Bull. Aust. Math. Soc. **81** (2010), 208–222
doi:10.1017/S0004972709000999

MAXIMAL ANNULI WITH PARALLEL PLANAR BOUNDARIES IN THE THREE-DIMENSIONAL LORENTZ–MINKOWSKI SPACE

JUNCHEOL PYO

(Received 29 March 2009)

Abstract

We prove that maximal annuli in $\mathbb{L}^3$ bounded by circles, straight lines or cone points in a pair of parallel spacelike planes are part of either a Lorentzian catenoid or a Lorentzian Riemann's example. We show that under the same boundary condition, the same conclusion holds even when the maximal annuli have a planar end. Moreover, we extend Shiffman's convexity result to maximal annuli; but by using Perron's method we construct a maximal annulus with a planar end where a Shiffman-type result fails.

2000 *Mathematics subject classification*: primary 53C42; secondary 53C50, 53A10.
Keywords and phrases: Lorentzian catenoid, maximal annulus surface, Lorentzian Riemann's example.

1. Introduction

In 1956, Shiffman [18] proved simple but beautiful theorems on minimal surfaces lying between two horizontal planes. Let M be a minimal annulus in $\mathbb{R}^3$, P_1, P_2 horizontal planes such that $\partial M = C_1 \cup C_2$ and $C_i \subset P_i$ for all $i = 1, 2$. First, Shiffman's circle theorem: for any horizontal plane P between P_1 and P_2, $M \cap P$ is a circle whenever C_1, C_2 are circles. Second, Shiffman's convexity theorem: for any horizontal plane P between P_1 and P_2, $M \cap P$ is a convex Jordan curve whenever C_1, C_2 are convex Jordan curves. Fang [3] generalized Shiffman's circle theorem when a minimal annulus is bounded by a circle and a straight line in parallel planes. In the case where both curves are straight lines, they must be parallel. Moreover Fang and Wei [6] proved that a minimal annulus with one planar end, bounded by straight lines or circles in a pair of parallel planes, is part of a Riemann's example. On the other hand, Shiffman's convexity theorem does not hold when the minimal annulus has a planar end. Fang and Hwang [5] constructed a minimal annulus with a planar end bounded by a circle and a strictly convex noncircular Jordan curve such that its intersection with a horizontal plane is a nonconvex Jordan curve.

This work was supported in part by KOSEF R17-2008-001-01001-0.

By analogy with minimal surfaces in $\mathbb{R}^3$, López *et al.* [16] proved that (i) only Lorentzian catenoids and Lorentzian Riemann's examples are foliated by circles in parallel planes. And they proved a theorem similar to that of Enneper (see [17]), that is, (ii) on a maximal spacelike surface foliated by pieces of circles, the planes containing these circles must be parallel. By (ii), we can rewrite (i) as follows: only Lorentzian catenoids and Lorentzian Riemann's examples are foliated by pieces of circles. Also, they proved a theorem similar to Shiffman's circle result: (iii) given a maximal annulus M bounded by two parallel planar circles, the intersection of M with a plane parallel to the boundary circles is again a circle. Hence, the maximal annulus is part of a Lorentzian catenoid or a Lorentzian Riemann's example.

In this paper we extend Fang, Hwang and Wei's works to the maximal version (see [3, 5, 6]). We have organized the present paper as follows.

In Section 2 we review some well-known facts on the Lorentz–Minkowski space. In particular, we refer the readers to the result in López *et al.* [16].

In Section 3 we consider maximal annuli bounded by parallel planar curves which have constant curvature. We prove the Lorentzian version of Shiffman's circle theorem.

THEOREM 1 (See Theorem 3.1). *A maximal annulus, bounded by straight lines, circles or cone points in a pair of parallel planes, is part of a Lorentzian catenoid or a Lorentzian Riemann's example. If both curves are straight lines, they must be parallel.*

In Section 4 we consider maximal annuli with a planar end bounded by parallel planar curves of constant curvature.

THEOREM 2 (See Theorem 4.1). *A maximal annulus with a planar end, bounded by straight lines, circles or cone point in a pair of parallel planes, is part of a Lorentzian Riemann's example.*

In Section 5 we prove the Lorentzian version of Fang and Hwang's theorem. More precisely, we have the following theorem.

THEOREM 3 (See Theorem 5.2). *We construct a maximal annulus with a planar end bounded by a circle and a strictly convex noncircular Jordan curve such that its intersection with a horizontal plane is a nonconvex Jordan curve.*

2. Preliminaries

Let $\mathbb{L}^3$ be the three-dimensional Lorentz–Minkowski space, that is, the real vector space $\mathbb{R}^3$ endowed with the Lorentz–Minkowski metric $\langle , \rangle$, where $\langle , \rangle = dx_1{}^2 + dx_2{}^2 - dx_3{}^2$ and x_1, x_2, x_3 are the canonical coordinates of $\mathbb{R}^3$. We say that a vector $v \in \mathbb{R}^3 - \{0\}$ is spacelike, timelike or lightlike if $|v|^2 = \langle v, v\rangle$ is positive, negative or zero, respectively. The zero vector 0 is spacelike by convention. A plane in $\mathbb{L}^3$ is spacelike, timelike or lightlike if the normal vector of the plane is timelike, spacelike or lightlike, respectively. An immersed surface $\Sigma \subset \mathbb{L}^3$ is called spacelike if every tangent plane is a spacelike. An immersed spacelike surface Σ is called *maximal* if the mean curvature is zero everywhere.

Near a regular point of a maximal surface, a unit normal vector field can be considered as a *Gauss map*

$$N : \Sigma \to \mathcal{H}^2 = \{(x_1, x_2, x_3) \in \mathbb{L}^3 : {x_1}^2 + {x_2}^2 - {x_3}^2 = -1\},$$

where $\mathcal{H}^2$ is the hyperbolic sphere in $\mathbb{L}^3$ with constant intrinsic curvature identically -1. Denote by $\overline{\mathbb{C}}$ the extended complex plane $\mathbb{C} \cup \{\infty\}$. Let the stereographic projection σ for $\mathcal{H}^2$ be defined by

$$\sigma : \overline{\mathbb{C}} - \{|z| = 1\} \to \mathcal{H}^2, \quad z \mapsto \left(\frac{2\,\mathrm{Im}(z)}{|z|^2 - 1}, \frac{-2\,\mathrm{Re}(z)}{|z|^2 - 1}, \frac{|z|^2 + 1}{|z|^2 - 1}\right),$$

where $\sigma(\infty) = (0, 0, 1)$, that is, $\sigma(z)$ is the intersection of $\mathcal{H}^2$ and the line joining the point $(\mathrm{Re}(z), \mathrm{Im}(z), 0)$ and 'the north pole' $(0, 0, 1)$ of $\mathcal{H}^2$. It is well known that σ is conformal in the natural manner. Here $\mathcal{H}^2$ has two connected components

$$\mathcal{H}^2_+ := \mathcal{H}^2 \cap \{x_3 \geq 1\} \quad \text{and} \quad \mathcal{H}^2_- := \mathcal{H}^2 \cap \{x_3 \leq -1\}.$$

Since Σ is of zero mean curvature, the coordinate functions x_1, x_2, x_3 are harmonic functions and hence it admits a Weierstrass representation (see [14] for details).

THEOREM 2.1 (Weierstrass representation of maximal surface in $\mathbb{L}^3$). *Any maximal spacelike surface in $\mathbb{L}^3$ is represented as*

$$X(p) = \mathrm{Re} \int^p \left(\frac{1}{2}(1 + g^2)\eta, \frac{i}{2}(1 - g^2)\eta, g\eta\right) = \mathrm{Re} \int^p (\omega_1, \omega_2, \omega_3), \qquad p \in D \tag{2.1}$$

where D is a domain in $\mathbb{C}$, and η (respectively, g) is holomorphic 1-form (respectively meromorphic function) on D such that $g^2\eta$ is holomorphic 1-form on D and that $|g(\zeta)| \neq 1$ for $\zeta \in D$. Moreover:

(a) *the Gauss map N is given by $N(\zeta) = \sigma(g(\zeta))$;*
(b) *the induced metric is given by $ds = (|1 - |g|^2||\eta|/2)$;*
(c) *the Gauss curvature is given by $K = [4|dg|/|1 - |g|^2|^2|\eta|]^2$.*

REMARK 2.2. Many properties of maximal surfaces are similar to minimal surfaces. Contrary to the case of minimal surfaces, maximal surfaces have naturally arising singularities due to the geometry of the Gauss map. And since the Gauss curvatures of maximal surfaces are always nonnegative, so every maximal surface is stable.

Following Calabi [1] (for a general method, see [2]), every nonplanar complete maximal surface has singularities. Hence, many authors has studied singularities intensively (see [7, 8, 15, 19]). Let $X : \mathcal{D} \to \mathbb{L}^3$ be a continuous map defined on an open disc $\mathcal{D}$, q be a interior point of $\mathcal{D}$, and suppose that X is a maximal immersion on $\mathcal{D} - \{q\}$. Let z be a conformal parameter on $\mathcal{D} - \{q\}$ associated to the metric $ds^2 = \lambda^2(z)|dz|^2$ induced by X, where $\lambda(z) > 0$ for any $z \in z(\mathcal{D} - \{q\})$. Define q to be an *isolated singularity* of X if for any sequence $\{q_n\} \subset \mathcal{D} - \{q\}$ tending to q, the limit $\lim_{n\to\infty} \lambda(z(q_n))$ vanishes. In this case, we say that $X(D)$ is a maximal surface with a singularity at $X(q)$. There are two kinds of isolated singularities called branch points and conelike singularities.

In the case where $\mathcal{D} - \{q\}$ endowed with a induced complex structure is conformally a once punctured disc, then q is called a *branch point*. This means that $\eta = 0$ near q, η is a holomorphic 1-form of Weierstrass representation and the surface cannot be embedded.

Suppose now that $\mathcal{D} - \{q\}$ is conformally to an annulus $\{z \in \mathbb{C} : 0 < r < |z| < 1\}$. If X can be extended continuously to

$$C_0 = \{z \in \mathbb{C} \mid 0 < r < |z| \leq 1\} \quad \text{with } X(\{|z| = 1\}) = X(q).$$

In this case we call q a *conelike singularity*, $P_0 = X(\{|z| = 1\}) = X(q)$ is called a *cone point*, and the surface is embedded near the cone point. At the cone point, maximal surfaces are naturally extended.

LEMMA 2.3 (Extension for a cone point in $\mathbb{L}^3$ [7]). *Let $X_0 : C = \{r < |z| < 1\} \to \mathbb{L}^3$ be an embedded maximal surface with cone point $P_0 = X_0(\{|z| = 1\})$, then the following holds.*

Let the Weierstrass data (g, η) of X_0 satisfy that g is injective and $|g| = 1$ on $\{|z| = 1\}$ and $\eta \neq 0$ on $\{|z| = 1\}$. The surface X_0 reflects analytically about $\{|z| = 1\}$ to the mirror surface. More precisely, let $J(z) = 1/\overline{z}$ denote the inversion about $\{|z| = 1\}$, the mirror surface X_0^* has the Weierstrass data $(J^*g = 1/\overline{g},\ J^*\phi = -\overline{\phi})$ and satisfies $X_0^* = -X_0 + 2P_0$, where $P_0 = X_0(\{|z| = 1\})$. Moreover, for any spacelike plane Π contains P_0 the Lorentzian orthogonal projection $\pi : X_0 \to \Pi$ is a local homeomorphism and near P_0, X_0 is asymptotic to the half light cone with vertex at P_0.

A *circle* in $\mathbb{L}^3$ is defined to be a planar curve with nonzero constant curvature. Therefore, there are three different types of circles in $\mathbb{L}^3$ since there are three different types of planes in $\mathbb{L}^3$. In this paper, however, circles are the same as in $\mathbb{R}^3$ since we focus only on spacelike planes in $\mathbb{L}^3$. *Straight lines* in $\mathbb{L}^3$ are defined as similarly.

We introduce Lorentzian Riemann's examples.

THEOREM 2.4 [16]. *Let $X : M \to \mathbb{L}^3$ be a spacelike conformal nonplanar maximal immersion of a Riemann surface M. If $X(M)$ is foliated by pieces of Euclidean circles in parallel planes with normal Euclidean vector $v = (0, 0, 1)$, then, up to scaling and linear isometries in $\mathbb{L}^3$, the Gauss map g of X satisfies:*

(1) $dg/dz = g$; *or*
(2) $(dg/dz)^2 = g(g^2 + 2rg + 1)$, *where* $r \in \mathbb{R}$.

We call the first case a *Lorentzian catenoid* and the second case a *Lorentzian Riemann's example*.

Now we consider a connected component of the outside of a Euclidean ball. This connected component is conformally equivalent to a punctured disc and the metric has

https://doi.org/10.1017/S0004972709000999 Published online by Cambridge University Press

a pole at the puncture. The connected component is called an *end*. The asymptotic behaviour of an end is similar to an end of minimal surfaces (see [17] for details). A similar result in the Lorentzian setting can be found in [11]. Also a different approach to an end by Klyachin can be found in [13]. We omit the proof.

LEMMA 2.5. *Let $X : \mathcal{D}\backslash\{0\} \to \mathbb{L}^3$ be an embedded end of the maximal surface with vertical limit normal and the Weierstrass data (g, η), then the following holds.*

The order of pole of ω_i for all $i = 1, 2, 3$ is two and the end X is asymptotic to the following:

$$(x_1, x_2, x_3) = (\alpha r^{-1} \cos\theta, \alpha r^{-1} \sin\theta, \beta \log r),$$

on a neighbourhood of 0, where $z = re^{i\theta}$, we have $\alpha \in \mathbb{R}\backslash\{0\}$ and $\beta \in \mathbb{R}$.

An end is called a *planar end* (respectively, *catenoidal end*) if $\beta = 0$ (respectively, $\beta \neq 0$) and it is asymptotic to a horizontal plane (respectively, a vertical half Lorentzian catenoid).

3. Maximal annuli in a slab

By Lorentzian isometry we can denote a spacelike plane

$$\Pi = \Pi_t = \{(x_1, x_2, x_3) \mid x_3 = t\}$$

and a slab $S(a, b) = \{(x_1, x_2, x_3) \mid a \leq x_3 \leq b\}$. By homothety we also assume that $S(a, b) = S(-1, 1)$.

THEOREM 3.1. *Let $A \subset S(-1, 1)$ be a compact maximal annulus in a slab whose set of singularities consists of a finite (possibly empty) set of conelike singularities. Suppose that $A(1) = A \cap \Pi_1$ and $A(-1) = A \cap \Pi_{-1}$ are straight lines, circles or cone points.*

(1) *If both $A(1)$ and $A(-1)$ are circles, then $A(t) = A \cap \Pi_t$ is a circle or cone point for all $-1 < t < 1$. In particular, A is embedded and the number of cone points is at most one.*
(2) *If $A(1)$ or $A(-1)$ is a straight line, the other is a circle and A is embedded, then $A(t) = A \cap \Pi_t$ is a circle or a cone point for all $-1 < t < 1$.*
(3) *If both $A(1)$ and $A(-1)$ are straight lines and A is embedded, then $A(t) = A \cap \Pi_t$ is a circle or a cone point for all $-1 < t < 1$.*
(4) *If $A(1)$ is a straight line or a circle and $A(-1)$ is a cone point (in the case where $A(1)$ is a straight line, we also assume that A is embedded), then $A(t) = A \cap \Pi_t$ is a circle or a cone point for $-1 < t < 1$.*

In order to prove the Theorem 3.1, we need some lemmas.

LEMMA 3.2. *Let $A \subset S(-1, 1)$ be a properly immersed maximal annulus such that both $A(1)$ and $A(-1)$ are circles or straight lines, then A can be conformally parameterized by*

$$X : A_R - C \to \mathbb{L}^3,$$

where $A_R = \{z \in \mathbb{C} : 1/R \le |z| \le R\}$ for all $1 < R < \infty$ and the set C is determined as follows.

If $A(1)$ and $A(-1)$ are both circles, then $C = \emptyset$; if $A(1)$ is a straight line and $A(-1)$ is a circle (respectively, $A(1)$ is a circle and $A(-1)$ is a straight line), then $C = \{p : |p| = R\}$ (respectively, $C = \{q : |q| = 1/R\}$); if $A(1)$ and $A(-1)$ are both straight lines, then $C = \{p, q : |p| = R, |q| = 1/R\}$.

In any case, the Gauss map g of A has neither zero nor pole in the interior of A_R, and it can be extended to a neighbourhood of A_R. Moreover, the extended g has either zero or pole order two at p and q.

PROOF. Since A is a proper maximal annulus, the conformal structure of the interior of A is equivalent to interior of $A_R = \{z \in \mathbb{C} : 1/R \le |z| \le R\}$ for some $1 < R < \infty$, and a conformal harmonic immersion $X : A_R - C \to S(-1, 1)$, where C is a subset of ∂A_R and $A(\{|z| = R\} - C) = A(1)$, $A(\{|z| = 1/R\} - C) = A(-1)$. In particular, the third coordinate function X_3, which is harmonic with $X3|_{(\{|z|=R\}-C)} = 1$, $X3|_{(\{|z|=1/R\}-C)} = -1$ and $-1 < X3|_{\mathrm{Int}(A_R)} < 1$, can be extended to whole A_R such that $X3|_{\{|z|=R\}} = 1$, and $X3|_{\{|z|=1/R\}} = -1$. By the existence and uniqueness of the Dirichlet problem, $X_3 = (1/\log R)\log|z|$, we have for any $-1 < t < 1$, that $A(t) = A \cap \Pi_t$ is the image $X(\{z \in A_R : |z| = R^t\})$.

First, g cannot have zeros or poles in $\mathrm{Int}(A_R)$, the interior of A_R. Suppose not, then the preimage of $A(t)$ for t has at least four rays at a zero or a pole by the harmonicity of maximal surfaces. However, the preimage of $A(t)$ is a circle since $X_3 = (1/\log R)\log|z|$. So there are no zeros and poles in the interior.

It remains only to prove that on the boundary of A, that is, the Gauss map N is not perpendicular to the $x_1x_2 = xy$-plane. Since boundaries are composed with a circle or a straight line, the projection of the boundary into the xy-plane satisfies the sphere condition, inner or outer. There is well-defined normal direction at every boundary point. Near any boundary point p, N has a vertical normal, the surface is a graph over a small open disc $D \subset P_1$ with p on ∂D, assuming that $p \in A(1)$. Then we can write by the maximal surface equation. We write $(x, y, z = x_3) \in A$, where $x_3 = z(x, y)$ satisfies

$$(1 - z_y^2)z_{xx} + 2z_x z_y z_{xy} + (1 - z_x^2)z_{yy} = 0, \quad z_x^2 + z_y^2 < 1.$$

Since X_3, the third coordinate function of A, is harmonic, by the maximum principle we have for any $(x, y) \in D$ that $z(x, y) < 1 = z(p)$. Define a uniformly elliptic operator on a smaller domain if necessary,

$$Lu = (1 - u_y^2)u_{xx} + 2u_x u_y u_{xy} + (1 - u_x^2)u_{yy}, \quad u_x^2 + u_y^2 < 1.$$

Then z satisfies $Lz = 0$. By the Hopf boundary point lemma

$$\frac{\partial z}{\partial \nu} > 0,$$

where ν is the outward normal to ∂D at p. However, this means that the normal is not vertical. This contradiction proves that N is never vertical on the boundary of A. Hence, $g \neq 0$ or ∞.

If $A(1)$ is a straight line, by Lorentzian isometry we can assume that $A(1)$ is parallel to the y-axis in $\mathbb{L}^3$. Then the normal vector of A along the $A(1)$ stays in the xz-plane. Let $C_1 = C \cap \{|z| = R\}$, g is real on $\{|z| = R\} - C_1$. Using the Schwarz reflection principle, g can be extended to $\{R < |z| < R^3\}$ by $\widetilde{g}(z) = \overline{g(R^2/\overline{z})}$ for all $R < |z| < R^3$. So we obtain a maximal surface

$$\mathcal{A} = \overline{X} : \{1/R < |z| < R^3\} - C_1 \to S(-1, 3).$$

Since X is properly immersed, the extended surface $\overline{X}(\{1/R < |z| < R^3\} - C_1)$ is also properly immersed and contains a complete maximal annular end. Since the Gaussian curvature of a maximal surface is always nonnegative, by [10, Huber's theorem] (or see [19, Appendix]), the annular end of $\mathcal{A}$ conformally equivalent to a punctured disc and the Gauss map of $\mathcal{A}$ can be extended to the puncture. Hence, $C_1 = \{p\}$ is singleton and g is either zero or infinite, unless the length of the straight line is finite. Hence, $\mathcal{A}$ has a vertical limit end, by Lemma 2.5, at the p has zero of order two. If $A(-1)$ is a straight line, we apply the same process. □

Now, we derive the Lorentzian Shiffman function in terms of Weierstrass data. First we calculate the planar curvature of each $A(t) = A \cap \Pi_t$ for all $-1 \le t \le 1$. At any point of $A(t)$, let ψ be the angle between the tangent vector and the positive x-axis. By Lemma 3.2, $g \neq 0, \infty$ in the interior of A_R, so the unit normal vector is $g/|g|$, and $\phi = \arg g = \operatorname{Im}(\log g) = \psi - \pi/2$. We note that the function ϕ can be multivalued but harmonic. Now suppose that s is the arc length parameter of the curve $A(t)$, and $X^{-1}(A(t)) = \{z : |z| = r = R^t\}$, write $z = re^{i\theta}$, then the curvature of $A(t)$ is

$$\begin{aligned} \kappa = \psi_s = \phi_s = \frac{d}{ds}\operatorname{Im}(\log g) = \operatorname{Im}\left(\frac{d}{ds}\log g\right) \\ = \operatorname{Im}\left(\frac{g'}{g}\frac{dz}{d\theta}\frac{d\theta}{ds}\right) = \operatorname{Im}\left(\frac{g'}{g} izr^{-1}\Lambda^{-1}\right) = r^{-1}\Lambda^{-1}\operatorname{Re}\left(z\frac{g'}{g}\right). \end{aligned} \tag{3.1}$$

Here we use the fact that on the curve $\{z : |z| = r = R^t\}$,

$$\frac{dz}{d\theta} = ire^{i\theta}, \quad ds = \Lambda|dz| = r\Lambda\, d\theta.$$

By a direct calculation, we have the Lorentzian Shiffman function:

$$u := r\Lambda\frac{\partial\kappa}{\partial\theta} = \operatorname{Im}\left[\frac{1}{2}\frac{|g|^2+1}{|g|^2-1}\left(z\frac{g'}{g}\right)^2 - z\frac{d}{dz}\left(z\frac{g'}{g}\right)\right]. \tag{3.2}$$

LEMMA 3.3. *Let A and C be as in Lemma 3.2, and let u be the Lorentzian Shiffman function as (3.2). Then u can be continuously extended on the set C and $u = 0$ on the boundary ∂A.*

PROOF. Let

$$\begin{aligned} U(z) &= \left[-\frac{1}{2}\left(z\frac{g'(z)}{g(z)}\right)^2 - z\frac{d}{dz}\left(z\frac{g'(z)}{g(z)}\right)\right] + \left[\left(1 - \frac{1}{1-|g|^2}\right)\left(z\frac{g'(z)}{g(z)}\right)^2\right] \\ &= \Phi(z) + \Psi(z). \end{aligned}$$

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We claim that both Φ and Ψ are C^∞ complex functions near any point of the set C. The claim is proved, then since $u(z) = \operatorname{Im} U(z)$ is smooth near z_0, $u(z)$ can be continuously extended to p.

Let $z_0 = p$ or q. By Lemma 3.2, the extended Gauss map $\widetilde{g}$ has a zero or a pole of order two. Let us assume that $g(z_0) = 0$.

First, we show that Φ is a C^∞ complex function near each point of the set C. Let $\zeta = z - z_0$, we have

$$\widetilde{g}(z) = (z - z_0)^2 h(z) = \zeta^2 h(z_0 + \zeta),$$

where h is a holomorphic function and $h(z_0) \neq 0$. For convenience, write g instead of $\widetilde{g}$, then

$$z\frac{g'(z)}{g(z)} = \frac{2z_0}{z - z_0} + 2 + z\frac{h'(z)}{h(z)} = \frac{a_{-1}}{\zeta} + \sum_{k=0}^{\infty} a_k \zeta^k, \quad a_{-1} = 2z_0.$$

We also have

$$\left(z\frac{g'(z)}{g(z)}\right)^2 = \frac{a_{-1}^2}{\zeta^2} + \frac{2a_{-1}a_0}{\zeta} + \sum_{k=0}^{\infty} b_k \zeta^k,$$

and

$$z\frac{d}{dz}\left(z\frac{g'(z)}{g(z)}\right) = -\frac{a_{-1}z_0}{\zeta^2} - \frac{a_{-1}}{\zeta} + (\zeta + z_0)\sum_{k=1}^{\infty} k a_k \zeta^{k-1}.$$

Since $a_{-1} = 2z_0$, we have $a_{-1}^2 - 2a_{-1}z_0 = 0$. The expression

$$z\frac{h'(z)}{h(z)} = (\zeta + z_0)\sum_{k=0}^{\infty} \frac{1}{k!}\left(\frac{h'}{h}\right)^{(k)}(z_0)\zeta^k = z_0\frac{h'(z_0)}{h(z_0)} + \sum_{k=1}^{\infty} c_k \zeta^k,$$

implies

$$a_0 = 2 + z_0\frac{h'(z_0)}{h(z_0)}.$$

Now we calculate the value a_0. The Weierstrass representation for the extended surface $\mathcal{S}$ is

$$\phi_1 = \frac{1}{\log R}\frac{1}{2z}\left(\frac{1}{\widetilde{g}} + \widetilde{g}\right)dz, \quad \phi_2 = \frac{1}{\log R}\frac{i}{2z}\left(\frac{1}{\widetilde{g}} - \widetilde{g}\right)dz, \quad \omega_3 = \frac{1}{\log R}\frac{1}{z}\,dz.$$

For simplicity, we write g instead of $\widetilde{g}$. Let us choose a loop γ around z_0 small enough so that the inside of γ has only one element of the set C. By the well-definedness of an extended maximal surface and

$$\int_\gamma (\omega_1, \omega_2, \omega_3) = \overrightarrow{0},$$

we have

$$\int_\gamma \frac{1}{zg} = \int_\gamma \frac{g}{z} = 0.$$

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Then

$$0 = \lim_{z \to z_0} \left(\frac{(z - z_0)^2}{zg(z)} \right)' = \lim \left(\frac{1}{zh(z)} \right)'$$
$$= -\frac{1}{z_0^2 h(z_0)} - \frac{h'(z_0)}{z_0 h^2(z_0)}.$$

Finally, we have

$$a_0 = 2 + z_0 \frac{h'(z_0)}{h(z_0)} = 1.$$

Hence,

$$\Phi(z) = -\frac{1}{2} \frac{a_{-1}^2 - 2a_{-1}z_0}{\zeta^2} - \frac{a_{-1}a_0 - a_{-1}}{\zeta}$$
$$- \frac{1}{2} \sum_{k=0}^{\infty} b_k \zeta^k - (\zeta + z_0) \sum_{k=0}^{\infty} k a_k \zeta^{k-1} = \sum_{k=0}^{\infty} d_k \zeta^k$$

is holomorphic near $z = z_0$.

Now we consider the function $\Psi(z)$. Since

$$|g(z)|^2 = |z - z_0|^4 |h(z)|^2 = |\zeta|^4 |h(z)|^2 \quad \text{and} \quad \zeta^2 \left(z \frac{g'(z)}{g(z)} \right)^2$$

is holomorphic, it follows that

$$\frac{1}{\zeta^2} \left(1 - \frac{1}{|g|^2 - 1} \right) = \frac{1}{\zeta^2} \sum_{k=1}^{\infty} |g|^{2k} = \zeta^2 \sum_{k=1}^{\infty} |\zeta|^{4(k-1)} |h(z)|^{2k}$$

is a smooth function near the z_0. Thus, $\Psi(z)$ is a smooth function near z_0, and so $U(z)$ is also smooth. Since $u|_{\partial A - C} = 0$ and u can be continuously extended to C, $u = 0$ on the ∂A. □

LEMMA 3.4. *The Lorentzian Shiffman function u can be smoothly extended on the conelike singularities.*

PROOF. Let $X : \{r < |z| < 1\}$ imply that $\mathbb{L}^3$ has cone point at $X(\{c := |z| = 1\})$ with Weierstrass data (g, ϕ). By Möbius transformation on c, we can assume that the curve c is $\mathrm{Re}(z) = 0$ the involution J is $J(z) = -\bar{z}$, and the Weierstrass data of the mirror surface are $(1/\bar{g}, -\bar{\phi})$. Write $g = e^{w(z)}$, then we have $w(-\bar{z}) = -\overline{w(z)}$.

The Lorentzian Shiffman function u extends to c if and only if

$$V(z) := \mathrm{Im} \left(\frac{1}{|g^2 - 1|} \left(\frac{d \log g}{dz} \right)^2 \right)$$

extends to c.

We claim that $V(z)$ can be smoothly extended to c.

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Take $z_0 \in i\mathbb{R}$ and let the Taylor series of the function

$$w(z) = \sum_{m=0}^{\infty} a_m(z_0)(z - z_0)^m.$$

Since $w(-\overline{z}) = -\overline{w(z)}$, we have $(-1)^m a_m(z_0) = -\overline{a_m(z_0)}$, that is, $\mathrm{Re}(a_{2n}(z_0)) = 0$ and $\mathrm{Im}(a_{2n+1}(z_0)) = 0$ for all $n \in \mathbb{N} \cup \{0\}$. Since g is injective near a conelike singularity, we have $a_1(z_0) \neq 0$, for any $z_0 \in c$.

Now we have

$$|g(z)| - 1 = e^{\mathrm{Re}(w(z))} - 1 = \mathrm{Re}(w(z))\widetilde{H}_1(\mathrm{Re}(w(z))),$$

where $H_1(z) = (e^z - 1)/z$ and $z \in \mathbb{C}$. Since the coefficient of the function is $w(z)$, we deduce that $\mathrm{Re}w(z) = \mathrm{Re}(z)V_1(z)$, where V_1 is a suitable smooth function around c, and because of $a_0(z_0) \neq 0$ for all z_0 in the compact set c, we have $|V_1|_c \geq \epsilon > 0$. Thus,

$$|g(z)| - 1 = \mathrm{Re}(z)H_1(z),$$

where the smooth function is $H_1(z)$ with $|H_1|_c \geq \epsilon' > 0$. By similar argument, we have

$$\mathrm{Im}\left(\left(\frac{d \log g}{dz}\right)^2\right) = \mathrm{Re}(z)H_2(z),$$

where H_2 is a smooth function around c. Hence, around c,

$$V(z) = \frac{H_2(z)}{H_1(z)(1 + |g(z)|)}$$

is a smooth function. □

Proof of Theorem 3.1. Let us show that $A : M \to \mathbb{L}^3$ bounded by two cone points P_1 and P_{-1} is not possible. If not, by Lemma 2.3 successive reflections about cone points, we have a complete maximal annulus $\widetilde{A} : \widetilde{M} \to \mathbb{L}^3$ with infinitely many conelike singularities such that $\widetilde{A}$ is a translation invariant. The quotient of $\widetilde{M}$ under the holomorphic translation induced by above translation gives a torus T, and the Weierstrass data $(\omega_1, \omega_2, \omega_3)$ of $\widetilde{A}$ can be induced on T. Furthermore, ω_j is holomorphic, and so $\omega_j = \lambda_j \tau_0$ for all $j = 1, 2, 3$, where $\lambda_j \in \mathbb{C}$ and τ_0 is a nonzero holomorphic 1-form on T. Because $\omega_1^2 + \omega_2^2 - \omega_3^2 = 0$ and the associated maximal immersion is singly periodic, it is not hard to see that $\lambda_j = r_j \lambda$, where $r_j \in \mathbb{R}$, $\lambda \in \mathbb{C}$ and $r_1^2 + r_2^2 - r_3^2 = 0$. In particular, $\widetilde{A}$ lies in a lightlike straight line, which is impossible.

First, both $A(1)$ and $A(-1)$ are circles. By Lemma 3.2, we find a conformal annulus A_R and that the set C is empty. By Lemmas 3.3 and 3.4, the Lorentzian Shiffman function u is a smooth in the interior of A_R and u satisfies

$$\triangle_A u = 2Ku, \quad u|_{\partial A_R} = 0.$$

Since every maximal surface is stable, the first eigenvalue of Jacobi operator is positive. Hence, $u \equiv 0$ and Π_t is a circle or a conelike singularity, for any $-1 < t < 1$. Moreover, A is part of the Lorentzian catenoid or a Lorentzian Riemann's example.

So the maximal annulus is embedded. Because Lorentzian Riemann's examples can have at most one cone point without planar end, the maximal annulus has at most one cone point.

Second, $A(1)$ is a straight line and $A(-1)$ is a circle. By Lemma 3.2, the function u is smooth near $A(1)$ and zero on the $A(1)$. The same argument in the first case still holds. The third case is similar to the second case.

Finally, either $A(1)$ or $A(-1)$ is a cone point and the other is a circle or straight line. Using the Lemma 2.3, we obtain maximal annulus bounded by circles or straight lines. So it is a previous case. The theorem is complete. □

COROLLARY 3.5. *Let $A(1)$ and $A(-1)$ be nonparallel straight lines to each other. Then $\Gamma = A(1) \cup A(-1)$ cannot bound a properly embedded maximal annulus in $S(-1, 1)$.*

4. Maximal annuli with a planar end in a slab *I*

In this section, we consider maximal annuli with an end. This gives a characterization of Lorentzian Riemann's examples.

THEOREM 4.1. *Let $A \subset S(-1, 1)$ be an embedded maximal annulus with a planar end in a slab whose set of singularities consists of a finite (possibly empty) set of conelike singularities. Suppose that $A(1) = A \cap \Pi_1$ and $A(-1) = A \cap \Pi_{-1}$ are straight lines, circles or cone points, except they are bounded by two cone points, then $A(t) = A \cap \Pi_t$ is a circle or cone point for any $-1 < t < 1$, except at the height of the end where the intersection is a straight line. Consequently, A is part of a Lorentzian Riemann's example, so if the boundary consists of two straight lines, then the lines must be parallel.*

LEMMA 4.2. *Let $A \subset S(-1, 1)$ be a maximal annulus with a planar end and both $A(1)$ and $A(-1)$ consist of circles or straight lines, then A can be conformally parameterized by*

$$X : A_R - C - \{z_e\} \to \mathbb{L}^3,$$

where $A_R = \{z \in \mathbb{C} : 1/R \le |z| \le R\}$ for all $1 < R < \infty$ and the set C is determined as follows.

For $|p| = R$ and $|q| = 1/R$, we have $C = \{p, q\}$ if $A(1)$ and $A(-1)$ are straight lines; $C = \{p\}$ (respectively, $C = \{q\}$) if $A(1)$ is a straight line and $A(-1)$ is a circle (respectively, $A(1)$ is a circle and $A(-1)$ is a straight line); and $C = \emptyset$ otherwise.

In any case, the Gauss map g of A has neither zero nor pole in the interior of A_R, and g can be extended to a neighbourhood of A_R such that the extended g has either zero or pole of order two at z_e, p and q.

PROOF. Since the Gaussian curvature of a maximal surface is always nonnegative. By Huber's theorem, the conformal domain of a maximal surface is $A_R - C - \{z_e\}$. By Lemma 2.5, the Gauss map g has zero or pole of order two at z_e. For the rest parts of lemma are proved in the same way as in Lemma 3.2. □

PROOF OF THEOREM 4.1. Either $A(1)$ or $A(-1)$ is a cone point, using Lemma 2.3, the maximal annulus can be extended to maximal surface bounded by circles or straight lines. By the Lemmas 3.3, 3.4 and 4.2, the Lorentzian Shiffman function u can be smoothly extended to the set C, the end z_e and cone points. Then u satisfies

$$\triangle_A u = 2Ku, \quad u|_{\partial A_R} = 0.$$

By the stability of Jacobi operator, $u \equiv 0$. So the theorem is complete. □

5. Maximal annuli with a planar end in a slab II

First, we extend the Shiffman's convexity theorem to Lorentzian space.

THEOREM 5.1. *Let $A \subset S(-1, 1)$ be a properly immersed maximal annulus where $A(1)$ and $A(-1)$ consist of convex Jordan curve, then $A \cap \Pi_t$ is a strictly convex Jordan curve for every $-1 < t < 1$. In particular, A is embedded.*

PROOF. Let the angle function ψ, the planar curvature κ as (3.1). Define $h = \mathrm{Re}(z(g'/g)) = r\Lambda\kappa$ for any $r\Lambda > 0$. Here h is a harmonic and nonnegative on the boundary. By the strong maximum principle, h is strictly positive. Thus, $A \cap \Pi_t$ is locally strictly convex. Similar to a minimal surface [4], the period of the angle function ψ is exactly 2π. Hence, $A \cap \Pi_t$ is strictly convex. □

THEOREM 5.2. *We construct a maximal annulus $A \subset S(-1, 1)$ with a planar end and it satisfies the following properties: $A \cap \Pi_{t_0}$ is a nonconvex Jordan curve for some $t_0 \in (-1, 1)$, even when the boundary ∂A consists of a circle and a strictly convex real analytic Jordan curve.*

LEMMA 5.3. *Let $A \subset S(-1, 1)$ be a maximal annulus with a planar end, and the boundary ∂A consists of two Jordan curves lying in a pair of parallel planes which are the boundary of $S(-1, 1)$, then A can be conformally parameterized by*

$$X : A_R - \{z_e\} \to \mathbb{L}^3,$$

where $A_R = \{z \in \mathbb{C} : 1/R \le |z| \le R\}$ for all $1 < R < \infty$ and $1/R < |z_e| < R$.

Moreover, if $A \cap \Pi_t$ are strictly convex C^2 Jordan curves for all $-1 \le t \le 1$ except at $t_0 \in (-1, 1)$, the height of the end, then $A \cap \Pi_{t_0}$ is a straight line.

PROOF. Because A is a maximal annulus with a planar end, in the interior, as in Lemma 4.2, A is conformally equivalent to $A_R - \{z_e\}$ for suitable $1 < R < \infty$ and $1/R < |z_e| < R$. By the same argument and the Dirichlet problem $X_3 \equiv (1/\log R)\log|z|$, the planar curvature $\kappa(z) = |z|^{-1}\Lambda^{-1}\,\mathrm{Re}(z(g'/g))$. Since $z_e \neq 0$, then g'/g is meromorphic and has an isolated pole at $z_e \neq 0$, and $\Xi(z) = \mathrm{Re}(z(g'/g)) = |z|\Lambda\kappa(z)$ takes positive and negative values near z_e. So $\Xi^{-1}(0)$ is a nonempty set and a real analytic one-dimensional variety except for isolated points $\{p_i\} \subset \Xi^{-1}(0)$, at the p_i, so $D\Xi(p_i)$ is zero and at least four equal angular curves emit from p_i. However, $A \cap \Pi_t$ are strictly convex except t_0, so are $\Xi \neq 0$ except $|z| = |z_0|$.

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So $\Xi^{-1}(0) \subset \{z : |z| = t_0\}$ and $\Xi^{-1}(0)$ has no singularities and is a one-dimensional manifold without boundary. This means that $\Xi^{-1}(0) = \{z : |z| = t_0\} - \{z_e\}$. Hence, $\Xi(z) = \mathrm{Re}(z(g'/g)) = |z|\Lambda\kappa(z) \equiv 0$ on $\{z : |z| = t_0\} - \{z_e\}$. The only case is $\kappa \equiv 0$, so $X(\{z : |z| = t_0\} - \{z_e\})$ is a straight line. $\square$

We are going to construct a maximal surface by solving the exterior Dirichlet problem for the maximal surface equation. The variational problem of the area functional leads to the following divergence form of the maximal surface equation:

$$Q\nu = \mathrm{Div}\left(\frac{D\nu}{\sqrt{1 - |D\nu|^2}}\right),$$

with $|D\nu| \leqslant 1$. Because we use Perron's method, we define the subsolution and supersolution to the maximal surface equation.

Definition 5.4 (Subsolution and supersolution). Let Ω be a domain in the (x_1, x_2) plane. A $C^0(\Omega)$ function α is a *subsolution* (respectively, *supersolution*) in Ω if for every ball $B \Subset \Omega$ where $\overline{B} \subset \Omega$, and every function ν satisfying $Q\nu = 0$ in B and $\alpha \le \nu$ (respectively, $\alpha \ge \nu$) on ∂B, then we have $\alpha \le \nu$ (respectively, $\alpha \ge \nu$).

We follow the classical Perron's strategy (see [9]). (i) A subsolution (supersolution) in a domain Ω satisfies the strong maximum principle. (ii) Let ν be a subsolution in Ω and B be a ball strictly contained in Ω. Denote by $\overline{\nu}$ the solution in B satisfying $\overline{\nu} = \nu$ on the boundary ∂B. We define the *solution lifting* of ν in $B \Subset \Omega$ ($\overline{B} \subset \mathrm{Int}\,\Omega$) by

$$V(x) = \begin{cases} \overline{\nu}(x) & \text{for } x \in B \\ \nu(x) & \text{for } x \in \Omega - B. \end{cases}$$

Then the function V is also subsolution in Ω. (iii) If ν_1 and ν_2 are subsolutions (respectively supersolutions) to the maximal surface equation, using the maximum principle, $\sup\{\nu_1, \nu_2\}$ (respectively $\inf\{\nu_1, \nu_2\}$) is a subsolution (respectively supersolution) to the maximal surface equation.

For a continuous function φ defined on $\partial\Omega$, define S_φ to be the set of subsolutions to the maximal surface equation which are $C^0(\overline{\Omega})$ and equal to φ on the boundary $\partial\Omega$. The guarantee of S_φ is nonempty and existence of a supersolution ν^+, the function $\mu(x) = \sup_{\nu \in S_\varphi} \nu(x)$ solves the Dirichlet problem

$$\begin{cases} Q\mu = 0 & \text{in } \Omega \\ \mu = \varphi & \text{on } \partial\Omega. \end{cases}$$

This is a classical argument by iterating the solution lifting in small balls.

Proof of Theorem 5.2. Choose a C, a Lorentzian catenoid with cone point in $\Pi_0 = x_1x_2$-plane. Let $C^+ = C \cap \{(x_1, x_2, x_3) \in \mathbb{L}^3 : x_3 \ge 1\}$ and $D_1 \subset \Pi_1$ be a disc with $\partial D_1 = C \cap \Pi_1$, let C^- be a reflection of C^+ with respect to Π_1, then C^- is a

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graph of the function $v : \Pi_0 - D \to \mathbb{R}$, where D is a vertical translation of D to the plane Π_0.

Kim and Yang [12] construct maximal surfaces asymptotic to the Lorentzian catenoid with genus k. Let $\mathcal{M}_1$ be a Kim and Yang's example with genus one. We cut $\mathcal{M}_1$ at a sufficient large height. Then we gain an annular end E such that E is a graph and has noncircular real analytic strictly convex boundary lying in a horizontal plane. Because $C^- \cap \Pi_{-1}$ is a circle with enclose a disc D_{-1}, we can translate the end E in such a way that $\partial E \subset \Pi_{-1} \backslash D_{-1}$ and $E \cap C^- = \emptyset$. Denote by $B_{-1} \subset \Pi_{-1} \backslash D_{-1}$ the closed bounded convex domain bounded by ∂E in Π_{-1}. Let B be the vertical translation of B_{-1} to Π_0 and use the notation $\Omega = \Pi_0 - (D \cup B)$. The annular end E is a graph as $w : \Pi_0 - B \to \mathbb{R}$ such that $w \equiv -1$ on ∂B.

On Ω, we have a subsolution (respectively, supersolution) $\nu^- = \sup\{v, -1\}$ (respectively, $\nu^+ = \inf\{1, w\}$) to the maximal surface equation. They satisfy the boundary condition $\nu^\pm = \varphi$ on $\partial\Omega$, where $\varphi : \partial\Omega \to \mathbb{R}$ is the function

$$\begin{cases} \varphi = 1 & \text{on } \partial D \\ \varphi = -1 & \text{on } \partial B. \end{cases}$$

Hence, $\mu(x) = \sup_{\nu \in S_\varphi} \nu(x)$ solves the Dirichlet problem

$$\begin{cases} Q\mu = 0 & \text{in } \Omega \\ \mu = \varphi & \text{on } \partial\Omega. \end{cases}$$

So the graph of μ is a maximal surface A bounded by a circle and a noncircular real analytic strictly convex Jordan curve in Π_1 and Π_{-1}, respectively. Since $-1 \leq \mu \leq 1$, we have $A \subset S(-1, 1)$ with an end. So the end must be planar. Let $t_0 \in (-1, 1)$ be the height of the end, the intersection curve $A \cap \Pi_{t_0}$ is not a straight line. Suppose not $A \cap \Pi_{t_0}$ is straight line denote A_s be a subannulus of A bounded by a circle and a straight line. By Theorem 3.1, A_s is part of a Lorentzian Riemann's example, thus A is also part of a Lorentzian Riemann's example. This contradicts the boundary condition of A. Hence, by Lemma 5.3, there exist $t \in (-1, 1)$ such that $A \cap \Pi_t$ is a nonconvex Jordan curve. □

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JUNCHEOL PYO, Department of Mathematics, Seoul National University,
Seoul 151-742, Korea
e-mail: jcpyo@snu.ac.kr