derivation then to provide the basis of the proof of Theorem 2 of Branford (1986).

Theorem 2 of Branford (1986) and its proof have not to my knowledge appeared elsewhere.