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Karamata Renewed and Local Limit Results

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Abstract. Connections between behaviour of real analytic functions (with no negative Maclaurin series coefficients and radius of convergence one) on the open unit interval, and to a lesser extent on arcs of the unit circle, are explored, beginning with Karamata's approach. We develop conditions under which the asymptotics of the coefficients are related to the values of the function near 1; specifically, $a(n) \sim f(1-1/n)/\alpha n$ (for some positive constant α), where $f(t) = \sum a(n)t^n$. In particular, if $F = \sum c(n)t^n$ where $c(n) \geq 0$ and $\sum c(n) = 1$, then f defined as $(1-F)^{-1}$ (the renewal or Green's function for F) satisfies this condition if F' does (and a minor additional condition is satisfied). In come cases, we can show that the absolute sum of the differences of consecutive Maclaurin coefficients converges. We also investigate situations in which less precise asymptotics are available.

This paper concerns asymptotics of Maclaurin series coefficients. It was motivated, in at least two ways, by probabilistic questions, and by the elegant method of Karamata in proving the Hardy–Littlewood theorem, *e.g.*, [T, 7.53, p. 227 ff.]).

Let $f = \sum a(n)t^n$ be a Maclaurin series (or the function corresponding to the Maclaurin series) with radius of convergence at least 1, and with all the coefficients nonnegative. We say that f is *weakly momentous* if $\lim_{t\uparrow 1} f(t^k)/f(t)$ exist and are not zero for $k = 2, 3$, and $f(1) = \infty$; if the same properties also hold for f', then f is *momentous* (the notation $t\uparrow 1$ means t approaches 1 along the real interval $(0,1)$). We say that f satisfies LLT (a local limit property) if $a(n) \sim h(1-1/n)/n$ (*i.e.*, $na(n)/h(1-1/n) \to 1$ as the integer $n \to \infty$) for some weakly momentous h (generically, we can take $h = cf$ for some constant c). The basic questions considered in this paper are criteria for such series to be LLT, and then specifically to answer questions concerning the renewal (or Green's) functions which arise as $f = (1-F)^{-1}$ where $F = \sum_{n=1}^{\infty} c(n)t^n$ with $c(n) \geq 0$ and $\sum c(n) = 1$.

The primary motivation is suggested by the terminology. Let X be a Markov chain with a state o. Start the process at discrete time zero, at state o. Set $a(n) = \Pr(X^n = o)$. Here $\Pr(X^n = o)$ is probability-speak[1] for the likelihood that the particle is at state o at time n, *i.e.*, at exactly n iterations of the process (we could use any other state than o). A local limit formula in this context is an asymptotic formula for $a(n)$ (as $n \to \infty$), and of course, if we form $f = \sum a(n)t^n$, the definition above (applied to f) seems to be a more specific version of the probabilistic one. There is an enormous literature on local limit theorems in probability.

If we take as our Markov chain the stationary discrete random walk on the nonnegative integers given by $\Pr(X = k) = c(k)$ (*i.e.*, the particle moves from state l to $k + l$ in one unit of time with likelihood $c(k)$), and form $F = \sum c(n)t^n$, then we obtain its corresponding Green's (or renewal) function $f = (1-F)^{-1}$. It is easy to see that f has radius of convergence 1 and all its Maclaurin series coefficients are nonnegative;

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[1] Honouring the centennial of the birth of George Orwell—when I submitted this paper originally.

moreover, they go to zero if and only if the process is interesting. The Maclaurin series coefficients, $\{a(n)\}$, of $f = \sum a(n)t^n$ have an obvious interpretation: $a(n)$ is the likelihood that the particle (starting at position zero and time zero) *ever* lands at n.

A rough outline of the paper is as follows. Section 1 deals with momentous functions, based on Karamata's argument. Sections 2 and 3 develop criteria and consequences for the series to be LLT, mostly in terms similar to those of the original Hardy–Littlewood theorem, *i.e.,* the behaviour of f along $(0, 1)$. Sections 4 and 5 deal with properties of coefficients transferred by convolution; Sections 6 and 7, with a generalization of momentous (*imitation momentous*); and Sections 8 and 9 discuss the behaviour of these functions on the boundary of the unit disk and a consequence for summability of the absolute differences of the coefficients. Section 10 exploits all the previous sections to obtain results on the coefficients of the Green's functions, including that, if F' satisfies LLT and $\alpha(F') > 0$, then its Green's function, $(1-F)^{-1}$, satisfies LLT. The Appendix contains a sufficient condition for some power of a power series to have increasing coefficients.

In more detail, Section 1 discusses momentous functions, and shows that these are more or less the same functions considered by Karamata, specifically of the form $(1-t)^{-\alpha}L$ where L is slowly varying. The α is recoverable from the definition above, via $\alpha(f) = -\ln_2 \lim_{t\uparrow 1} f(t^2)/f(t)$. For each momentous function there exists a probability measure on $[0, 1]$, depending only on $\alpha(f) \equiv \alpha$, whose moments are $\{\lim_{t\uparrow 1} f(t^{k+1})/f(t)\}_k$ (Proposition 1.2), and these are used to reprove elaborations of Karamata's version of the Hardy–Littlewood theorem. Moreover, sufficient for f to be momentous (provided all the Maclaurin coefficients are nonnegative) is that $f'(t) \sim (1-t)^{-1}f(t)c$ (as $t\uparrow 1$) for a positive constant c (Corollary 1.8). (This is almost necessary; the correct necessary and sufficient conditions are discussed.)

Section 2 deals with characterizations of f satisfying LLT. For example, suppose that $f = \sum a(n)t^n$ is momentous, $\alpha(f) > 0$, and $d_f(n) := \max\{0, a(n) - a(n+1)\}$ satisfies $d_f(n) = \boldsymbol{O}(a(n)/n)$. Then $a(n) \sim f(1-1/n)/n\Gamma(\alpha)$ (Proposition 2.6). When $\alpha(f) = 0$, the conclusion is that $a(n) \sim f'(1-1/n)/n^2$. These properties are the LLT results we are seeking, and we give other sets of conditions that yield them. We also give necessary and sufficient conditions for a momentous function to be LLT, in the form of a modified limit ratio property VRT: for all $M\colon \mathbf{N} \to \mathbf{N}$ such that $M(n) = \boldsymbol{o}(n)$, $\lim a(n + M(n))/a(n)$ exists (Theorem 2.7). However, this can be difficult to verify in specific examples. A weakened form of monotonicity of the coefficients, in the presence of momentous, is also sufficient for LLT.

Section 3 examines the behaviour of LLT and momentous functions under Hadamard operations. This section mostly concerns examples and some of the asymptotic summation formulas that can be derived therefrom. Although it is trivial that the ordinary product (convolution) of momentous Maclaurin series is still momentous, the corresponding result for Hadamard products fails utterly. However, if one of them is LLT and the obvious additional condition on the growth holds, then the outcome *is* momentous (Corollary 3.3).

Section 4 concerns transfer of properties under convolution: if f has property P (usually a condition on its coefficients) and g is suitably conditioned, then the product fg should have property P. If f is LLT and g is momentous, it is not true that fg is LLT; however, with additional hypotheses on g (and sometimes f, such as

$\alpha(f) > 0$), there is a theorem. The motivation, of course, is the fact that the Green's function satisfies $((1-F)^{-1})' = F' \cdot (1-F)^{-2}$ and we can deduce properties of the coefficients of the derivative of the Green's function which easily transfer back to those of the Green's function.

The brief Section 5 is so because of the paucity of results on the coefficients of perturbed power series. Sections 6 and 7 discuss a generalization of momentous, called *imitation momentous*, namely that $f'(t) = \boldsymbol{O}((1-t)^{-1} f(t))$ as $t\uparrow 1$. Many of the results on momentous functions have analogues in this context; for example, if f is imitation momentous and satisfies VRT, then both $(f, t^N) = \boldsymbol{O}(f'(1-1/N)/N^2)$ and $f'(1-1/N)/N^2 = \boldsymbol{O}((f, t^N))$ (this relation is denoted $(f, t^N) \approx f'(1-1/N)/N^2$).

Section 8 deals with the behaviour of these functions on the unit circle (with 1 deleted). For example, if $f = \sum a(n)t^n$ has only nonnegative coefficients and radius of convergence 1, and $|a(n) - a(n-1)| = \boldsymbol{O}(a(n)/n)$, then f is defined on $\mathbf{T} \setminus \{1\}$ and $|f(e^{i\theta})| = \boldsymbol{O}(f(1-\theta))$ as $\theta\downarrow 0$. This yields estimates of the form $(f, t^N) = \boldsymbol{O}(f(1-1/N)/N)$. This is exploited in Section 9 to give sufficient conditions under which $(1-t)(1-F)^{-1}$ has absolutely summable coefficients (that is, the set of absolute values of consecutive differences of the coefficients of the Green's function is summable).

In Section 10, we can finally prove decisive results on Green's functions. For example, if F' is LLT and $\alpha(F') > 0$, then $(1-F)^{-1}$ is LLT. In other words, there is a local limit theorem for the coefficients of the Green's function. There are corresponding results dealing with VRT. Section 11 gives explicit conditions on the coefficients of F under which F' and $(1-F)^{-1}$ satisfy the momentous-like properties discussed in the rest of the paper.

Finally, the appendix deals with a quantitative version of one of the characterizations of LLT. In Section 2, *inter alia*, we show that $\alpha \equiv \alpha(f) > 0$ and f LLT is equivalent to

$$\left| (f, t^N) - \frac{\alpha \sum_{n<N} (f, t^n)}{N} \right| = \boldsymbol{o}((f, t^N)).$$

If we replace the error estimate, $\boldsymbol{o}((f, t^N))$, by the stronger $\boldsymbol{O}((f, t^N)/N^r)$ for some $r > 0$, then the conclusion is that there exists a positive integer k such that f^k has its coefficients increasing, after deleting an initial segment. The "N^r" is close to being best possible, *e.g.*, the result fails if the error term is $\boldsymbol{O}((f, t^N)/(\ln N)^m)$ for any choice of $m > 0$.

Notation Throughout the paper, $f(t) \sim g(t)$ (sometimes abbreviated $f \sim g$) means that $\lim_{t\uparrow 1} f(t)/g(t) = 1$. The same symbol is used for sequences, *e.g.*, if $a, b \colon \mathbf{N} \to \mathbf{R}^+$, then $a(n) \sim b(n)$ means $\lim_{n\to\infty} a(n)/b(n) = 1$. We use the notation $f(t) \approx g(t)$ (and similarly for sequences) if both $f(t) = \boldsymbol{O}(g(t))$ and $g(t) = \boldsymbol{O}(f(t))$ as $t\uparrow 1$.

If $f \equiv f(t)$ is analytic in a neighbourhood of zero (typically with radius of convergence 1), we denote the coefficient of t^n in its Maclaurin series expansion by (f, t^n). For the most part, we are dealing with real analytic functions, hence the use of t as the name of the variable; when we discuss f as a function on the open unit disk, D, of course, we use the variable z.

1 Momentous Behaviour

Let $f = \sum a(n)t^n$ be a power series with radius of convergence 1, and with $a(n)$ all real and nonnegative. The theorem of Hardy and Littlewood says that if $f(t) \sim (1-t)^{-1}$ (*i.e.*, $\lim_{t\uparrow 1}(1-t)f(t) = 1$), then $\sum_{n=0}^{N} a(n) \sim N$, exactly as if f were $(1-t)^{-1}$. From this, one can deduce similar results for other behaviours at 1, and results of this kind can be found, for example in Titchmarsh [T]. However, these have an *ad hoc* flavour. Here we provide a unified approach to this type of result.

Let $f = \sum a(n)t^n$ be a Maclaurin series with radius of convergence 1 and $a(n) \geq 0$ for all integers n. We say that f is *weakly momentous* if *both* of the following conditions hold:

(i) $\sum a(n) = \infty$;

(ii) for all positive integers k, the limits

$$\lim_{t\uparrow 1} \frac{f(t^{k+1})}{f(t)} = M_k(f)$$

exist and are not zero.

We say that f is *momentous* if

(iii) f is weakly momentous and *either*:

(a) $M_1(f) < 1$ *or*

(b) $M_1(f) = 1$ and f' is weakly momentous.

It is *not* true that f being weakly momentous implies that f' is; a simple example is provided by $f = \sum t^{2^n}$. As is well known, $f \sim \ln_2(1/(1-t))/t$, so that $M_k(f)$ all exist and equal 1, yet $(1-t)f'$ does not converge (although it is bounded above and below away from zero) as $t\uparrow 1$. (This is one of the standard examples showing that l'Hôpital's rule does not apply to anti-derivatives.)

More drastic failure to be weakly momentous arises when the growth of the coefficients is subexponential but not polynomial, *e.g.*, when $f(t) = \exp(1-t)^{-1}$ (the coefficients are obviously nonnegative). In this case, the $M_k(f)$ exist but are zero for $k \geq 1$.

When f is weakly momentous, we define $\alpha \equiv \alpha(f)$ via $\alpha = -\ln_2 M_1(f)$. Clearly $\alpha \geq 0$, and in the exceptional case (iii)(b), $\alpha = 0$.

The first result yields in particular that if f is weakly momentous, then $f \sim (1-t)^{-\alpha} \cdot L(1/(1-t))$ where L is slowly varying [F, p. 268 ff.], so that ostensibly nothing new has been defined. (Note that such a function is automatically weakly momentous.)

Lemma 1.1 *Suppose that $L\colon [0,1) \to \mathbf{R}^+$ is a function increasing to ∞ such that for all positive integers k, the limits*

$$W_k := \lim_{t\uparrow 1} \frac{L(t^{k+1})}{L(t)}$$

exist and are all nonzero. Then $W_k = (k+1)^{-\alpha}$ where $\alpha = -\ln_2 W_1$. Moreover, W_k exists and equals $(k+1)^{-\alpha}$ for all nonnegative real numbers k.

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Proof Let $q = a/b$ be a rational number with a and b positive integers. The claim is that $\lim_{t\uparrow 1} L(t^a)/L(t^b)$ exists, depends only on q (not on the choice of a and b), and is not zero. Since

$$\frac{L(t^a)}{L(t^b)} = \frac{\frac{L(t^a)}{L(t)}}{\frac{L(t^b)}{L(t)}},$$

and both numerator and denominator have nonzero limits, the left side has a limit, and it is W_{a-1}/W_{b-1}. Next, if $a'/b' = a/b$ and a' and b' are positive integers, then $a'b = ab'$. Now $L((t^{b'})^a)/L(t^{b'^b})$ has a limit, W_{a-1}/W_{b-1}. The expression is the same as $L((t^b)^{a'})/L(t^{b^{b'}})$, whose limit exists and is $W_{a'-1}/W_{b'-1}$, so that the limit of the displayed expression depends only on q.

We can thereby define the function $F\colon \mathbf{Q}^{++} \to \mathbf{R}^{+}$ via $F(q) = W_{a-1}/W_{b-1} = \lim_{t\uparrow 1} L(t^a)/L(t^b)$ for any positive a and b with $q = a/b$. Next, we observe that F is multiplicative; it is sufficient to show $W_{k-1}W_{l-1} = W_{kl-1}$ for positive integers k and l, and this is immediate from the factorization $L(t^{kl})/L(t) = (L((t^k)^l)/L(t^k)) \cdot (L(t^k)/L(t))$.

In addition, F is order-reversing (that is, if $0 < q < q'$ are positive rationals, then $F(q) \geq F(q')$). To see this, find a positive integer M so that both qM and $q'M$ are integers, and observe that since L is increasing, $L(t^{q'M}) \leq L(t^{qM})$ (since $t < 1$). Dividing both by $L(t^M)$ and taking the limits, we see that $F(q') \leq F(q)$.

Let S denote the additive subgroup consisting of the logarithms of elements of the positive rational numbers. We may define $G\colon S \to \mathbf{R}$ via $G(s) = -\log F(\exp s)$. The preceding shows that G is an additive mapping from S to $\mathbf{R}$ with the additional property that G is order-preserving. The latter guarantees that G is continuous (with regard to the topology on S inherited from $\mathbf{R}$), hence G extends to a real linear map $\mathbf{R} \to \mathbf{R}$. This is of the form $r \mapsto ur$ for some real u, and so $-\log F(\exp s) = us$. Hence $F(q) = 1/q^u$. Restricting F to the integers yields the desired result, with $\alpha = u$.

The preceding argument showed that W_k exists for rational k, and since the rationals are order dense in the reals (every real number is the supremum and the infimum of sequences of rational numbers) and $k \mapsto W_k$ is continuous and order-reversing, the result follows. ∎

In fact, it is sufficient to require merely that W_{k-1} be defined for a set of ks whose greatest common divisor is 1 (for such a finite set, every sufficiently large integer can be expressed as a sum of elements of the set, and the multiplicative group generated by the set is dense in $\mathbf{R}^{++}$). In particular, it is sufficient that merely W_1 and W_2 (corresponding to $k = 2, 3$) exist. However, it is possible to construct examples wherein for one prime p, W_{p-1} exists, but not for any other prime.

Obviously, in Proposition 1.2(ii) below, p is permitted to be complex-valued (to see this, work with Re p and Im p).

Proposition 1.2 *Suppose that f is weakly momentous, and let K be a real analytic function with radius of convergence* 1*, such that $f \sim K$ as $t\uparrow 1$. Let $p\colon [0,1] \to \mathbf{R}$ denote a continuous function.*

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(i) $\lim_{t\uparrow 1} \dfrac{\sum a_n t^n p(t^n)}{K(t)} := \psi(p)$ *exists;*

(ii) *there exists a unique probability measure* μ_α ($\alpha \equiv \alpha(f)$) *such that for all* p, $\psi(p) = \int_0^1 p(x)\, d\mu_\alpha(x)$;

(iii) *for all positive integers* k, $\int_0^1 x^k \, d\mu_\alpha(x) = M_k(f) = (k+1)^{-\alpha}$, *and* μ_α *depends only on* α, *not* f;

(iv) *if* $\alpha = 0$, *then* μ_α *is the point mass at* 1; *otherwise,* $d\mu_\alpha = (-\ln t)^{\alpha-1}/\Gamma(\alpha)\, d\lambda$ *where* λ *is Lebesgue measure on* $[0,1]$.

Proof We first observe that as $f \sim K$, it follows that $M_k(K)$ exist and equal $M_k(f)$ for each k. If $p(t) = t^k$, we can write

$$\frac{\sum a_n t^n p(t^n)}{K(t)} = \frac{\sum a_n t^n p(t^{nk})}{K(t)} = \frac{\sum a_n t^{n(1+k)}}{K(t^{k+1})} \cdot \frac{K(t^{k+1})}{K(t)} = \frac{f(t^{k+1})}{K(t^{k+1})} \cdot \frac{K(t^{k+1})}{K(t)}.$$

As t increases to 1, each of the two factors converges, so the limit exists, and is M_k.

Hence the assertion holds if p is a polynomial. Let $q\colon [0,1] \to \mathbf{R}$ be any (not necessarily continuous) bounded function, say with $|q(t)| \le L$. Then

$$\limsup_{t\uparrow 1} \frac{|\sum a_n t^n q(t^n)|}{K(t)} \le \limsup L \cdot \frac{\sum a_n t^n}{K(t)} = L.$$

It follows that the assignment of p to the limit is uniformly continuous (with regard to the uniform norm from $[0,1]$) on polynomials. Hence it extends to the closure of the polynomials, *i.e.,* $C([0,1])$, proving (i).

The assignment $p \mapsto \lim \sum a_n t^n p(t^n)/K(t)$ is denoted ψ. Thus $\psi\colon C([0,1]) \to \mathbf{R}$ is a linear functional, and moreover, it sends the constant function 1 to 1. Obviously, it is positive (since the coefficients are all nonnegative), so ψ is a normalized positive linear functional (of course, this would imply that it is uniformly continuous, but we still had to use the uniform continuity argument above). By the Riesz representation theorem, there exists a unique probability measure μ on $[0,1]$ such that $\psi(p) = \int p\, d\mu$ for all continuous functions p. This yields (ii).

Applying the linear functional to monomials, we saw above that $\psi(t^{k+1}) = M_k(f)$, so that the moments of the representing measure μ are $\{(k+1)^{-\alpha}\}$. Since a measure on $[0,1]$ is determined by its moments (as the polynomials are dense in $C([0,1])$), $\mu \equiv \mu_\alpha$ depends only on α. This gives us (iii).

The moments of the point mass measure $\delta_{\{1\}}$ are all 1, hence if $\alpha = 0$, it follows that $\mu_\alpha = \delta_{\{1\}}$. For $\alpha > 0$ and $k \ge 1$, set $F(k) = \int_0^1 t^{k-1}(-\ln t)^{\alpha-1}\, dt/\Gamma(\alpha)$, whenever the latter is defined. When $k = 1$, the substitution $u = -\ln t$ converts the integral into the usual expression for $\Gamma(\alpha)$; hence $F(1)$ exists and equals 1. Moreover, since the integrands are all nonnegative, it follows easily that $F(k)$ exists for all $k \ge 1$.

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For $l \geq 1$, we have:

$$\begin{aligned} F(kl) &= \frac{1}{\Gamma(\alpha)} \int_0^1 t^{kl-1}(-\ln t)^{\alpha-1}\, dt \\ &= \frac{1}{\Gamma(\alpha)} \int_0^1 (t^k)^{l-1} \frac{(-\ln t^k)^{\alpha-1}}{k^{\alpha-1}} t^{k-1}\, dt \\ &= \frac{1}{k^{\alpha-1}\Gamma(\alpha)} \int_0^1 (t^k)^{l-1}(-\ln t^k)^{\alpha-1}\, \frac{d(t^k)}{k} \\ &= \frac{1}{k^\alpha} F(l). \end{aligned}$$

Setting $l = 1$, we obtain $F(k) = 1/k^\alpha$, as desired (the moments are $\{F(k+1)\}_{k=0}$).

This argument shows that the indicated measures have the necessary moments. However, it completely obscures the derivation of the Radon–Nikodym derivative of μ_α, so we also give a quick outline of how to obtain the $(-\ln t)^{-\alpha}$ term.

We are free to select a function f with $\alpha(f) = \alpha$, since the measure depends only on α; we choose $f = (1-t)^{-\alpha}$. Select $0 < a \leq b < 1$ and set $E = [a, b]$. Then it is easy to see that if the limit as $t \uparrow 1$ of

$$\frac{\sum_{t^n \in E} a(n) t^n}{(1-t)^{-\alpha}}$$

exists, the limit must be $\mu_\alpha(E)$. Now $a \leq t^n \leq b$ is equivalent to $\ln b/\ln t < n < \ln a/\ln t$. Moreover, it is an easy consequence of Stirling's formula (applied to the Gamma function) that $a(n) \sim n^{\alpha-1}/\Gamma(\alpha)$. Since $a > 0$ and $b < 1$, we can replace the coefficients by this asymptotic form, and the sum in the display by the corresponding integral. Thus the displayed expression is asymptotic to

$$\frac{\frac{1}{\Gamma(\alpha)} \int_{\ln b/\ln t}^{\ln a/\ln t} x^{\alpha-1} t^x\, dx}{(1-t)^{-\alpha}}.$$

Now write $t = 1 - s$ (so $s \to 0$). Then $t^x = (1-s)^x = (1-s)^{\frac{1}{s}\cdot sx}$. Since $\ln t \sim -s$ and $-sx$ is bounded above and below away from zero, we have that $t^x \sim e^{-sx}$. Hence the expression is asymptotic to

$$\frac{\frac{1}{\Gamma(\alpha)} \int_{-\ln b/s}^{-\ln a/s} x^{\alpha-1} e^{-sx}\, dx}{s^{-\alpha}}.$$

Substituting $u = e^{-sx}$ and reversing endpoints of integration we find that all the s terms cancel, and we are left with $\int_a^b (-\ln u)^{\alpha-1}\, du/\Gamma(\alpha)$, whence this is $\mu_\alpha(E)$. ■

Now we obtain the classical theorem of Hardy and Littlewood, à la Karamata.

Theorem 1.3 *Let $f = \sum a(n)t^n$ be a weakly momentous function with $\alpha(f) = \alpha$, and let $r > 1$ be a positive real number. Then*

$$\sum_{n=0}^{N} a(n) \sim \frac{f(e^{-1/N})}{\Gamma(\alpha+1)} \quad \text{and} \quad \sum_{n=N/r}^{N} a(n) \begin{cases} \sim \dfrac{f(e^{-1/N})(1-\frac{1}{r^\alpha})}{\Gamma(1+\alpha)} & \text{if } \alpha > 0, \\ = \boldsymbol{o}(\sum_{n=0}^{N} a(n)) & \text{if } \alpha = 0. \end{cases}$$

Proof In the first case, define $g\colon [0,1] \to \mathbf{R}$ via $g(t) = 0$ if $t \leq 1/e$, and $g(t) = 1/t$ otherwise. Then g is piecewise continuous, and the set of point(s) of discontinuity has measure zero with regard to μ_α (even when $\alpha = 0$). Let χ_m be the indicator function of $[1/e - 1/m, 1/e + 1/m]$. Then g is a pointwise limit of continuous functions q_m for which $0 \leq q_m - g \leq \chi_m$, so that ψ (the linear functional implemented via μ_α) extends to g, and it follows from Proposition 1.2(i) that $\lim_{t\uparrow 1} \sum a(n)t^n g(t^n)/f(t)$ exists and equals $\int g\, d\mu$. Now we let N be a large integer, and set $t = e^{-1/N}$. Then $\sum a(n)t^n g(t^n)/f(t)$ becomes

$$\frac{\sum_{n=0}^{N} a(n)e^{-n/N}e^{n/N}}{f(e^{-1/N})}.$$

(Observe that $t^n < 1/e$ if and only if $n > N$.) The displayed expression is just $\sum_0^N a(n)/f(e^{-1/N})$. Thus

$$\sum_{n=1}^{N} a(n) \sim f(e^{-1/N}) \int g\, d\mu_\alpha.$$

The integral depends only on g and α. If $\alpha = 0$, the measure is point mass at 1, so the integral is 1. Otherwise, we see that everything applies in the special case that $f_0 = (1-t)^{-\alpha}$, where we can calculate the other two terms. Explicitly, the term on the left (with f_0 replacing f) is $((1-t)^{-1}f_0, t^N)$, and this is just $((1-t)^{-1-\alpha}, t^N)$, which we already know to be asymptotic with $N^\alpha/\Gamma(1+\alpha)$. Now $f_0(e^{-1/N}) \sim N^\alpha$, and it follows that $\int g\, d\mu_\alpha = 1/\Gamma(1+\alpha)$, which is the first result. (Alternatively, $\int g\, d\mu_\alpha = \int_{e^{-1}}^{1} x^{-1}(-\ln x)^{\alpha-1}\, dx/\Gamma(\alpha) = \int_0^1 (-\ln x)^{\alpha-1}\, d(-\ln x)/\Gamma(\alpha) = 1/(\alpha\Gamma(\alpha))$.)

Now consider the second result. If $\alpha = 0$, $\sum_0^{N/r} a(n) \sim f(e^{-r/N})$ and $\sum_0^N a(n) \sim f(e^{-1/N})$. The ratio is thus asymptotic to $f(e^{-r/N})/f(e^{-1/N})$ which tends to 1 (as $\alpha = 0$, $M_k = 1$ for all k). Hence the difference tends to zero in comparison with $\sum_0^N a(n)$. If $\alpha > 0$, the two terms are respectively $f(e^{-r/N})/\Gamma(1+\alpha)$ and $f(e^{-1/N})/\Gamma(1+\alpha)$, and their ratio thus tends to $r^{-\alpha}$. Hence their difference behaves as $(1 - r^{-\alpha})\sum_0^N a(n) \sim f(e^{-1/N})(1 - r^{-\alpha})$. This yields the rest of the second statement. ■

Suppose that the f is asymptotic to the analytic function K; then K is weakly momentous (and $\alpha(K) = \alpha(f)$, and $f(e^{-1/N}) \sim K(e^{-1/N})$. Then the occurrences of f on the right sides in Theorem 1.3 can be replaced by K, so the asymptotics depend only on K.

Lemma 1.4 *Suppose that $f = \sum a(n)t^n$ is weakly momentous, $\alpha \equiv \alpha(f)$. Let s be a nonzero complex number, and let r be a positive real number less than 1. If $\alpha \neq 0$ and* $\operatorname{Re} s \geq 0$, *then*

$$\sum_{n=1}^{N} a(n)n^s \sim \frac{N^s f(e^{-1/N})}{\Gamma(\alpha)(s+\alpha)};$$

$$\sum_{n=1}^{N} \frac{a(n)}{n^s} \sim \frac{f(e^{-1/N})}{N^s\Gamma(\alpha)(\alpha - s)} \quad \text{if } \alpha > \operatorname{Re} s;$$

$$\sum_{n=rN}^{N} \frac{a(n)}{n^s} \sim \begin{cases} \dfrac{f(e^{-1/N})}{N^s} \cdot \dfrac{1 - r^{\alpha - s}}{\Gamma(\alpha)(\alpha - s)} & \text{if } \alpha - s \text{ is not purely imaginary or zero,} \\ \dfrac{f(e^{-1/N})}{N^s} \cdot \dfrac{\ln(1/r)}{\Gamma(\alpha)} & \text{if } \alpha = s; \end{cases}$$

$$\frac{N^s \sum_{n=rN}^{N} \frac{a(n)}{n^s}}{f(e^{-1/N})} \to 0 \quad \text{if } \alpha - s \text{ is purely imaginary and not zero.}$$

If $\alpha = 0$ and $\operatorname{Re} s > 0$, *then*

$$\frac{N^s \sum_{n=rN}^{N} \frac{a(n)}{n^s}}{f(e^{-1/N})} \to 0.$$

Remark Since $e^{-1/N} \sim 1 - 1/N$, all occurrences of the former can be replaced by the latter in the formulæ above.

Proof In all cases, the result is based on $\sum a(n)t^n p(t^n)/f(t) \to \int p \, d\mu_\alpha$ for p with at most two discontinuities in $(0, 1)$. In the first case, set $p(t) = (-\ln t)^s/t$ for $e^{-1} < t \leq 1$ and zero otherwise. Then $p(e^{-1/N}) = 0$ if $e^{-n/N} < e^{-1}$, *i.e.*, $n > N$. It remains to evaluate the integral:

$$\int p \, d\mu_\alpha = \frac{1}{\Gamma(\alpha)} \int_{e^{-1}}^{1} \frac{(-\ln t)^{s+\alpha-1}}{t} \, dt = \frac{1}{\Gamma(\alpha)} \cdot \frac{u^{s+\alpha}}{s+\alpha}\bigg|_0^1 = \frac{1}{\Gamma(\alpha)} \cdot \frac{1}{s+\alpha}.$$

In the cases with denominator n^s, set $p(t) = t^{-1}(-\ln t)^{-s}$ if $e^{-1} \leq t \leq e^{-r}$ and zero otherwise. For the situation in which $\alpha > \operatorname{Re} s$, interpret r as zero. Then $p(t^n)\,|_{t=e^{-1/N}}$ is not zero only if $e^{-1} \leq e^{-n/N} \leq e^{-r}$, *i.e.*, $rN \leq n \leq N$. It remains to evaluate the integral.

$$\int p \, d\mu_\alpha = \frac{1}{\Gamma(\alpha)} \int_{e^{-1}}^{e^{-r}} \frac{(-\ln t)^{\alpha-s-1}}{t} \, dt = \frac{1}{\Gamma(\alpha)} \int_r^1 u^{\alpha-s-1} \, du.$$

If $\alpha \neq s$, $\int_r^1 u^{\alpha-s-1} \, du = (1 - r^{\alpha-s})/(\alpha - s)$. If now $\alpha - s$ is purely imaginary, the integral is zero, yielding the special case above. If $\alpha = s$, the integral is just $\ln(1/r)$.

If $\alpha = 0$ and $\operatorname{Re} s > 0$, then $\mu_\alpha = \delta_1$. As p is continuous at 1, the results apply, and since $p(1) = 0$, we obtain the final result. ■

The following result will be used later on. It bears a resemblance to [H, Theorem 100, p. 157].

Proposition 1.5 *Suppose that $f = \sum a(n)t^n$ is momentous and $\alpha \equiv \alpha(f) > 0$. Let s be a complex number such that $\operatorname{Re} s > \alpha$. Then*

$$\sum_{n>N} \frac{a(n)}{n^s} \sim \frac{f(e^{-1/N})}{N^s} \cdot \frac{1}{(s-\alpha)\Gamma(\alpha)}.$$

Proof Set $p(t) = t^{-1}(-\ln t)^{-s}$ if $0 < t < e^{-1}$, and zero otherwise. The claim is that p belongs to $\mathrm{L}^1(\mu_\alpha)$. From $\operatorname{Re} s > \alpha$,

$$\Gamma(\alpha) \cdot \int p\, d\mu_\alpha = \lim_{\epsilon \to 0} \int_\epsilon^{e^{-1}} \frac{dt}{t \cdot (-\ln t)^{s+1-\alpha}} = \lim_{M\to\infty} \int_1^M \frac{du}{u^{s+1-\alpha}}$$
$$= \lim_{M\to\infty} \frac{u^{\alpha-s}}{\alpha - s}\Big|_1^M = \lim_{M\to\infty} \frac{1}{\alpha-s}\Big(\frac{1}{M^{s-\alpha}} - 1\Big) = \frac{1}{s-\alpha}.$$

Hence $\sum a(n)t^n p(t^n)/f(t) \to (s-\alpha)^{-1}$. However,

$$\sum a(n)t^n p(t^n)\big|_{t=e^{-1/N}} = \sum_{n>\infty} a(n)(N/n)^s. \qquad \blacksquare$$

Now we show that if f is momentous (as opposed to weakly momentous), then so are all of its derivatives.

Proposition 1.6 *If f is momentous, then so are all of its derivatives. When this occurs, $M_k(f') = M_k(f)/(k+1)$ and $\alpha(f') = \alpha(f) + 1$. Moreover, if $\alpha \equiv \alpha(f) > 0$, then $(1-t)^{-1}f \sim f'/\alpha$.*

Proof Suppose that $\alpha \equiv \alpha(f) > 0$. We note that $\alpha((1-t)^{-1}f)) = 1 + \alpha$. Then with $s = 1$, Lemma 1.4 yields

$$\begin{aligned}
\big((1-t)^{-1}tf', t^N\big) &= \sum_0^N a(n)n \sim Nf(e^{-1/N})\frac{1}{\Gamma(\alpha)(\alpha+1)} \\
&\sim (1-t)^{-1}f|_{t=e^{-1/N}}\frac{1}{\Gamma(\alpha)(\alpha+1)} \\
&\sim \frac{\alpha\Gamma(1+\alpha((1-t)^{-1}f))}{\Gamma(\alpha)(\alpha+1)} \sum_0^N((1-t)^{-1}f, t^n) \\
&= ((1-t)^{-2}f, t^N)\frac{\Gamma(\alpha+2)}{\Gamma(\alpha)(\alpha+1)} \\
&= \alpha \cdot ((1-t)^{-2}f, t^N).
\end{aligned}$$

Thus $(1-t)^{-1}tf'(t) \sim \alpha \cdot (1-t)^{-2}f(t)$ as $t\uparrow 1$. It follows that $f' \sim \alpha \cdot (1-t)^{-1}f$ as $t\uparrow 1$, and now l'Hôpital's rule yields that $M_k(f')$ exist and equal $M_k(f)/(k+1)$. In particular, f' is weakly momentous; however, $\alpha(f') = -\ln_2 M_1(f') = 1 - \ln_2 M(f) = 1 + \alpha(f)$, so that $\alpha(f') > 0$, and the process can be repeated for all subsequent derivatives.

In case $\alpha = 0$, then the definition of momentous entails that f' be weakly momentous; again by l'Hôpital's rule, this forces $\alpha(f') = 1 + \alpha(f) = 1 > 0$, so that the preceding paragraph now applies to f'. ∎

Proposition 1.7 *Suppose that $f\colon (0,1) \to \mathbf{R}^+$ is C^1, $f(t) > 0$ for all t in $(0,1)$, and in addition, either $f'(t) \sim c(1-t)^{-1}f(t)$ for some $c > 0$ or $f'(t) = \boldsymbol{o}((1-t)^{-1}f(t))$ (corresponding to $c = 0$) as $t\uparrow 1$. Then for all positive real l,*

$$\lim_{t\uparrow 1} \frac{f(t^l)}{f(t)} \text{ exists and equals } l^{-c}.$$

Proof Write $f'/f = c(1-t)^{-1} + p$, *i.e.*, $p = f'/f - c(1-t)^{-1}$ (where $c = 0$ if the condition $f' = \boldsymbol{o}((1-t)^{-1}f(t))$ holds). Obviously, p is continuous on $(0,1)$, and $p(t) = \boldsymbol{o}((1-t)^{-1})$, *i.e.*, $(1-t)p(t) \to 0$ as $t\uparrow 1$.

Since the differential equation $f' = (c(1-t)^{-1}+p)f$ has continuous coefficient(s), there exists a unique solution up to a constant multiple, $f = K(1-t)^{-c}\exp(D^{-1}p)$, where K is a (positive) constant, and $D^{-1}p$ is normalized to satisfy $D^{-1}p(\frac{1}{2}) = 0$.

Set $q = D^{-1}p$; the claim is that $q(t) - q(t^l) \to 0$ as $t\uparrow 1$. We may assume that $l > 1$. Obviously q is C^1, so we apply the mean value theorem to obtain $q(t) - q(t^l) = (t - t^l)q'(t_0)$ for some t_0 with $t^l \leq t_0 \leq t$. Now $t - t^l = (1-t)\cdot(t(1-t^l)/(1-t))$, and the latter factor is bounded above by l. Moreover, $1 - t < 1 - t_0$, so we have

$$\begin{aligned} |(t-t^l)q'(t_0)| &\leq l|(1-t)q'(t_0)| \\ &\leq l|(1-t_0)q'(t_0)| = l|(1-t_0)p(t_0)|. \end{aligned}$$

As $t\uparrow 1$, so does $t^l\uparrow 1$, thus $t_0\uparrow 1$. Hence the right-hand side tends to zero as $t\uparrow 1$, yielding the claim. Finally,

$$\begin{aligned} \frac{f(t^l)}{f(t)} &= \frac{K(1-t^l)^{-c}\exp(q(t^l))}{K(1-t)^{-c}\exp(q(t))} \\ &= \frac{(1-t^l)^{-c}}{(1-t)^{-c}} \cdot \exp(q(t^l) - q(t)) \\ &\to l^{-c} \cdot 1 \text{ as } t\uparrow 1 \end{aligned}$$

∎

Notice that l'Hôpital's rule is not applicable here, since we do not know at the outset that either $\lim f(t^l)/f(t)$ or $\lim f'(t^l)/f'(t)$ exists. In any event, we obtain the converse of Proposition 1.6.

Corollary 1.8 *Suppose that f is a Maclaurin series with only nonnegative coefficients and radius of convergence* 1*, and moreover, $f(1) = \infty$. If either there exists $c > 0$ such that $f' \sim c(1-t)^{-1} f$ as $t \uparrow 1$, or $f'(t) = \boldsymbol{o}((1-t)^{-1} f(t))$ as $t \uparrow 1$, then f is weakly momentous.*

Corollary 1.9 *Suppose that $f = \sum a(n)t^n$ has only nonnegative coefficients, $\sum a(n) = \infty$ and $a(n) = \boldsymbol{O}(1/n)$. Then f is weakly momentous and $\alpha(f) = 0$.*

Proof From $na(n) = \boldsymbol{O}(1)$, it follows that

$$tf'(t) = \sum a(n)t^n \leq K \sum t^n = K(1-t)^{-1}.$$

Hence $f'(t) = \boldsymbol{o}((1-t)^{-1} f(t))$ (as $t \uparrow 1$). ■

2 Local Limit Theorems

In the previous section, we examined the asymptotic behaviour of sums of the form $\sum_1^N a(n)$. What we really want are estimates for the individual $a(N)$.

If f has increasing coefficients (or more subtly, increasing after an initial segment), then we can write $f = (1-t)^{-1} f_0$ where $f_0 = \sum b(n)t^n$ has no negative coefficients. Moreover, it is easy to check that f is weakly momentous if and only if $M_k(f_0)$ exist (it can happen that $f_0(1) < \infty$, so f_0 would not be weakly momentous), and in this case, $M_k(f) = M_k(f_0)/(k+1)$, and $\alpha(f) = 1 + \alpha(f_0)$. Applying Theorem 1.3, we see that

$$a(N) = \sum_{n=0}^{N} b(n) \sim \frac{f_0(e^{-1/N})}{\Gamma(1+\alpha(f_0))} \sim \frac{f(e^{-1/N})}{N\Gamma(\alpha(f))}.$$

However, this has very limited application: it applies only when the coefficients are monotone increasing. More generally, if the coefficients are monotone (increasing or decreasing) and f is momentous, then the corresponding asymptotic formula applies [F, Theorem 5, p. 423]. This will be subsumed by Proposition 2.1 below.

A more intricate and general method of "isolating" the coefficients (*i.e.,* in order to obtain an expression for $a(N)$ via the sum $\sum_{n=0}^N a(n)$), involves the drop. Let $f = \sum_n a(n)x^n$ be a power series with nonnegative coefficients (and radius of convergence 1, although this plays no role in the definition). We define the *drop* of f at position n via

$$d_f(n) = \begin{cases} a(n) - a(n+1) & \text{if } a(n) \geq a(n+1), \\ 0 & \text{else.} \end{cases}$$

In other words, $d_f(n)$ is zero if the n-th coefficient is less than the $n+1$-st, but the absolute value of the difference if $a(n)$ exceeds $a(n+1)$. We extend the definition so that d_f is defined on subsets, via $d_f(S) := \sum_{s \in S} d_f(s)$. We will be notationally abusive in that $d_f(\{n\}) = d_f(n)$. In the special case that S is an interval of integers $S = \{k, k+1, \dots, l\}$, we use the notation $d_f([k,l])$. Note that the drop of such an interval is *not* generally $a(k) - a(l+1)$ (as it would be if $\{a(n)\}$ were monotone decreasing), but is the sum of the drops, so can be considerably larger.

Normally, the function f is understood, so the notation d_f is simplified to d. We say $f = \sum a(n)t^n$ (or its sequence of coefficients) satisfies the *limit ratio property* (LRT for short) if $\lim a(n)/a(n+1) = 1$. Obviously, this property is preserved under derivatives and anti-derivatives.

Here is a fairly complicated hypothesis which guarantees that coefficients can be isolated:

$$(\dagger) \qquad \lim_{s\downarrow 0} \limsup_{N\to\infty} \frac{\sum_{i=0}^{[sN]} d([N-i,N+i])}{sNa(N)} = 0.$$

Note the limit as s goes to zero always exists, as the limsups are decreasing in s in any event.

Proposition 2.1 *Suppose that f is weakly momentous and satisfies* $(\dagger)$. *If $\alpha(f) = 0$, then*

$$\frac{Na(N)}{\sum_0^N a(n)} \to 0.$$

If f is momentous and satisfies $(\dagger)$, *then*

$$\frac{Na(N)}{\sum_0^N a(n)} \to \alpha,$$

and

$$a(N) \sim \begin{cases} \dfrac{f(e^{-1/N})}{N\cdot\Gamma(\alpha)} & \text{if } \alpha > 0, \\[2ex] \dfrac{f'(e^{-1/N})}{N^2} & \text{if } \alpha = 0. \end{cases}$$

Proof We show the convergence to α, from which all the other parts of the statement follow. For each positive real number s, define

$$E(s) = \limsup_{N\to\infty} \frac{\sum_{i=0}^{[sN]} d([N-i,N+i])}{sNa(N)}.$$

The hypothesis is that $E(s) \to 0$ as $s \to 0$. For each $s > 0$, for all sufficiently large N, $\sum_{i=0}^{[sN]} d([N-i,N+i]) \le 2E(s)sNa(N)$. Now fix s. For $i > 0$,

$$(2.1) \qquad a(N-i) - d([N-i,N-1]) \le a(N) \le a(N+i) + d([N,N+i])$$

Summing the right inequality over $i \le [sN]$, we obtain $sNa(N) \le \sum_N^{[N(1+s)]} a(i) + \sum_0^{[sN]} d\big([N,N+i]\big)$. Thus for all sufficiently large N, $(1-2E(s))sNa(N) \le \sum_N^{[N(1+s)]} a(i)$. Divide this by $s\sum_0^N a(i)$, and we obtain

$$(1-2E(s))\frac{Na(N)}{\sum_0^N a(n)} \le \frac{\sum_N^{[N(1+s)]} a(i)}{s\sum_0^N a(i)}.$$

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If $\alpha = 0$, the right side has limsup equalling zero, by Theorem 1.3. By the same result, if $\alpha > 0$, the limit of the right side as $N \to \infty$ is $((1+s)^\alpha - 1)/s$. Letting $s \to 0$, very conveniently, the latter converges to $((1+s)^\alpha)'|_{s=0} = \alpha$. Since $E(s) \to 0$, we deduce $\limsup \frac{Na(N)}{\sum_0^N a(n)} \leq \alpha$.

The reverse inequality follows from applying virtually the same process to the left inequality in (2.1). Fix s again. With $\alpha > 0$, we have

$$\sum_{[N(1-s)]}^{N} a(i) - \sum_{0}^{[sN]} d([N-i, N]) \leq sNa(N).$$

For N sufficiently large, this yields

$$\sum_{[N(1-s)]}^{N} a(i) \leq (1 + 2E(s))sNa(N).$$

Dividing by $s\sum_0^N a(i)$ and applying Theorem 1.3 as before yields

$$\limsup Na(N) \sum_{0}^{N} a(i) \geq (1 + 2E(s))^{-1}(1 - (1-s)^\alpha)/s,$$

and now we let $s \to 0$.

If $\alpha > 0$, then $a(N) \sim \alpha \sum_0^N a(n)/N$, and by Theorem 1.3, this is asymptotically $\alpha f(e^{-1/N})/N\Gamma(1+\alpha) = f(e^{-1/N})/N\Gamma(\alpha)$.

Finally, if f is momentous but $\alpha = 0$, we consider f'. By the earlier result, f' is momentous and $\alpha(f') = 1$. It is very easy to check that the condition (†) applies as well to f'—just note that if $na(n) \leq (n+1)a(n+1)$, then $d_{f'}(n-1) = 0$, otherwise $d_{f'}(n-1) = n(a(n) - a(n+1)) - a(n+1) \leq nd_f(n)$. Hence the preceding applies to tf', and thus $Na(N) \sim f'(e^{-1/N})/N$, whence $a(N) \sim f'(e^{-1/N})/N^2$. ■

The asymptotics of $a(N)$ obtained in this result can be combined in a single statement, avoiding division into two cases. Applying Proposition 1.6, in case $\alpha > 0$,

$$a(N) \sim \frac{f(e^{-1/N})}{N\Gamma(\alpha)} = \frac{(1-t)^{-1} f|_{t=e^{-1/N}}}{N^2\Gamma(\alpha)} \sim \frac{f'(e^{-1/N})}{N^2\alpha\Gamma(\alpha)} = \frac{f'(e^{-1/N})}{N^2\Gamma(1+\alpha)}.$$

The last term is the asymptotic value of $a(N)$ if $\alpha = 0$, so that in any case, $a(N) \sim f'(e^{-1/N})/(N^2\Gamma(1+\alpha))$. If the sequence, power series, or function satisfies the conclusion of Proposition 2.1, we say that it satisfies LLT.

If we replace $d_f(n) = \max\{0, a(n) - a(n+1)\}$ by the corresponding condition measuring the increases, $u_f(n) := \max\{0, a(n+1) - a(n)\}$ ("u" for *up*, in contrast to the original "d" for *down*) and replace d_f by u_f in the definition of (†), the same conclusion holds. In particular, this result applies if either $a(n)$ is increasing ($d(n) = 0$), or decreasing ($u(n) = 0$).

A result in the Appendix (Proposition A.3) considers what happens when the convergence to α is sufficiently fast.

Another variation on (†) which suffices to prove the local limit theorem is

$$\lim_{s\downarrow 0}\limsup_N \frac{d\big(\,[\,[N(1-s)],[N(1+s)]\,]\,\big)}{a(N)}=0.$$

It is easy to check that this implies (†). A weakening of (†) for which the proof still works is the complicated averaged version

$$\lim_{s\downarrow 0}\limsup_N \frac{\sum_0^{[sN]} d(N-i,N+i])}{\sum_{i=-[sN]}^{[sN]} a(N+i)}=0.$$

A somewhat different condition is a quantitative version of the ratio limit hypothesis. We require yet another form of the moment condition. We say that $f=\sum a(n)t^n$ (or its corresponding sequence of coefficients) is *strongly momentous* if $\lim a(n)/a(n+1)=1$ (LRT) and for each positive integer k, the fractions $a(n)/a((k+1)n)$ converge to a nonzero limit (denoted $N_k(f)$). Obviously, f is strongly momentous if and only if f' is, and of course this applies to all higher derivatives. Note that we do not impose the condition $\sum a(n)=\infty$ in this definition. This is a formally weaker condition than "regularly varying sequence" [GH, Definition 1.37, p. 54], although it turns out to be more or less equivalent.

Lemma 2.2 *Suppose that $f=\sum a(n)t^n$ satisfies $a(n)\geq 0$ for all n, $d_f(n)=\boldsymbol{o}(a(n))$, $\sum a(n)=\infty$, and the radius of convergence is 1. Then for any positive integer D, the function $f_D:=\sum_{n=0}^{\infty} a(nD)t^{nD}$ satisfies $f_D\sim D^{-1}f$ as $t\uparrow 1$. In fact, $f_{(0,D)}:=\sum_n(\min_{0\leq i<D}\{a(nD+i)\})t^{nD}$ also satisfies $f_{(0,D)}\sim D^{-1}f$.*

Proof For each integer $i=0,1,2,\dots,D-1$, define $f_{D,i}=\sum_{n=0}^{\infty} a(nD+i)t^{nD+i}$, so that $f_{D,0}=f_D$ and $f=\sum_{i<D} f_{D,i}$. Also define $f_{D,D}=\sum a((n+1)D)t^{(n+1)D}$; of course, $f_{D,D}=f_D-a(0)$. From $a(n+1)\geq a(n)-d(n)$, we deduce that $\liminf a(n+1)/a(n)\geq 1$. Thus for $t>0$, and $0\leq i\leq D-1$, $f_{D,i+1}(t)\geq f_D+q_i(t)$ where $q_i=\sum b_i(n)t^n$ with $|b_i(n)|=\boldsymbol{o}(a(n))$. In particular, $q_i(t)\sim\boldsymbol{o}(f(t))$ as $t\uparrow 1$. Since $f=\sum_{i<D} f_i$, it follows that each $f_i(t)\to\infty$ as $t\uparrow 1$, and so $a(0)=\boldsymbol{o}(f_{D,D}(t))$ (as $t\to 1$). It follows immediately that $f_D\sim f_{D,i}$ for each i, and we are done. Concerning $f_{(0,D)}$, we note that $\liminf a(n+1)/a(n)\geq 1$ is sufficient to yield $f_{(0,D)}\sim f_D$. ■

If we replace $d_f(n)=\boldsymbol{o}(a(n))$ by $u_f(n)$, the same conclusion holds. In particular, this result applies if either $a(n)$ is increasing ($d(n)=0$), or decreasing ($u(n)=0$), or satisfies LRT (equivalent to $|a(n+1)-a(n)|=\boldsymbol{o}(a(n))$, or what amounts to the same thing, *both* $d(n)=\boldsymbol{o}(a(n))$ and $u(n)=\boldsymbol{o}(a(n))$).

Proposition 2.3 *Let $f=\sum a(n)t^n$ be a strongly momentous function with $\sum a(n)=\infty$. Then f and all its derivatives are momentous, and $M_k(f)=N_k(f)/(k+1)$.*

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Proof From LRT, we infer $d_f(n) = \boldsymbol{o}(a(n))$, so that Lemma 2.2 may be applied. We note that for k fixed, $a(n)/a(n+k) \to 1$, and so:

$$
\begin{aligned}
f(t^{k+1}) &= \sum a(n) t^{n\cdot(k+1)} \\
&\sim N_k \sum a(n\cdot(k+1)) t^{n\cdot(k+1)} \\
&\sim \frac{N_k}{k+1} \sum a(n) t^n \\
&= \frac{N_k}{k+1} f(t)
\end{aligned}
$$

(The second line uses $\sum a(n) = \infty$.) ∎

Lemma 2.4 *Suppose that $f = \sum a(n)t^n$ satisfies*

$$\limsup_{n\to\infty} \frac{n d_f(n)}{a(n)} = K < \infty.$$

If k is an integer exceeding K, then after removing an initial segment, the coefficients of the k-th derivative of f are increasing. Conversely, if the coefficients of the k-th derivative are eventually increasing, then the displayed condition holds.

Proof For all sufficiently large n, $nd_f(n)/a(n) < k$. Replacing n by $n+k$, we have $(n+k)d_f(n+k)/a(n+k) < k$. This translates to $(n+k)a(n+k) > na(n+k-1)$. However, $d_{f^{(k)}}(n) = \max\left\{0, \left((n+k-1)!/n!\right)\left(na(n+k-1)-(n+k)a(n+k)\right)\right\} = 0$. Hence the coefficients of $f^{(k)}$ are eventually increasing. The converse is obtained by reversing this argument. ∎

Proposition 2.5 *Suppose that h is weakly momentous, and $f = \sum_1^\infty h(e^{-1/n})t^n/n$. Then f is momentous and*

$$\begin{cases} f \sim \Gamma(\alpha)h & \text{if } \alpha(h) > 0, \\ f' \sim (1-t)^{-1}h & \text{if } \alpha(h) = 0. \end{cases}$$

Proof Set $f_1 = tf' = \sum h(e^{-1/n})t^n$. As h is weakly momentous, $h(e^{-1/n}) \sim \Gamma(1+\alpha)\sum^n (h, t^j) = \Gamma(1+\alpha)((1-t)^{-1}h, t^n)$. Thus $f_1 \sim \Gamma(1+\alpha)(1-t)^{-1}h$ (their coefficients are asymptotic). Hence $f' \sim \Gamma(1+\alpha)(1-t)^{-1}h$, and the rest follows from Proposition 1.6. ∎

Proposition 2.6 *Suppose that f is momentous and $d_f(n) = \boldsymbol{O}(a(n)/n)$. Then f is LLT.*

Proof There exists an integer k such that $f_1 := f^{(k)}$ has eventually increasing coefficients. Since f is momentous, so is any derivative. Replacing an initial segment of the coefficients of f_1 by zeroes does not affect the asymptotic behaviour near 1, so that the perturbed function is momentous and has increasing coefficients. Thus f_1 satisfies the local limit theorem, so that $(f_1, t^N) \sim f_1(e^{-1/N})/N\Gamma(k+\alpha(f))$. As $a(n) \sim (f_1, t^N)/N^k$, we deduce $a(n) \sim c f_1(e^{-1/N})/N^{k+1}$, and since $f^{(k)} \sim (1-t)^{-1} f^{(k-1)}$, we can continue until either the zeroth derivative (if $\alpha(f) > 0$) or the first (if $\alpha = 0$) is reached. ■

Attempts to weaken this "local" condition, say to $d_f(n) \sim \boldsymbol{O}(a(n)/n^\delta)$ with $\delta < 1$, or even $d_f(n) \sim \boldsymbol{O}(\ln(n+1)a(n)/n)$, fail. Parameters in the upcoming Example 4.1 can be set to obtain counterexamples.

We say a sequence of nonnegative real numbers $\{a(n)\}$ satisfies a *very superior limit ratio theorem* (VRT) if for every sequence $M\colon \mathbf{N} \to \mathbf{Z}$ such that $M(n) = \boldsymbol{o}(n)$,

$$\lim_{n\to\infty} \frac{a(n+M(n))}{a(n)} = 1.$$

(where $a(-k) = 0$ by convention). It is easy to see that this is equivalent to the conditions wherein we restrict M to be positive-valued, or to be negative-valued, respectively (with the displayed limit still holding). By abuse of notation, we also say that f satisfies VRT if $f = \sum a(n)t^n$ and $\{a(n)\}$ satisfies VRT.

We note that f satisfies VRT if and only if f' does. In particular, if g is momentous and $f = \sum g(1-1/n)t^n/n$ then f satisfies VRT. Other related examples occur as a consequence of Lemma 2.8.

Theorem 2.7 *Suppose that the power series $f = \sum a(n)t^n$ has nonnegative coefficients and radius of convergence equal to* 1. *Then f satisfies LLT if and only if f is momentous and $\{a(n)\}$ satisfies VRT.*

Proof If f satisfies LLT and $\alpha \equiv \alpha(f) > 0$, then $\Gamma(\alpha)a(n) \sim f(1-1/n)/n$, and it easily follows that $\{a(n)\}$ satisfies VRT. If $\alpha = 0$, differentiate f and the result easily pulls back to the original f.

If f is momentous and satisfies VRT, then so does its derivative; hence we may assume that $\alpha \equiv \alpha(f) \geq 1$. Now set $b(n) = na(n)\Gamma(\alpha)/f(1-1/n)$. It suffices to show that $b(n) \to 1$. If not, there exists $\delta > 0$ such that $|b(n) - 1| > \delta$ for infinitely many n. Without loss of generality, we may assume that $b(n) > 1 + \delta$ for infinitely many n (if not, replace $b(n)$ by $1/b(n)$).

Index these n as $n_1 < n_2 < \cdots$, and define for each i the integers $m(i)$ and $M(i)$, respectively, as

$$m(i) = \inf\{m \mid b(k) \geq 1 + \delta/3 \text{ for all } k \text{ such that } m \leq k \leq n_i\};$$
$$M(i) = \sup\{m \mid b(k) \geq 1 + \delta/3 \text{ for all } k \text{ such that } n_i \leq k \leq m\}.$$

Set I_i to be the interval $[m(i), M(i)]$. Note that n_i belongs to I_i, and moreover,

$$\frac{b(n_i)}{b(m(i)-1)}, \frac{b(n_i)}{b(M(i)+1)} > \frac{1+\delta}{1+\delta/3} > 1 + \frac{\delta}{2}.$$

From the VRT hypothesis, this forces $(M(i) - m(i))/n_i$ not to converge to zero — just observe that $b(s)/b(t) = (a(s)/a(t)) \cdot (s/t) f(1 - 1/t)/f(1 - 1/s)$. Hence there exists a positive real number κ and a subset $J \subset \mathbf{N}$ such that for each j in J,

$$\frac{M(j) - m(j)}{n_j} > \kappa.$$

For each j, at least one of $(M(j) - n_j)/n_j$ and $(n_j - m(j))/n_j$ exceeds $\kappa/2$, and set $N_j = n_j(1+\kappa/2)$ if the former and $N_j = n_j$ otherwise. In the latter case, $m(j)/n_j < 1-\kappa/2$. Thus if $1 > \gamma > (1+\kappa/2)^{-1}$, it follows that $\sum_{N_j\gamma}^{N_j} b(n) \geq N_j \cdot (1+\delta/2)(1-\gamma)$.

Let γ be any number less than 1, to be determined later. We claim that

$$\limsup_{N\to\infty} \frac{1}{N} \sum_{n=N\gamma}^{N} \frac{a(n)\Gamma(\alpha)}{f(1-\frac{1}{n})} \leq \frac{\gamma^\alpha(1-\gamma^\alpha)}{\alpha}.$$

To see this, we note that for $N\gamma \leq n \leq N$, we have $f(1 - 1/N\gamma) \leq f(1 - 1/n) \leq f(1 - 1/N)$. Thus

$$\begin{aligned}\frac{1}{N}\sum_{n=N\gamma}^{N} b(n) &\leq \frac{N\Gamma(\alpha)}{Nf(1-1/N\gamma)} \sum_{n=N\gamma}^{N} a(n)\\ &\sim \frac{N\Gamma(\alpha) f(1-1/N)(1-\gamma^\alpha)}{Nf(1-1/N\gamma)\Gamma(1+\alpha)}\\ &\sim \frac{\gamma^\alpha f(1-1/N)(1-\gamma^\alpha)}{f(1-1/N)\alpha}\\ &= \frac{\gamma^\alpha(1-\gamma^\alpha)}{\alpha}.\end{aligned}$$

The second last line comes from $(1 - 1/N\gamma)^\gamma \sim 1 - 1/N$, so that $f(1 - 1/N\gamma) \sim \gamma^{-\alpha} f(1 - 1/N)$.

By the results of the last two paragraphs, provided $\gamma > (1 + \kappa/2)^{-1}$,

$$\frac{\gamma^\alpha(1-\gamma^\alpha)}{\alpha} \geq \gamma(1+\delta/2).$$

As $\gamma \uparrow 1$, the left side goes to zero, but the right side does not. Hence if we choose γ sufficiently close to 1, we obtain a contradiction. ■

The VRT property is probably worth studying on its own. For a sequence $\{a(n)\}$, recall that the difference sequence $\{\Delta a(n)\}$ is defined via $\Delta a(n) = a(n) - a(n-1)$.

Lemma 2.8 *If the sequence of positive real numbers $\{a(n)\}$ satisfies $|\Delta a(n)| = \boldsymbol{O}(a(n)/n)$, then it satisfies VRT.*

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Proof Let K be the constant in the big Oh expression. For any k,

$$\begin{aligned}|a(n)-a(n+k)| &\leq \sum_{i=1}^{k-1} |a(n+i)-a(n+i-1)| \\ &\leq \frac{K}{n}\sum_{i=1}^{k-1} a(n+i).\end{aligned}$$

Inductively, we see that $\sum_{i=1}^{k-1} a(n+i) \leq (k+kK/n)a(n)$, and the VRT property follows. ■

Corollary 2.9 *Suppose that $f\colon (0,1) \to \mathbf{R}^+$ is C^1 and satisfies $f' \geq 0$ and $f'(t) = \boldsymbol{O}((1-t)^{-1}f(t))$ (as $t\uparrow 1$). Then the functions defined as $g_0 = \sum_{n\geq 1} f(1-1/n)t^n/n$ and $g_1 = \sum f(1-1/n)t^n$ satisfy VRT.*

Proof Since $tg_0' = g_1$, it is sufficient to show g_1 satisfies VRT; we verify the sufficient condition of Lemma 2.8, with $a(n) = f(1-1/n)$:

$$\begin{aligned} f\Big(1-\frac{1}{n+1}\Big) - f\Big(1-\frac{1}{n}\Big) \\ &= \frac{f'\big(1-\frac{1}{n'}\big)}{n(n+1)} \quad \text{for some real } n' \text{ in the interval } [n, n+1] \\ &\leq \frac{f'\big(1-\frac{1}{n+1}\big)}{n(n+1)} \\ &\leq \frac{Kf\big(1-\frac{1}{n+1}\big)}{n} \leq \frac{K'f\big(1-\frac{1}{n+1}\big)}{n+1}.\end{aligned}$$

Hence $|\Delta a(n)| = \boldsymbol{O}(a(n)/n)$. ■

The condition $f'(t) = \boldsymbol{O}((1-t)^{-1}f(t))$ will be discussed in detail in Section 6. It is frequently difficult to verify VRT, so we consider other criteria for a momentous function to satisfy LLT.

Proposition 2.10 *Suppose that $f = \sum a(n)t^n$ is momentous with $\alpha(f) > 0$, and there exists $M\colon \mathbf{N} \to \mathbf{N}$ such that $M(n) = \boldsymbol{o}(n)$ and for all $u \geq M(n)$, $a(n) \geq a(n+u)$. Then f is LLT.*

Proof By removing an initial segment if necessary, we may assume that $M(n) \leq n/2$ for all $n \geq 3$. Abbreviate $\alpha(f)$ to α. Select a real number $\rho > 1$ and positive real number ϵ such that $\epsilon \ll \rho - 1$. There exists $k \equiv k(\epsilon)$ such that $n \geq k$ entails $M(n) \leq \epsilon n$. For (large) N, set $S_N = \{n \in \mathbf{N} \mid n + M(n) \leq N\}$. Then $[0, 2N/3] \cap \mathbf{N} \subseteq S_N$.

Suppose that $N/2 > k(\epsilon)$; then for all $j > N/2$, $M(j) \leq \epsilon j$, whence $j + M(j) \leq (1+\epsilon)j$. Thus $[0, N(1+\epsilon)^{-1}] \cap \mathbf{N} \subseteq S_N$.

Consider first

$$\begin{aligned}\sum_{N}^{\rho N} a(n) &= \sum_{(1+\epsilon)N}^{\rho N} a(n) + \sum_{N}^{(1+\epsilon)N} a(n)\\ &\le (\rho-(1+\epsilon))Na(N) + \sum_{N}^{(1+\epsilon)N} a(n)\\ &< N(\rho-1)a(N) + \sum_{N}^{(1+\epsilon)N} a(n).\end{aligned}$$

Thus,

$$\begin{aligned}a(N) &> \frac{1}{(\rho-1)N}\sum_{N}^{\rho N} a(n) - \frac{1}{(\rho-1)N}\sum_{N}^{(1+\epsilon)N} a(n)\\ &\sim \frac{1}{(\rho-1)N\Gamma(1+\alpha)}\Big(f(1-1/\rho N)(1-\rho^{-\alpha})\\ &\qquad - f(1-1/(1+\epsilon)N)(1-(1+\epsilon)^{-\alpha})\Big) \text{ as } N\uparrow\\ &\sim \frac{1}{(\rho-1)N\Gamma(1+\alpha)} f(1-1/N)\big(\rho^{\alpha}-1-((1+\epsilon)^{-\alpha}-1)\big).\end{aligned} \tag{2.2}$$

Also, $\sum_{N/\rho}^{N/(1+\epsilon)} a(n) \ge N((1+\epsilon)^{-1}-\rho^{-1})a(N)$, so
(2.3)

$$\begin{aligned}N((1+\epsilon)^{-1}-\rho^{-1})a(N) &\le \sum_{N/\rho}^{N/(1+\epsilon)} a(n)\\ &\sim \frac{1}{\Gamma(1+\alpha)} f(1-1/N(1+\epsilon))\big(1-((1+\epsilon)\rho^{-1})^{\alpha}\big)\\ &\sim \frac{f(1-1/N)\big((1+\epsilon)^{\alpha}-\rho^{-\alpha}\big)(1+\epsilon)^{2\alpha}}{\Gamma(1+\alpha)}.\end{aligned}$$

As N increases, we can permit ϵ to decrease towards zero; thus from (2.2), we deduce

$$\liminf_{N\uparrow\infty} \frac{Na(N)\Gamma(1+\alpha)}{f(1-1/N)} \ge \frac{\rho^{\alpha}-1}{\rho-1}.$$

As $\rho\downarrow 1$, the right-hand side tends to α (as $\alpha > 0$). Hence

$$\liminf_{N\uparrow\infty} \frac{Na(N)\Gamma(1+\alpha)}{f(1-1/N)} \ge \alpha.$$

On the other hand, from (2.3), we similarly deduce

$$\limsup \frac{Na(N)\Gamma(1+\alpha)}{f(1-1/N)} \le \frac{1-\rho^{-\alpha}}{1-\rho^{-1}}.$$

Again, the limit on the right side as $\rho\downarrow 1$ is α, and the two inequalities together yield $a(n) \sim f(1-1/N)/N\Gamma(\alpha)$; in particular, f is LLT. ■

3 Hadamard Powers and Products

In this section, we catalogue the asymptotic behaviours of LLT functions obtained by taking Hadamard products and powers (the latter is defined below). A particular case is that of Green's (or renewal) functions, *i.e.*, $f = (1-F)^{-1}$ where $F = \sum_1^\infty c(n)t^n$ with $c(n) \geq 0$, $\sum c(n) = 1$, and $\sum nc(n) = \infty$ (if the last condition fails, the problems are trivial). If $(1-F)^{-1}$ is LLT, then we see what the asymptotic estimates have to be. Sections 4 and 10 deal with sufficient conditions for $(1-F)^{-1}$ to be LLT.

Proposition 3.1 *Let $f = \sum a(n)t^n$ be LLT. Then f is strongly momentous, and in particular, all derivatives are momentous.*

Proof If $\alpha > 0$, then $a(N)/a(N\cdot(k+1)) \sim (k+1)f(e^{-1/N})/f(e^{-1/N\cdot(k+1)})$, and since f is weakly momentous, the latter converges to a nonzero number. Similarly, if $\alpha = 0$, then f' is weakly momentous, and the corresponding ratios of coefficients converge. ■

We wish to show that if $g = \sum a(n)t^n$ is momentous and $f = \sum c(n)t^n$ is LLT, then their Hadamard product, $f \circ g$, is momentous provided $\sum a(n)c(n) = \infty$ (*e.g.*, if $\alpha(f) + \alpha(g) > 1$). The special case below yields this result.

Lemma 3.2 *Suppose that $g = \sum a(n)t^n$ is momentous with $\alpha(g) \equiv \alpha > 0$, and f is momentous with $\alpha(f) \equiv \beta > 0$. Then the function $h = \sum_{n\geq 1} a(n)f(1-1/n)t^n$ is momentous, $h \sim fg \cdot c$ (as $t\uparrow 1$) where $c = \Gamma(\alpha+\beta)/\Gamma(\alpha)$, and $\alpha(h) = \alpha+\beta$.*

Proof Denote by $b(n)$ the expression $a(n)f(1-1/n)$. Let m be a positive integer, and set $\gamma = 2^{-1/m}$, so that $\gamma^m = 1/2$. For m fixed, given $\epsilon > 0$, there exists a positive integer $K \equiv K(m,\epsilon)$ such that $M > K$ entails each of the following:

$$\Bigl|\sum_{M\gamma}^{M} a(n) - \frac{(1-\gamma^\alpha)g(1-1/M)}{\Gamma(\alpha+1)}\Bigr| < \epsilon\frac{g(1-1/M)}{\Gamma(\alpha+1)};$$

$$\Bigl|\frac{f(1-1/M\gamma^{-k})}{f(1-1/M)} - \gamma^{-k\beta}\Bigr| < \epsilon f(1-1/M\gamma^{-k}) \quad \text{for each of } k = 1,2,\ldots,m;$$

$$\Bigl|\frac{g(1-1/M\gamma^{-k})}{g(1-1/M)} - \gamma^{-k\alpha}\Bigr| < \epsilon g(1-1/M\gamma^{-k}) \quad \text{for each of } k = 1,2,\ldots,m.$$

For $N > 2K$ consider, for $1 \le k \le m$,

$$\begin{aligned}(3.1)\quad \Gamma(\alpha+1)\sum_{n=N\gamma^k}^{N\gamma^{k-1}} b(n) &\le f(1-1/N\gamma^{k-1})\sum_{N\gamma^k}^{N\gamma^{k-1}} a(n)\\ &\le (1+\epsilon)^2 f(1-1/N)g(1-1/N\gamma^{k-1})\gamma^{(k-1)\beta}(1-\gamma^\alpha)\\ &\le (1+\epsilon)^3\gamma^{(k-1)(\beta+\alpha)}(1-\gamma^\alpha)f(1-1/N)g(1-1/N).\end{aligned}$$

Similarly, we obtain

$$(3.2)\qquad \Gamma(\alpha+1)\sum_{n=N\gamma^k}^{N\gamma^{k-1}} b(n) \ge (1-\epsilon)^3\gamma^{k(\beta+\alpha)}(1-\gamma^\alpha)f(1-1/N)g(1-1/N).$$

Note the appearance of k rather than $k-1$ in the exponent. Adding (3.1) and (3.2) over $k = 1,\dots,m$ and dividing by $f(1-1/N)g(1-1/N)(1-\gamma^\alpha)$, we obtain

$$(3.3)\qquad (1-\epsilon)^3\sum_{k=1}^m \gamma^{k(\beta+\alpha)} \le \frac{\Gamma(\alpha+1)\sum_{N/2}^N b(n)}{fg(1-1/N)(1-\gamma^\alpha)} \le (1+\epsilon)^3\sum_{k=1}^m \gamma^{(k-1)(\beta+\alpha)}.$$

The sum on the right is $(1-\gamma^{m(\alpha+\beta)})/(1-\gamma^{\alpha+\beta}) = (1-2^{-m(\alpha+\beta)})/(1-\gamma^{\alpha+\beta})$, and that on the left is $\gamma^{\alpha+\beta}$ times this. The ratio of the right side to the left side is thus $(1+\epsilon)^3(1-\epsilon)^{-3}\gamma^{-(\alpha+\beta)}$, and of course as $m \to \infty$, $\gamma \to 1$. Multiplying through by $(1-\gamma^\alpha)$ (terms were put in the denominator because otherwise the inequalities in (3.3) would not fit on one line), we obtain (allowing $\epsilon \to 0$),

$$\begin{aligned}\Gamma(\alpha+1)\sum_{N/2}^N b(n) &\sim fg(1-1/N)(1-2^{-(\alpha+\beta)})\lim_{\gamma\uparrow 1}\frac{1-\gamma^\alpha}{1-\gamma^{\alpha+\beta}}\\ &= fg(1-1/N)(1-2^{-(\alpha+\beta)})\frac{\alpha}{\alpha+\beta}.\end{aligned}$$

Having an asymptotic estimate for $\sum_{N/2}^N b(n)$, we easily derive the corresponding asymptotic estimate for $\sum_1^N b(n)$. We expand, for fixed l and all sufficiently large N, $\sum_1^N b(n) = \sum_{i=0}^l \sum_{N/2^{i+1}}^{N/2^i} b(n) + \sum_1^{N/2^l} b(n)$, and observe that the last sum goes to zero in l (as $N \to \infty$) compared with $\sum_1^N b(n)$. Adding the resulting geometric series and discarding the last sum yields

$$(3.4)\qquad \Gamma(\alpha+1)\sum_1^N b(n) \sim fg(1-1/N)\frac{\alpha}{\alpha+\beta}.$$

Since fg is momentous, it follows that $(1-t)^{-1}h$ is strongly momentous, and thus LLT, and therefore h is momentous. Since $((1-t)^{-1}fg, t^N) \sim fg(1-1/N)/\Gamma(\alpha+\beta+1)$, the right-hand side gives us $h \sim fg\cdot\Gamma(\alpha+\beta)/\Gamma(\alpha)$. In particular, $\alpha(h) = \alpha(f)+\alpha(g) = \alpha+\beta$. ∎

Corollary 3.3 *Suppose that $g = \sum a(n)t^n$ is momentous, $f = \sum c(n)t^n$ is LLT, and $\sum a(n)c(n) = \infty$. Then the Hadamard product $H := g \circ f$ is momentous and $\alpha(H) = \alpha(g) + \alpha(f) - 1$.*

Proof Set $J = t(t(tH')')' = \sum n^3 a(n)c(n)t^n$; rewrite this as $\sum na(n) \cdot n^2 c(n)$, and set $G = tg' = \sum na(n)$ and $F = t(tf')' = \sum n^2 c(n)$, so that $J = G \circ F$. Obviously, $\alpha(G) \geq 1 > 0$ and $\alpha(F) \geq 2 > 1$. Also, F is momentous, and thus $n^2 c(n) \sim F(1 - 1/n)/n\Gamma(\alpha(F)+1) = f_0(1-1/n)/\Gamma(\alpha(F)+1)$, where $f_0 = tf'$ (so $\alpha(f_0) \geq 1$). As Hadamard products respect the equivalence relation $\sim$ on sequences of coefficients (unlike convolution), we deduce from the preceding result that $((1-t)^{-1}J, t^N) \sim ((1-t)^{-1}f_0 G, t^N)c$ for a positive constant c. Since $H''' \sim J$ as $t\uparrow 1$, it follows that H''' is momentous. A simple application of l'Hôpital's rule yields that if j has only nonnegative Maclaurin coefficients and $j(1) = \infty$, and j' is momentous, then so is j. Applying this (three times), we obtain that H is momentous. Since $J \sim f_0 G$, $\alpha(J) = 2 + \alpha(f) + \alpha(g)$, and as $\alpha(H) = \alpha(H''') - 3$, the result follows. ■

The condition $\sum a(n)c(n) = \infty$ of the corollary is guaranteed if in particular, $\alpha(f) + \alpha(g) > 1$. The use of the third derivative is to get around the possibility that in the preceding lemma, α or β is zero.

Lemma 3.4 *Suppose that $f = \sum c(n)t^n$ is momentous, $\alpha(f) \leq r+1$, and if $\alpha(f) = r+1$, then $\sum n^r/f(1-1/n) = \infty$. Then $h := \sum t^n n^r/f(1-1/n)$ is LLT and $\alpha(h) = r + 1 - \alpha(f)$.*

Proof We first show that h is strongly momentous. For k an integer,

$$(h, t^{kN})/(h, t^N) = k^r f(1-1/kN)/f(1-1/N) \sim k^{r-\alpha(f)},$$

whence h is strongly momentous. If $\alpha(h) < r+1$, it is easy to check that $h(1) = \infty$, so that in any case, h is momentous. Finally, if $a(n) = (h, t^n)$, then a simple computation reveals that $\Delta a(n) = \boldsymbol{O}(a(n)/n)$, so that h is LLT. ■

Now we obtain results about the asymptotic behaviour of Hadamard powers and Hadamard products of some power series. Suppose that $f = \sum a(n)t^n$ and $g = \sum b(n)t^n$ are convergent real Maclaurin series. Let β be a real number. Define the *β Hadamard power of f* to be the power series $f^{(\beta)} := \sum a(n)^\beta t^n$, and define the *Hadamard product* of f and g to be $f \circ g := \sum a(n)b(n)t^n$. If $a(n)$ and $b(n)$ are reasonable, we can obtain asymptotic estimates for $f^{(\beta)}$ and $f \circ g$ near 1.

To begin with, let $\{a(n)\}$ be a strongly momentous sequence. In our case, $f = \sum a(n)t^n$, and we use the previously-defined notation $N_k(f)$. Let β be a real number, and consider the sequence $\{a(n)^\beta\}$, and the corresponding power series $f^{(\beta)}$. Then of course $N_k(f^{(\beta)})$ exists and equals $N_k(f)^\beta$, and $\lim a(n)^\beta/a(n+1)^\beta = 1$; so $f^{(\beta)}$ is strongly momentous. We must also assume $\sum a(n)^\beta = \infty$ (this can often be avoided by differentiation).

For a generic power series $h = \sum c(n)t^n$ for which the $M_k \equiv M_k(h)$ all exist, let $\alpha(h)$ denote $-\ln_2 M_1(h)$. From Proposition 2.3, we have that $M_k(f)$ all exist and

$M_k(f) = N_k(f)/(k+1)$. Since $f^{(\beta)}$ also satisfies the hypotheses, and we obtain that $M_k(f^{(\beta)}) = N_k(f)^\beta/(k+1)$, and thus $\alpha(f^\beta) = \beta\cdot(\alpha-1)+1$, where $\alpha \equiv \alpha(f)$. We must also impose the condition $\beta\cdot(\alpha-1)+1 \geq 0$.

Hadamard Powers of f Now we discuss the behaviour of $f^{(\beta)}$ near 1. In the following computations, we use that $z/\Gamma(z+1) = 1/\Gamma(z)$, and $\Gamma(z)\cdot\Gamma(1-z) = \pi/(\sin \pi z)$.

Case 1 $(\beta\cdot(\alpha-1)+1>0)$ We have that

$$a(n)^\beta = (f^{(\beta)}, x^n) \sim f^{(\beta)}(e^{-1/n})/(n\cdot\Gamma(\beta\cdot(\alpha-1)+1)).$$

Subcase 1.1 $(\alpha > 0)$ Then $a(n) \sim f(e^{-1/n})/(n\cdot\Gamma(\alpha))$. Substituting this into the previous asymptotic formula yields

$$f^{(\beta)}(e^{-1/n}) \sim \frac{f(e^{-1/n})^\beta}{n^{\beta-1}}\cdot\frac{\Gamma(\beta\cdot(\alpha-1)+1)}{\Gamma(\alpha)^\beta}.$$

So by comparing coefficients,

$$f^{(\beta)}(t) \sim (1-t)^{\beta-1} f(t)^\beta\cdot\frac{\Gamma(\beta\cdot(\alpha-1))}{\Gamma(\alpha)^\beta}.$$

Subcase 1.2 $(\alpha = 0)$ This forces $\beta < 1$ (it could be negative). Here $a(n) \sim f'(e^{-1/n})/n^2$, and substituting this into the expression in Case 1, we obtain

$$f^{(\beta)}(e^{-1/n}) \sim \frac{f'(e^{-1/n})^\beta\Gamma(2-\beta)}{n^{2\beta-1}}.$$

So by the same reasoning as before, which enables us to replace $e^{-1/n}$ by t,

$$f^{(\beta)}(t) \sim (1-t)^{2\beta-1} f'(t)^\beta\Gamma(2-\beta).$$

Case 2 $(\beta(\alpha-1)+1=0)$ This forces $\beta = 1/(1-\alpha)$ (so in particular, $\alpha \neq 1$). This time $a(n)^\beta \sim (f^{(\beta)})'(e^{-1/n})/n^2$.

Subcase 2.1 $(\alpha > 0)$ Equating the expressions, we obtain in similar fashion to the preceding cases

$$(f^{(\beta)})'(t) \sim \left(\frac{f(t)}{\Gamma(\alpha)}\right)^{\frac{1}{1-\alpha}}(1-t)^{\frac{1}{1-\alpha}-2}.$$

Subcase 2.2 $(\alpha = 0)$ This forces $\beta = 1$, so that $f^{(\beta)} = f$.

Let us consider the special case that $\beta = -1$. This forces the hypothesis $\sum 1/a(n) = \infty$. In Subcase 1.1, $\alpha > 0$ and $2-\alpha > 0$, *i.e.,* $0 < \alpha < 2$, and we obtain

$$f^{(-1)}(t) \sim \frac{(1-t)^{-2}}{f(t)}\cdot\Gamma(2-\alpha)\Gamma(\alpha).$$

The last factor simplifies. In Subcase 1.2 we have that $\alpha = 0$, so

$$f^{(-1)}(t) \sim \frac{(1-t)^{-3}}{f'(t)}.$$

Finally, in Subcase 2.1, we have that $1 - \alpha = -1$, whence $\alpha = 2$, so

$$(f^{(-1)})'(t) \sim \frac{(1-t)^{-3}}{f(t)}.$$

Hadamard Products of f and g Now we discuss Hadamard products. We assume that both $\{a(n)\}$ and $\{b(n)\}$ are strongly momentous and satisfy LLT, and we also require $\sum a(n)b(n) = \infty$. We notice that the sequence of coefficients of $f \circ g$, *i.e.*, $\{f(n)g(n)\}$, is strongly momentous, and it is easy to verify that $N_k(f \circ g) = N_k(f) \cdot N_k(g)$, so that $M_k(f \circ g) = M_k(f) \cdot M_k(g) \cdot (k+1)$. It follows that $\alpha(f \circ g) = \alpha(f) + \alpha(g) - 1$. This last condition requires an apparently extra hypothesis — for all the results to apply, we require all the relevant αs to be nonnegative, hence we must impose the condition that $\alpha(f) + \alpha(g) \geq 1$, which is admittedly fairly strong. There are now a similar set of cases for the Hadamard product as for the Hadamard power.

Case 1 $(\alpha(f), \alpha(g), \alpha(f \circ g) > 0)$ The last condition is equivalent to $\alpha(f)+\alpha(g) > 1$. On one hand, we have $a(n)b(n) = (f \circ g, x^n) \sim (f \circ g)(e^{-1/n})/(n\Gamma(\alpha(f \circ g)))$, and on the other, $a(n)b(n) = (f, x^n)(g, x^n) \sim fg(e^{-1/n})/(n^2\Gamma(\alpha(f))\Gamma(\alpha(g)))$. We obtain

$$(f \circ g)(t) \sim (1-t)f(t)g(t)\frac{\Gamma(\alpha(f)+\alpha(g)-1)}{\Gamma(\alpha(f))\Gamma(\alpha(g))}.$$

Case 2 $(\alpha(f) = 0$ but $\alpha(f \circ g) > 0)$ Here the extra hypothesis forces $\alpha(g) > 1$. The expression for $a(n)$ is now replaced by $f'(e^{-1/n})/n^2$, and we deduce as in all the earlier cases,

$$(f \circ g)(t) \sim (1-t)^2 f'(t)g(t)\frac{\Gamma(\alpha(g)-1)}{\Gamma(\alpha(g))} = \frac{(1-t)^2 f'(t)g(t)}{\alpha(g)-1}.$$

Case 3 $(\alpha(f \circ g) = 0$ but $\alpha(f), \alpha(g) > 0)$ Now we have that $\alpha(g) = 1 - \alpha(f)$ (so of course $\alpha(f) < 1$). This yields

$$(f \circ g)'(t) \sim \frac{f(t)g(t)}{\Gamma(\alpha(f))\Gamma(1-\alpha(f))} = f(t)g(t) \cdot \frac{\sin \pi\alpha(f)}{\pi}.$$

Case 4 $(\alpha(f \circ g), \alpha(f) = 0)$ This forces $\alpha(g) = 1$, and the usual manipulations yield

$$(f \circ g)'(t) \sim (1-t)f'(t)g(t).$$

Of course, the case that $\alpha(f) = \alpha(g) = 0$ cannot be analyzed directly by these methods; however, we may differentiate $f \circ g$ to obtain $f \circ (tg')$, and this is covered by Case 4. We deduce

$$(f \circ g)''(t) \sim (1-t)f'(t)g'(t).$$

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Some of the results above are rather startling. For example, in Case 1, we have that $(1-t)^{-1} f \circ g(t) \sim f(t)g(t)c$ (where c is the quotient of the values of Γ). Both sides satisfy the conditions necessary for the local limit theorem to arise, and so we obtain (taking the n-th coefficients),

$$\sum_{i=0}^{n} a(i)b(i) \sim c \sum_{i=0}^{n} a(i)b(n-i).$$

Similarly, Case 4 yields the surprising

$$n \sum_{i=0}^{n} a(i)b(i) \sim \sum_{i=0}^{n} ia(i)b(n-i).$$

Cases 2 and 3 also yield slightly more complicated forms.

Green's Function Suppose that $F(t) = \sum c(n)t^n$ where $c(n) \geq 0$ and $\sum c(n) = 1$. Among other things, we wish to obtain asymptotic estimates for $((1-F)^{-1}, t^N)$ under optimal circumstances—specifically, this is asymptotic to a constant multiple of $(n^2 c(n))^{-1}$ (when $\alpha(F') > 0$).

Set $h(n) = 1 - \sum_{i=0}^{n} c(n)$, so that $h(n)\downarrow 0$, and $K(t) := (1-t)^{-1}(1-F)(t) = \sum h(n)t^n$ has nonnegative coefficients. Set $H(n) = \sum_{i=0}^{n} h(n)$, so that

$$(1-t)^{-1}K(t) = (1-t)^{-2}(1-F)(t) = \sum H(n)t^n.$$

Now we assume that K satisfies the moments condition ($\lim_{t\uparrow 1} K(t^k)/K(t)$ exists for all positive integers k), which is of course equivalent to F' satisfying it. In any event, the early variant of the Hardy–Littlewood–Karamata theorem (on the sums of the coefficients) applies to K, and we obtain that $H(n) \sim K(e^{-1/n})/\Gamma(1+\alpha) \sim n(1-F)(1-1/n)/\Gamma(1+\alpha)$, where $\alpha = -\ln_2 M_1(K)$ as usual.

Set $f = (1-t)^{-1}K$, and form $f^{(-1)}(t)$, *i.e.*, the series whose n-th coefficient is $1/H(n)$. We require that $\sum 1/H(n) = \infty$. We observe that $H(n+1) - H(n) = h(n)\downarrow 0$, hence $\{H(n)\}$ satisfies LLT (since $H(n)$ is increasing). With $\beta = -1$ and $2 > \alpha(f) > 0$, we deduce:

$$f^{(-1)}(t) \sim \frac{(1-t)^{-2}}{f(t)} \cdot \Gamma(2-\alpha(f))\Gamma(\alpha(f)) \sim \frac{1}{1-F} c_F.$$

In particular, the behaviour of $f^{(-1)}(t)$ near 1 is, up to a constant multiple, that of $(1-F)^{-1}$. If the local limit theorem applies, then we deduce that asymptotically the n-th coefficient of $(1-F)^{-1}$ is a scalar multiple of $1/H(n)$.

Lemma 3.5 *Suppose $F = \sum c(n)t^n$ where $c(n) \geq 0$, $\sum c(n) = 1$, and $\sum nc(n) = \infty$. If both F' and $(1-F)^{-1}$ are LLT and $1 > \alpha \equiv \alpha(F') > 0$, then*

$$((1-F)^{-1}, t^N) \sim \frac{(1-\alpha)\sin \pi\alpha}{N^2 c(N)\pi} \sim \frac{\sin \pi\alpha}{H(N)\pi\alpha},$$

where $H(n) = \sum_{i \leq n}(1 - \sum_{j \leq i} c(i))$.

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Proof Since $(1-F)^{-1}$ is momentous, $\beta := \alpha((1-F)^{-1})$ is well defined, and $((1-F)^{-1})'(t) \sim \beta(1-t)^{-1}(1-F)^{-1}(t)$ (or $\boldsymbol{o}((1-t)^{-1}(1-F)^{-1}(t))$ if $\beta = 0$). This yields $F'(t) \cdot (1-F)^{-1}(t) \sim \beta(1-t)^{-1}$. If $\beta > 0$, then $\beta = 1-\alpha$ follows, and thus

$$((1-F)^{-1}, t^N) \sim \frac{(1-F)^{-1}\left(1-\frac{1}{N}\right)}{N\Gamma(1-\alpha)} \sim \frac{(1-\alpha)N}{N\Gamma(1-\alpha)F'\left(1-\frac{1}{N}\right)}.$$

Now $(F', t^N) \sim Nc(N)$ and also $(F', t^N) \sim F'(1-1/N)/N\Gamma(\alpha)$. Thus $F'(1-1/N) \sim N^2c(N)\Gamma(\alpha)$, so

$$((1-F)^{-1}, t^N) \sim \frac{1-\alpha}{N^2c(N)\Gamma(\alpha)\Gamma(1-\alpha)} = \frac{(1-\alpha)\sin\pi\alpha}{\pi N^2 c(N)}.$$

From the definition of α and l'Hôpital's rule, it is easy to see that $\beta = 1-\alpha$ whenever both are defined—in particular, $\beta \neq 0$ here.

Next, $(1-t)^{-2}(1-F) = \sum H(n)t^n$, and as $(1-t)^{-1}(1-F)$ has decreasing coefficients and is momentous, it is LLT. In particular, so is $(1-t)^{-2}(1-F)$, and obviously $\alpha((1-t)^{-2}(1-F)) \geq 1 > 0$. Since $(1-t)^{-2}(1-F)\cdot(1-F)^{-1} = (1-t)^{-2}$, we deduce $\alpha((1-t)^{-2}(1-F)) + 1 - \alpha = 2$, and thus $\alpha((1-t)^{-2}(1-F)) = 1+\alpha$.

Hence $H(N) = ((1-t)^{-2}(1-F), t^N) \sim (N^2(1-F)(1-1/N))/(N\Gamma(1+\alpha))$. Therefore $(1-F)(1-1/N) \sim H(N)\Gamma(1+\alpha)/N$. Plugging this into the first asymptotic expansion above yields the final result. ■

To describe what happens when $\alpha(f) = 2$ requires a little more work, which however, is quite interesting. The other apparent boundary case, $\alpha(f) = 0$ does not even occur here: in fact, $\alpha(f) \geq 1$ in general. Write $f = (1-t)^{-1}K$. Then $M_k(f) = M_k(K)/(1+k)$, so that $\alpha(f) = 1+\alpha(K)$, whence $\alpha(f) \geq 1$. The case to be considered is $\alpha(f) = 2$, *i.e.*, $\alpha(K) = 1$. In this case (Subcase 2.1),

$$(f^{(-1)})'(t) \sim \frac{\Gamma(2)}{f(t)} \cdot (1-t)^{-3} = ((1-t)^3 f(t))^{-1} = (1-t)^{-1}(1-F)^{-1}(t).$$

Now $((1-F)^{-1})' = F'(1-F)^{-2}$, so that by l'Hôpital's rule,

$$\frac{((1-F)^{-1})(t)}{(f^{(-1)})(t)} \sim \frac{((1-F)^{-1})'(t)}{(f^{(-1)})'(t)} \sim (1-t)(1-F)^{-1}(t)F'(t) = \frac{F'(t)}{K(t)}.$$

This goes to zero, as will follow from the behaviour of the coefficients:

$$((1-t)^{-1}F', t^n) = \sum_{i=1}^{n+1} ic(i),$$

while $((1-t)^{-1}K, t^n) = H(n)$, and we will show later that in this boundary case, $\sum_{i=1}^{n+1} ic(i)/H(n) \to 0$.

For example, if $F = 1 - t/\ln((1-t)^{-1})$, it can be checked that the coefficients are nonnegative and sum to 1. Obviously $(1-F)^{-1} = \ln((1-t)^{-1})/t = \sum t^n/(n+1)$. Hence $\alpha((1-F)^{-1}) = 0$, and it easily follows that $\alpha(K) = 1$ and $\alpha(f) = 2$.

In contrast, let us see what happens when $\alpha((1-F)^{-1}) = 0$, *i.e.*, $\alpha(K) = 1$. Then we obtain

$$((1-F)^{-1}, t^n) \sim \frac{F'(e^{-1/n})}{n^2 \cdot (1-F)^2(e^{-1/n})} \sim \frac{F'(e^{-1/n})}{H(n)} \cdot \frac{1}{H(n)}.$$

Since $\alpha(F') = \alpha(K) = 1$, we deduce $\sum_{i=1}^{n+1} ic(i) = \sum_{i=0}^{n}(F', t^i) \sim F'(e^{-1/n})$. By rearranging the left sum, we obtain $\sum_{i=1}^{n+1} ic(i) = 1 + H(n) - (n+1)h(n+1)$. This is asymptotic with $H(n) - nh(n)$. Since $nh(n)/H(n) \to \alpha(F') = 1$, we deduce that

$$\frac{F'(e^{-1/n})}{H(n)} \sim \frac{H(n) - nh(n)}{H(n)} \to 0.$$

In particular, the coefficients of $(1-F)^{-1}$ must be infinitely smaller than $1/H(n)$. It would be nice to come up with an expression for $(H(n) - nh(n))/H(n)$.

4 Convolutions

In this section, we discuss conditions under which properties of f transfer to the product fg (as functions, *i.e.,* convolution of the corresponding Maclaurin series) for reasonable g. For example, it is not always true that if f and g have nonnegative coefficients and f satisfies LRT, then the product fg does as well—however, if we impose modest conditions on g, then it is the case. If f satisfies VRT, more drastic conditions (often verifiable) are needed on g in order that fg be VRT. Of course, our aim is to get results on the coefficients of $(1-F)^{-1}$, via its derivative, $F' \cdot (1-F)^{-2}$, where F' plays the role of f.

A sequence $(b(n))$ or its corresponding Maclaurin series $\sum b(n)t^n$, *or* the corresponding function is *robust* if there exists K such that for all N, and all i in $[N/2, N]$, $(b(i)/b(N))^{\pm 1} \leq K$. One-sided versions of this are also of interest. Any LLT function obviously satisfies this condition.

A weaker condition than LLT which turns up in the next result is that of super-LLT: a momentous f is *super*-LLT if

$$\liminf \frac{\Gamma(\alpha)N(f, t^N)}{f(1-1/N)} \geq 1 \quad \text{when } \alpha(f) > 0;$$

$$\liminf \frac{N^2(f, t^N)}{f'(1-1/N)} \geq 1 \quad \text{when } \alpha(f) = 0.$$

This condition says that the coefficients of f are (asymptotically) larger than they should be. Similarly, *infra-LLT* can be defined, but this property never turns up here. An example of a super-LLT function that is not LLT is easy to construct.

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Example 4.1 We give a robust super-LLT sequence with LRT that is not LLT. Let $\{(A(i), B(i))\}$ be a sequence of ordered pairs of positive integers with the following properties:

(i) $\{A(i)\}$ is increasing and $\liminf B(i) = \infty$,
(ii) $B(i) + B(i+1) < A(i+1) - A(i)$,
(iii) $\sum_{i=0}^{M} B(i) = \boldsymbol{o}(A(M))$.

Define the function $F\colon \mathbf{R}^+ \to [0,2]$ as follows. On intervals of the form $[A(i) - B(i), A(i)]$ define F to be linear with value 1 at the left endpoint and 2 at the right endpoint. On intervals of the form $[A(i), A(i) + B(i)]$, F is linear with value 2 at the left endpoint and value 1 at the right. The second condition forces all these intervals to be disjoint from each other. At all other points, $F(x) = 1$.

We note that the graph of F is the straight line $y = 1$, except for some triangles, and the total area of the triangles up to $A(N)$, $\sum B(i)$, is small compared to $A(N)$. Moreover, the absolute values of the slope tends to zero (as $B(i) \to \infty$). Set $a(n) = F(n)$, and form $f = \sum a(n)t^n$. The slope condition ensures that f satisfies LRT, and robustness follows from $1 \leq a(n) \leq 2$.

Next, we note that $\sum_{n \leq N}(f, t^n) = N + \sum_{\{i \mid B(i) \leq N\}} B(i)$, and it is straightforward to check that $\sum(f, t^n) \sim N$, so that $(1-t)^{-1}f \sim (1-t)^{-2}$. Thus f is weakly momentous with $f \sim (1-t)^{-1}$, hence $\alpha(f) = 1$, so f is momentous. On the other hand, f is not LLT, as follows from Proposition 2.5.

This example can be modified to admit other properties, *e.g.,* allowing the peaks to be unbounded (this loses robustness). Moreover, if $\{b(n)\}$ is any LLT sequence, we can form the Hadamard product $\{F(n)b(n)\}$, resulting in a similarly super-LLT sequence that is not LLT. More generally, if h is *any* power series with nonnegative coefficients and $h \sim \boldsymbol{o}((1-t)^{-1})$, then $h + (1-t)^{-1}$ is at least super-LLT, and will fail to be LLT if the coefficients of h do not converge to zero. Of course, smoothing operations (convolution with nice functions) can often be used to eliminate the excess.

Proposition 4.2 *Let $f = \sum a(n)t^n$ be momentous with $\alpha \equiv \alpha(f) > 0$. Suppose that $g = \sum b(n)t^n$ is momentous.*

(i) *If $\alpha(f) < 1$ and the coefficients of f are eventually decreasing, then fg is super-LLT if either $\beta \equiv \alpha(g) > 0$, or if both $\beta = 0$ and the coefficients of f' are eventually increasing.*
(ii) *If $\alpha(f) > 1$ and the coefficients of f are eventually increasing, then fg is LLT.*

Proof In both cases (i) and (ii), the hypotheses guarantee that f is LLT. (i) First, assume that $\beta := \alpha(g) > 0$. Fix s in $(0,1)$, and for $0 \leq j \leq [s^{-1}] - 1$, define $S_j := \sum_{N(1-(j+1)s)+1}^{N(1-js)} a(N-i)b(i)$, so that $(fg, t^N) \geq \sum_{0 \leq j \leq [s^{-1}]-1} S_j$ (we have deleted the last piece and a bit from the entire sum). Set $\mathcal{A}_j$ to be $\min\{a(i) \mid jsN \leq i \leq (j+1)sN\}$. Then $S_j \geq \mathcal{A}_j \sum_{N(1-(j+1)s)+1}^{N(1-js)} b(i)$. Since $\{a(n)\}$ is eventually decreasing, for all sufficiently large N, $\mathcal{A}_j = a((j+1)sN)$, and this is asymptotically $f(1 - 1/(j+1)sN)/\Gamma(\alpha(j+1)sN)$. As f is momentous, $f((1 - 1/(j+1)sN)$ is asymptotically (in N—note that s remains fixed until near the end) $f(1-1/N)(j+1)^\alpha s^\alpha$. Also note that $\sum_{N(1-(j+1)s)+1}^{N(1-js)} b(i)$ is asymptotically (in N)

$$g(1-1/N)((1-js)^\beta - (1-(j+1)s)^\beta)/\Gamma(1+\beta),$$

as follows easily from Theorem 1.3 and the momentous property.

Combining these inequalities, we see that $\liminf(fg, t^N)$ is at least as large as the limit superior (as $N \to \infty$) of

$$\frac{f(1-1/N)g(1-1/N)}{\Gamma(\alpha)\Gamma(1+\beta)N} \cdot \sum_{j=0}^{[s^{-1}-1]} \frac{(1-js)^\beta - (1-(j+1)s)^\beta}{(1+j)s} s^\alpha (j+1)^\alpha \tag{4.1}$$

$$= \frac{fg(1-1/N)}{\Gamma(\alpha)\Gamma(1+\beta)N} \cdot \sum_{j=0}^{[s^{-1}-1]} \left((1-js)^\beta - (1-(j+1)s)^\beta\right)(j+1)^{\alpha-1} s^{\alpha-1}.$$

Thus

$$\liminf \frac{(fg, t^N)}{fg(1-1/N)/N\Gamma(\alpha+\beta))} \geq \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(1+\beta)} \cdot \sum_{j=0}^{[s^{-1}-1]} \left((1-js)^\beta - (1-(j+1)s)^\beta\right)(j+1)^{\alpha-1} s^{\alpha-1}.$$

Hence in the term on the right, we may now permit $s \to 0$. We observe that it depends only on α, β, and s, not on f, g, or N. So we could evaluate the expression by setting $f = (1-t)^{-\alpha}$ and $g = (1-t)^{-\beta}$. This requires having to show that the missing part of the sum is of no consequence, which even here is somewhat tedious.

Instead, we evaluate an integral. The sum is well approximated (*i.e.,* the ratio tends to 1 as $s \to 0$) by the integral

$$\int_0^{s^{-1}-1} \frac{(1-xs)^\beta - (1-(x+1)s)^\beta}{s} (x+1)^{\alpha-1} s^\alpha \, dx.$$

Set $u = 1-(x+1)s$; the integral becomes

$$-\int_0^{1-s} \frac{(u-s)^\beta - u^\beta}{s} \cdot (1-u)^{\alpha-1} \, du.$$

As $s \to 0$, the integral becomes improper. However it is easy to justify taking the limit through the integral. The quotient becomes the derivative, and the integral becomes

$$\beta \int_0^1 u^{\beta-1} \cdot (1-u)^{\alpha-1} \, du.$$

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The integral is one of many equivalent forms of the beta function (of α and β). The value of this improper (but convergent) integral is $\Gamma(\alpha)\Gamma(\beta)/\Gamma(\alpha+\beta)$, and since $\Gamma(1+\beta)/\beta = \Gamma(\beta)$, the lim inf is bounded below by 1. This says that fg is super-LLT (as a product of momentous functions is obviously momentous).

Before dealing with the case of $\beta = 0$, we prove (ii).

(ii): There exists n_0 such that the sequence $(a(n))_{n\geq n_0}$ is increasing. Hence we can write $f = p + (1-t)^{-1}h$ where p is a polynomial with at most one negative coefficient (if it exists, it must be the one of highest degree) and h has only nonnegative coefficients and $(h, t^n) = 0$ for $n < n_0$. It is easy to verify that h is momentous. Then $fg = pg + (1-t)^{-1}hg$; set $H = (1-t)^{-1}hg$. Clearly H is momentous and has increasing coefficients, whence H is LLT. Therefore, $(H, t^N) \sim (f-p)(1-1/N)$ $g(1-1/N)/N(\Gamma(\alpha+\beta))$, and since p is a polynomial, $(f-p)(1-1/N) \sim f(1-1/N)$.

Let K be the maximum absolute coefficient of p, and let d be the degree of p. Obviously, $|(pg, t^N)| \leq K\sum_{i=0}^{d}(g, t^{N-i})$. Choose $r > 1$ and require N to be large enough that $N - d > N/r$.

From $\sum_0^N (g, t^i) \sim cg(1-1/N)$, we have $|(pg, t^N)| = \boldsymbol{O}(g(1-1/N))$; as $\alpha > 1$, it follows easily that $|(pg, t^N)| = \boldsymbol{o}((H, t^N))$. Thus $(H, t^N) \sim (fg, t^N)$, whence the latter is LLT.

(i): Now let $\beta = 0$. Consider $(fg)' = f'g + fg'$; the first summand is momentous and has increasing coefficients, hence is LLT; the result above applies to the second summand, so it is super-LLT, hence the sum is super-LLT. ∎

Remark If $\alpha = 0$, the result fails. An example can be constructed, based on the super-LLT non-LLT function given earlier, and $f = -\ln(1-t)/t = \sum t^n/(n+1)$.

The result in Proposition 4.2(ii) will be subsumed by several results in Section 10. The case that $\alpha = 1$ is oddly difficult (even if the coefficients are eventually increasing). A weaker result is available.

Lemma 4.3 *If $f = \sum a(n)t^n$ where $a(n) \geq 0$ and $a(n) \to a > 0$, and g is momentous, then $(fg, t^N) \geq \frac{1}{2}g(1-1/N)/\Gamma(\alpha(g))$.*

Proof Without loss of generality, $a = 1$, so we may write $a(n) = 1 + \xi(n)$ where $-1 \leq \xi(n) \to 0$. If $g = \sum b(n)t^n$, then $(fg, t^N) = \sum_0^N b(i) + \sum_0^N \xi(i)b(N-i)$. Let $\delta = \max_{\xi(m)}\{\xi(n)\}$. Choose $\epsilon > 0$. There exists $n_0 \equiv n_0(\epsilon)$ such that $i \geq n_0$ entails $|\xi(i)| < \epsilon$. Hence $|\sum_{i=n_0}^N \xi(i)b(N-i)| \leq \epsilon\sum_0^{N-n_0} b(i)$. Also,

$$\Big|\sum_{i=0}^{n_0-1} \xi(i)b(N-i)\Big| \leq \delta \sum_{N-n_0}^{N} b(i).$$

If $\alpha(g) = 0$, this last is $\boldsymbol{o}(g(1-1/N))$.

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If $\beta := \alpha(g) > 0$, and $r > 1$, for all sufficiently large N, $N/r < N - n_0$. Hence $\sum_{N-n_0}^{N} b(i) \leq g(1-1/N)(1-r^{-\beta})$. Select r so that $1 - r^{-\beta} < \Gamma(\beta)/4\delta$. Then

$$\Big|(fg, t^N) - \frac{g(1-1/N)}{\Gamma(\beta)}\Big| \leq g(1-1/N) + \epsilon g(1-1/N) \\ + \delta g(1-1/N)(1-r^{-\beta})/4\Gamma(\beta) \\ \leq g(1-1/N)/2\Gamma(\beta).$$

■

Lemma 4.4 *Suppose $j\colon (0,1) \to \mathbf{R}^{++}$ is a function satisfying the following properties:*

(i) *j is C^1, $j'(t) > 0$ for all t in $(1-\delta, 1)$ (for some $\delta > 0$), and $j'(e^{-1/n}) = \boldsymbol{o}(n^2 j(e^{-1/n}))$;*

(ii) *for all positive integers k, $\lim_{t\uparrow 1} j(t^{k+1})/j(t)$ exists and is nonzero;*

(iii) $\liminf_{t\uparrow 1} j(t) = \infty$.

Then $f := \sum j(e^{-1/n})t^n/n$ is LLT.

Proof First, the ratio of consecutive coefficients tends to 1, as follows from l'Hôpital and property (i). Hence the radius of convergence of f is 1. Item (ii) implies that f is strongly momentous. Set $f_0 = \sum j(e^{-1/n})t^n = tf'$. As tf' is thus strongly momentous, it is momentous. Moreover, j is eventually increasing, so that the coefficients of f_0 are eventually increasing, and thus f_0 is LLT. Create a new power series, f_1, by replacing an initial segment of f_0 by zero, so that the resulting series has increasing coefficients. Since $\alpha(f_0) \geq 1$ (as the coefficients are eventually increasing), $j(e^{-1/n}) \sim f_1(e^{-1/n})/n\Gamma(\alpha(f_1))$, and thus $j(e^{-1/n})/(n+1) \sim ((1-t)f_1)(e^{-1/n})/n\Gamma(\alpha(f_1))$. However, $(1-t)f_1$ has no negative coefficients (as the coefficients of f_1 are increasing), so that Proposition 2.5 applies. ■

Corollary 4.5 *If f and g are LLT, then so is their product, fg.*

Proof By two applications of the combination of Lemma 4.4 and Proposition 4.2, we can replace f and g by

$$f_0 = \sum_2^{\infty} f'(e^{-1/n})t^n/(n^2 - n)\Gamma(1 + \alpha(f))$$

and

$$g_0 = \sum g'(e^{-1/n})t^n/(n^2 - n)\Gamma(1 + \alpha(g)),$$

respectively, so that $(fg, t^n) \sim (f_0 g_0, t^n)$ (as $n \to \infty$). Consider $(f_0 g_0)'''$ (the third derivative). Each term in the Leibniz expansion contains a second or third derivative of one of f_0 or g_0. However, the coefficients of the second and third derivatives are of the form $f'(e^{-1/(n+2)})$ $(k = 0)$ or $nf'(e^{-1/(n+3)})$, and are therefore increasing. Thus, any product involving them with a power series with no negative coefficients

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has increasing coefficients. Hence $(f_0g_0)'''$ has increasing coefficients. Since fg is momentous, so is f_0g_0, and thus so is the third derivative. A momentous power series with increasing coefficients is LLT, and thus the third derivative is LLT. This easily entails that f_0g_0 is LLT (*e.g.*, by the preceding result), and therefore fg is. ∎

The following is a relatively easy (to prove) theorem about $(1-F)^{-1}$. It will be supplemented (§10) by more difficult results with different hypotheses. The appendix contains results about powers of functions having increasing coefficients.

Theorem 4.6 *Suppose that $F = \sum c(n)t^n$ (where $c(n) \geq 0$ and $\sum c(n) = 1$) satisfies:*

(i) *the coefficients of F'' are increasing, as are those of $(F')^k$, for some integer k;*

(ii) *F' is momentous.*

Then the conclusions of Lemma 3.5 apply, and moreover, the k-th derivative of $(1-F)^{-1}$ has increasing coefficients.

Remark If $(F')^k$ has increasing coefficients, then $k\alpha(F') \geq 1$, so that in particular, $\alpha(F') > 0$. Note that hypothesis (ii) together with F'' having increasing coefficients causes F' to satisfy LLT.

Proof We show that $((1-F)^{-1})^{(k)} = (F' \cdot (1-F)^{-2})^{(k-1)}$ has increasing coefficients; since it is momentous, it will then be LLT, and so $(1-F)^{-1}$ will be LLT.

The hypotheses in (i) yield that $F^{(l)}$ has increasing coefficients for all $l \geq 2$, and and the same is true for $(F')^{k-1}F''$. Expanding $(F' \cdot (1-F)^{-2})^{(k-1)}$ via the Leibniz formula yields that it is a sum of terms of the form $(F^{(d(1))})^{v(1)} \cdot (F^{(d(2))})^{v(2)} \cdots (1-F)^{-(1+\sum v(i))}$, where $\sum d(i)v(i) = k$, and we can assume $\{d(i)\}$ are distinct positive integers and $\{v(i)\}$ are positive integers. Each one of these terms contains a factor which has increasing coefficients; since all the terms have nonnegative coefficients, each term has increasing coefficients, and thus the sum does as well. ∎

The same technique works if hypothesis (i) is replaced by F''' and $(F')^2$ having increasing coefficients (this forces $\alpha(F') \geq 1/2$)—in this case, $((1-F)^{-1})'''$ has increasing coefficients. The drawback with Theorem 4.6 and its relatives is that it is not easy to decide whether some power of F' has increasing coefficients. This will be discussed briefly in the appendix.

We recall that for sequences a and b, the new sequence $a * b$ denotes the convolution (to avoid confusion with the Hadamard product, which is denoted $a \circ b$).

Lemma 4.7 *Let a, a', and b be sequences of nonnegative real numbers and suppose that for each integer $i \geq -1$, $b(N-i) = \boldsymbol{o}(a * b(N))$.*

(i) *If $a \sim a'$, then $a * b \sim a' * b$.*

(ii) *If a satisfies LRT, then so does $a * b$.*

Proof (i) $a * b(N) - a' * b(N) = \sum_{i=0}^{N} \big(a(N-i) - a'(N-i)\big) b(i)$. Given $\epsilon > 0$, there exists $m \equiv m(\epsilon)$ such that whenever $j \geq m$, we have $|a(j) - a'(j)| < \epsilon a(j)$.

Thus

$$\begin{aligned}|a*b(N)-a'*b(N)| &\leq \sum_{i=0}^{N-m}|a(N-i)-a'(N-i)|b(i)\\ &\qquad+\sum_{i=N-m+1}^{N}|a(N-i)-a'(N-i)|b(i)\\ &\leq \epsilon\sum_{i=0}^{N-m}a(N-i)b(i)+2\max\{a(j)\mid j\leq m\}\cdot\sum_{i=0}^{m-1}b(N-i)\\ &\leq \epsilon a*b(N)+2\max\{a(j)\mid j\leq m\}\cdot\sum_{i=0}^{m-1}b(N-i).\end{aligned}$$

For all N sufficiently large, each of $b(N),\dots,b(N-(m-1))$ is less than

$$\epsilon a*b(N)/2m\max\{a(j)\mid j\leq m\},$$

so $a*b(N)<2\epsilon a*b(N)$.

(ii) Define a' via $a'(n)=a(n+1)$; then LRT is equivalent to $a\sim a'$. By (i), $a*b\sim a'*b$, so $|a'*b(N-1)-a*b(N-1)|=\boldsymbol{o}(a*b(N-1))$. However, $a*b(N)=a'*b(N-1)+a(0)b(N)$, so that $|a*b(N)-a'*b(N-1)|=\boldsymbol{O}(b(N))$. The triangle inequality and the hypothesis on b (with $i=-1$ yields $|a*b(N)-a*b(N-1)|=\boldsymbol{o}(a*b(N-1))$. ■

It might be that if a satisfies VRT and b the condition in the lemma, then $a*b$ is VRT. If so, it would improve subsequent results substantially. Still useful is the following. The obstruction to all results of this type concerns the initial segment, hence the elaborate hypothesis.

Lemma 4.8 *Suppose that a and b are sequences of nonnegative real numbers such that a is VRT and there exists a function* $\mathcal{F}\colon(0,1)\to\mathbf{R}^+$ *such that* $\lim_{\delta\to 0}\mathcal{F}(\delta)=0$, *and for all sufficiently large N,*

$$\sum_{i=0}^{\delta N}a(i)b(N-i)\leq\mathcal{F}(\delta)a*b(N).$$

Then $a*b$ is VRT.

Proof Let $M\colon\mathbf{Z}^+\to\mathbf{Z}^+$ be such that $M(n)=\boldsymbol{o}(n)$; we will show (eventually) that $a*b(n+M(n))/a*b(n)\to 1$ as n increases (the same process works for $M\colon\mathbf{Z}^+\to-\mathbf{Z}^+$). Let δ be any positive real number less than 1. Define a new function $M_0\colon\mathbf{Z}^+\to\mathbf{Z}^+$ via $M_0(m)=\max\{M(m')\mid m\leq m'\leq m/\delta\}$.

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(a) First we show that $M_0(m) = \boldsymbol{o}(m)$. Given $\xi > 0$, there exists $k \equiv k(\xi)$ such that $n \geq k$ entails $M(n) \leq \xi n$. If $m \geq k$, then

$$\begin{aligned}\frac{M_0(m)}{m} &= \frac{\max\{M(m') \mid m \leq m' \leq m/\delta\}}{m}\\ &\leq \frac{\max\{\xi m' \mid m \leq m' \leq m/\delta\}}{m}\\ &\leq \frac{\xi m/\delta}{m} = \xi/\delta.\end{aligned}$$

Thus, for all sufficiently large m, we have that $M_0(m)/m \leq \xi/\delta$, that is

$$\limsup M_0(m)/m \leq \xi/\delta,$$

and since δ is fixed, $M_0(m) = \boldsymbol{o}(m)$.

(b) Next, we claim that the sequence $\{r(n)\}$ defined by

$$r(m) = \max\Big\{ \Big(\frac{a(m)}{a(m+j)}\Big)^{\pm 1} \;\Big|\; 0 \leq j \leq M(m)\Big\}$$

converges to 1. If not, there exists $\eta > 0$ and an infinite subset K of $\mathbf{N}$ such that for each k in K, either $r(k) > 1+\eta$ or $r(k) < 1/(1+\eta)$. Index the set $K = \{m_1, m_2, \dots\}$ where $m_i < m_{i+1}$. For each m_i, there exists a positive integer $j_i \leq m_i$ such that either $a(m_i)/a(m_i + j_i) > 1+\eta$ or $a(m_i)/a(m_i+j_i) < 1/(1+\eta)$. Define $M_1 \colon \mathbf{Z}^+ \to \mathbf{Z}^+$ via

$$M_1(m) = \begin{cases} M(m) & \text{if } m \notin K,\\ j_i & \text{if } m = m_i.\end{cases}$$

Then $M_1(m) \leq M(m)$, so $M_1(m) = \boldsymbol{o}(m)$. Thus $a(n)/a(n+M_1(n)) \to 1$, which contradicts the definition of j_i.

Now let δ be any small positive number, and consider the difference $a * b(N) - a * b(N + M(N))$. We can write this as

$$\sum_{i=0}^{\delta N}(a(i) - a(i+M(N)))b(N-i) + \sum_{i=\delta N}^{N}(a(i) - a(i+M(N)))b(N-i) \\ - \sum_{i=0}^{M(N)} a(i)b(N+M(N)-i).$$

We analyze the middle summand first. Define M_0 from M and δ as in (a); then observe that $M_0(i) \geq M(N)$ for all i in the interval $[[\delta N], N]$. By (a) and (b) above (the latter applied to $(M_0)_1$), given $\epsilon > 0$, for all sufficiently large N we have that $|a(i) - a(i+M(N))| < \epsilon a(i)$ for all i in $[[\delta N], N]$. Hence for N sufficiently large, the middle term is bounded in absolute value by $\epsilon \sum a(i)b(N-i)$, which is itself bounded above by $\epsilon a * b(N)$.

The left summand is bounded in absolute value by the larger of the two sums $\sum_{i=0}^{\delta N} a(i)b(N-i)$ and $\sum_{i=0}^{\delta N} a(i+M(N)b(N-i)$. The first one is bounded above (for sufficiently large N) by $\mathcal{F}(\delta)a * b(N)$ by hypothesis. If we choose N so large that $M(m) < \delta m$ for all $m \geq N$, then the second summand is bounded above by $\mathcal{F}(2\delta)a * b(N + M(N))$.

The third sum is an initial segment of $a * b(N+M(N))$, and if we assume the same condition as in the previous paragraph, that $M(m) < \delta m$ for all $m \geq N$, then the third sum is bounded above by $\mathcal{F}(\delta)a * b(N + M(N))$.

The outcome is that for all sufficiently large N, $|a * b(N) - a * b(N + M(N))|$ is bounded above by $(\epsilon + \mathcal{F}(\delta))a * b(N) + (\mathcal{F}(\delta) + \mathcal{F}(2\delta))a * b(N + M(N))$. Since this is true for $\delta > 0$ and $\epsilon > 0$ (we could have selected $\epsilon = \mathcal{F}(\delta)$), and both $\mathcal{F}(\delta)$ and $\mathcal{F}(2\delta)$ go to zero, we have that

$$\limsup \frac{|a * b(N) - a * b(N + M(N))|}{a * b(N) + a * b(N + M(N))} = 0.$$

It is easy to see that this forces $a * b(N)/(a * b(N + M(N)))$ to converge to 1. ■

Lemma 4.9 *Suppose that f is momentous with $\alpha := \alpha(f) > 0$, $g = \sum b(n)t^n$ is momentous, and $b(n) = \boldsymbol{O}(g(1-1/n)/n)$. Set $a(n) = f(1-1/n)/n$. Then the $\mathcal{F}$ of Lemma* 4.8 *exists and goes to zero as δ goes to zero.*

Proof Define the new sequence $B(i) = g(1-1/i)/i$ (setting $B(0) = 0$), so that $b(n) = \boldsymbol{O}(B(n))$, and define $G(t) = \sum B(n)t^n$. Then G is LLT. Define $f_0 = \sum_{n \geq 1} a(n)t^n$. Since f is momentous with $\alpha(f) > 0$, there $f_0 \sim Cf$ for some constant $C > 0$ and f_0 is LLT. In particular, $f_0 G$ is LLT, and $\alpha(f_0 g) \geq \alpha > 0$. Then $a * B(N) = (f_0 G, t^n)$, so $a * B(N) \sim c f_0 g(1-1/N)/N$ for some $c > 0$.

Let K be the constant in the big Oh expression.

$$\begin{aligned}
\sum_{i=0}^{\delta N} a(i)b(N-i) &\leq K \sum_{i=0}^{\delta N} \frac{a(i)g(1-\frac{1}{N-i})}{N-i} \\
&\leq K \frac{g(1-\frac{1}{N})}{N(1-\delta)} \sum_{i=0}^{\delta N} a(i) \\
&\sim K' \frac{f_0(1-\frac{1}{\delta N})g(1-\frac{1}{N})}{N(1-\delta)} \\
&\sim K' \delta^{\alpha} \frac{f_0 g(1-\frac{1}{N})}{N(1-\delta)}.
\end{aligned}$$

The sequence $\{a(n)\}$ satisfies the property that there exists a constant $D > 0$ such that for all sufficiently large N, for all i between $N/2$ and N, $a(i) \geq Da(N)$. Obviously, $a * b(N) \geq \sum_{N/2}^{N} a(i)b(N-i)$, and the latter is bounded below by

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$Da(N)\sum_0^{N/2} b(i)$, which is asympotic with $D'a(N)g(1-2/N) \sim D'' f_0 g(1-1/N)/N$. Hence we can choose F to be a small constant multiple of δ^α (or preferably, with exponent that is smaller than α but positive), which obviously goes to zero as δ does. ■

Proposition 4.10 *Suppose that f is LLT with $\alpha(f) > 0$, g is momentous, and $(g, t^N) = \boldsymbol{O}(g(1-1/N)/N)$. Then fg is LLT.*

Proof Let $f_0 = \sum_1^\infty f(1-1/n)t^n/n$, so that for suitable positive constant c, $f(t) \sim cf_0(t)$ and $(f, t^n) \sim c(f_0, t^N)$. By Proposition 4.10, $f_0 g$ is VRT; since it is also momentous, $f_0 g$ is therefore LLT. It suffices to show that $(f_0 g, t^N) \sim (fg, t^N)$. To that end, it suffices to show (by Lemma 4.7) that $(g, t^{N-i}) = \boldsymbol{o}((fg, t^N))$ for all i.

Since f is LLT, there exists a constant D such that for all sufficiently large N, for all $N/2 \leq j \leq N$, $(f, t^j) \geq D(f, t^N)$. It follows that $a * b(N) \geq D'a(N)\sum_0^{N/2} b(j) \sim D'' fg(1-1/N)/N$, and obviously $a*b(N+i) \geq D''' fg(1-1/N)/N$ for any i (for all sufficiently large N). Since $f(1-1/N)$ increases to infinity, the $b(N-i) = \boldsymbol{o}(a*b(N))$ condition is verified. ■

If we avoid momentous, and wish to conclude that the product satisfies VRT, here are conditions which guarantee it.

Proposition 4.11 *Suppose that a and b are sequences of nonnegative real numbers with the following properties:*

(i) *a is VRT;*
(ii) *there exist constants $0 < c < C$ such that for all sufficiently large N and all $N/2 \leq i \leq N$, $c \cdot b(N) \leq b(i) \leq C \cdot b(N)$;*
(iii) $\displaystyle\lim_{\delta\downarrow 0} \limsup_{N\uparrow\infty} \frac{\sum_0^{\delta N} a(n)}{\sum_0^{N/2} a(n)} = 0.$

Then $a * b$ satisfies VRT.

Remark The last condition is to avoid (in the LLT case) $\alpha(f) = 0$, *e.g.*, if $a(n) = 1/n$.

Proof We note that $a * b(N) \geq \sum_0^{N/2} a(i)b(N-i) \geq cb(N)\sum_0^{N/2} a(i)$. Similarly, the initial segment of $a * b(N)$ up to δN is bounded above by $Cb(N)\sum_0^{\delta N} a(i)$, and it follows that the condition of Lemma 4.8 is satisfied. ■

The initial segment condition of Lemma 4.8 is an annoying but necessary restriction on theorems of this type. We now describe two classes of examples illustrating this point.

Let E be a subset of $\mathbf{N}$; define $\pi_E \colon \mathbf{N} \to \mathbf{Z}^+$ via $\pi_E(n) = |E \cap \{1, 2, 3, \dots, n\}|$. We say *$E$ has measure zero* if $\pi_E(n) = \boldsymbol{o}(n)$. Let χ_E denote the characteristic (indicator) function of E. Define two Maclaurin series f_E and g_E via

$$f_E = \sum_{\mathbf{N}} \frac{\chi_E(n)}{n} t^n \quad g_E = \sum_{\mathbf{N}} \chi_E(n) t^n.$$

Obviously, $t f_E' = g_E$ and $((1-t)^{-1} g_E, t^N) = \pi_E(N)$.

Lemma 4.12

(i) *If $\sum \chi_E(n)/n$ diverges, then f_E is weakly momentous and $\alpha(f_E) = 0$.*

(ii) *The function g_E is weakly momentous if and only if for each of $k = 2$ and 3, the sequence $\{\pi_E(kN)/\pi_E(N)\}_N$ converges. If g_E is momentous and $\sum \chi_E(n)/n$ diverges, the limit of each such sequence is k, and g_E is momentous with $\alpha(g_E) = 1$. If for one of the k, $\lim \pi_E(kN)/\pi_E(N)$ exists but is not 1, then g_E is momentous.*

Proof (i) Lemma 1.9 applies directly. (ii) Denote the limits α_k. Obviously $\alpha_k \geq 1$. Consider $((1-t)^{-1} g_E(t^k)), t^N) = \pi_E(\lfloor N/k \rfloor)$. Define h by means of $h(t) = g_E(t^k)$. Replacing N by $\lfloor N/k \rfloor$ and observing that $k\lfloor N/k \rfloor$ is close to N, we have that $((1-t)^{-1} h, t^N) \sim \alpha_k((1-t)^{-1} g_E, t^N)$. As $\alpha_k > 0$, we thus have $(1-t)^{-1} h \sim \alpha_k (1-t)^{-1} g_E$ (as $t\uparrow 1$), and therefore, $h(t) \sim g_E(t)$. Hence $\lim_{t\uparrow 1} g_E(t^k)/g_E(t)$ exists and equals α_k. Thus g_E is weakly momentous. The converse is straightforward.

If g_E is weakly momentous and $\sum \chi_E(n)/n$ diverges, then f_E is weakly momentous with $\alpha(f_E) = 0$, and since $t f_E' = g_E$, it follows that f_E is momentous (from the definition), and thus $\alpha(g_E) = \alpha(f_E') = 1$. The rest follow from the results in Section 1. ∎

If we assume the hypotheses of Lemma 4.12(i) and (ii), then $\sum_{n \leq N} \chi_E(n)/n \sim f_E(1 - 1/N)$ and $\pi_E(N) = \sum_{n \leq N} \chi_E(n) \sim g_E(1 - 1/N) \sim f_E'(1 - 1/N)$. Now define two more functions, h_E and j_E, via

$$h_E = \sum_{\mathbf{N}} \frac{\pi_E(n)}{n} t^n \quad j_E = \sum_{\mathbf{N}} \frac{\pi_E(n)}{n^2} t^n.$$

Obviously $t h_E' = (1-t)^{-1} g_E$, whence h_E' is momentous with $\alpha(h_E) = 2$, and thus h_E is momentous with $\alpha(h_E) = 1$. If we assume that E has measure zero, then the coefficients of h_E converge to zero, so that in particular, $h_E(t) = \boldsymbol{o}((1-t)^{-1})$. On the other hand, since h_E is momentous, $\sum_1^N \pi_E(n)/n \sim h_E(1 - 1/N)$.

Since $t j_E' = h_E$, j_E is either convergent or momentous with $\alpha(j_E) = 0$. We eliminate the possibility that $j_E(1) < \infty$. Since $g_E'(t) \sim (1-t)^{-1} = t h_E'$, we have that $g_E' \sim h_E'$. By l'Hôpital's rule, $g_E \sim h_E$. In particular, $j_E' \sim g_E \sim f_E'$, and since $f_E(1) = \infty$, we can use l'Hôpital again, and thus deduce $j_E \sim f_E$.

Therefore, $((1-t)^{-1} j_E, t^N) \sim j_E(1 - 1/N) \sim f_E(1 - 1/N)$, so $\sum_1^N \pi_E(n)/n^2 \sim \sum_1^N \chi_E(n)/n$ (this can be interpreted as a Stieltjes integral).

An elementary consequence of the results above is the following. If f_E is momentous, then for any real $l > -1$,

$$\sum_{n \in E}^{N} n^l \sim \frac{N^l g_E\left(1 - \frac{1}{N}\right)}{l+1}.$$

For l a positive integer, this is an easy consequence of

$$l!\, t^l g^{(l)}(t) = \sum_{n \geq l} \chi_E(n) n! \,/(n-l)!\, t^n \sim \sum \chi_E(n) n^l t^n,$$

followed by use of $g^{(l)}(1 - 1/N)$ to determine the sum of the first N coefficients. For general real values of $l > -1$, we can either develop fractional derivatives in this context (!), or more simply use the Hadamard power results of Section 3 with h_E as the relevant LLT function.

Now we pose the following question:

$(*)$ Suppose that f_E is momentous, $q = \sum b(n)t^n$ is LLT, and $f_E(t) \sim q(t)$. Does it follow that $f_E'(t) \sim q'(t)$?

This can be extended to a form without referring to the set E. If j and q are LLT and $j(t) \sim q(t)$, then is it true that $j'(t) \sim q'(t)$? The answer is trivially yes if $\alpha(j) > 0$ (since $j' \sim \alpha(1-t)^{-1} j$), leaving the case that $\alpha(j) = 0$. This question restricts to $(*)$ upon noticing that $j_E \sim f_E$, $j_E' \sim f_E'$, and j_E is LLT. The question is motivated by classical results in number theory.

Let E be the set of prime numbers. An elementary argument [R, Theorem 2.3, p. 224] shows that $\sum_{p \leq N, p \in E} 1/p = \ln\ln(1 - 1/N) + c + \boldsymbol{O}(1/\ln N)$ where c is a constant. It follows that $f_E \sim \ln(\ln(2/(1-t)))$ (the 2 is there to guarantee that the latter function is analytic on the open unit disk). That g_E is momentous is a consequence of the prime number theorem: $\pi(kN)/\pi(N) \to k$ is immediate from $\pi(N) \sim N/\ln N$ (the prime number theorem is usually given in a stronger form, as an estimate of $\pi(N) - \mathrm{Li}(N)$). That $g_E \sim (\ln\ln(1/(1-t)))' = (1-t)^{-1}/\ln(1/(1-t))$ follows directly from the prime number theorem (without the error estimate). If $(*)$ were true, we could deduce $\pi(N) \sim N/\ln N$ from $\sum_{p \in E}^{N} 1/p \sim \ln\ln N$ and $\pi(kN)/\pi(N) \to 1$. In fact, we could also obtain $\pi(N) \sim N/\ln N$ if the error term in $\sum_{p \leq N} 1/p$ above were little oh rather than big Oh, by examining the coefficients of $((1-t)^{-1} f_E)' - (1-t)^{-2} f_E$.

Example 4.13 Let $m\colon \mathbf{N} \to \mathbf{N}$ be strictly increasing and satisfy $m(i+1)/m(i) \to 1$ (alternatively, m grows subexponentially). Define $G := \sum (m(i+1) - m(i)) t^{m(i)}$ and $g = \sum (m(i+1) - m(i)) t^{m(i)}/m(i)$. Then $tg' = G$ and $G \sim (1-t)^{-1}$; if $\sum (m(i+1) - m(i))/m(i)$ diverges, then g is momentous.

Let $f = \sum a(n)t^n$ be LLT with $\alpha := \alpha(f) > 0$. If $m(j+1) - m(j) = \boldsymbol{o}(f(1 - 1/m(i)))$ (or equivalently, $m(j+1) - m(j) = \boldsymbol{o}(m(j) a(m(j)))$), then fG is LLT; if additionally, $\sum (m(i+1) - m(i))/m(i)$ diverges, then fg is LLT.

Proof First, we show $G \sim (1-t)^{-1}$. For a positive integer n, define $s(N) := \max\{m(i) \mid m(i) \leq N\}$. Obviously $s(N) \leq N$, so if $s(N)/N$ fails to converge to 1, there exist $\delta > 0$ and infinitely many positive integers $N_1 < N_2 < N_3 < \cdots$ such that $s(N_u) \leq (1-\delta)N_u$. If we write $s(N_u) = m(k(u))$, then $m(k(u)+1) > N_u$ by definition. Hence for all u, $m(k(u)+1)/m(k(u)) > 1/(1-\delta)$, contradicting

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$m(i+1)/m(i) \to 1$. Hence $s(N) \sim N$ and thus $(1-t)^{-1}G(t) \sim (1-t)^{-2}$, so $G(t) \sim (1-t)^{-1}$. In particular, G is momentous and $\alpha(G) = 1$.

If $\sum(m(i+1) - m(i))/m(i)$ diverges, we show g is momentous. Since $g' \sim G \sim (1-t)^{-1}$ and $g(t)\uparrow\infty$ as $t\uparrow 1$, we have that $g'(t) = \boldsymbol{o}((1-t)^{-1}g(t))$; thus g is weakly momentous and $\alpha(g) = 0$. Assume until further notice that $\alpha < 1$.

Now assume that there exists $c > 0$ such that for all n and all $0 \le j \le n$, $a(n) \le a(n-j)$ (*e.g.*, if a is monotonic non-increasing, then $c = 1$). We consider the sum over the initial segment,

$$\sum_{i=0}^{\delta N} a(i)(G, t^{N-i}) = \sum_{\{j | N(1-\delta) \le m(j) \le N\}} a(N - m(j)) \cdot (m(j+1) - m(j)).$$

Define $s(N)$ as above, and let $s^+(N)$ denote $m(l+1)$ if $s(N) = m(l)$.

We have, with the exception of one value of j,

$$a(N - m(j)) \cdot (m(j+1) - m(j)) \le c \sum_{u=1}^{m(u+1)-m(j)} a(N - m(j+1) + l),$$

and the exceptional value is bounded above by $a(0)(s^+(N) - s(N))$. Thus,

$$\begin{aligned}\sum_{\{j | N(1-\delta) \le m(j) \le N\}} & a(N - m(j)) \cdot (m(j+1) - m(j)) \\ & \le a(0)((s^+(N) - s(N))) + c\sum_{0}^{\delta N} a(i) \\ & \sim c'(m(l+1) - m(l)) + c\frac{f(1 - \frac{1}{\delta N})}{\Gamma(1+\alpha)} \\ & \sim c'(m(l+1) - m(l)) + c\delta^{\alpha}\frac{f(1 - \frac{1}{N})}{\Gamma(1+\alpha)}.\end{aligned}$$

Since $m(l) \le N < m(l+1)$ and $m(l+1) - m(l) = \boldsymbol{o}(m(l+1))$ (for N and therefore l sufficiently large), the initial segment is $\boldsymbol{o}(\delta^{\alpha})f(1 - 1/N)$. On the other hand, $(fG, t^N) \ge c^{-1}a(N)\sum^N (G, t^i) \sim c''Na(N) \sim c'''f(1 - 1/N)$. Thus the sum over the initial segment satisfies the conditions of Lemma 4.8, and so fG is VRT, and being momentous, is therefore LLT.

Now drop the monotonicity of the coefficients of f. Let $f_0 = \sum_1^{\infty} f(1-1/n)t^n/n$, so that $(f, t^N) \sim c_{\alpha}(f_0, t^N)$ for some nonzero constant. Since $\alpha < 1$, the coefficients of f_0 are eventually decreasing, so we can find f_1 whose coefficients are decreasing with $(f_1, t^N) \sim (f_0, t^N)$, and thus $(f_1, t^N) \sim (f, t^N)$. By the earlier results, $f_1 G$ is LLT. Also, the condition $b(N) = \boldsymbol{o}(a * b(N))$ is formally weaker than the initial segment condition. Thus $(f_1 G, t^N) \sim (fG, t^N)$, which (together with $f(t) \sim f_1(t)$ which is a consequence of their coefficients being asymptotic) entails that fG is LLT.

If $\alpha > 1$, fG is LLT. If $\alpha(f) = 1$, then for $\epsilon > 0$, for all sufficiently large n, and all i with $n/2 \le i \le n$, we have that $|(a(i)/a(n))^{\pm 1} - 1| < \epsilon$. The estimate of the initial segment sum will work, except at a bounded number of places, and this simply adds a bounded multiple of $m(l+1) - m(l)$ to the upper bound, not affecting the outcome.

Finally, if $\sum (m(i+1) - m(i))/m(i)$ diverges, consider $(fg)' = fg' + f'g$. Since $tfg' = fG$, fg' is LLT by the preceding. On the other hand, $\alpha(f') > 1$, so $f'g$ is also LLT. The sum of two LLT functions is obviously LLT. ■

For the following class of examples, it did not seem possible to obtain corresponding results applying to functions of the form f_E, where E is a reasonable subset of $\mathbf{N}$. The first few terms in the expansion play an unreasonably large role, as is clear from the proofs.

Example 4.14 Let $h \colon \mathbf{R}^+ \to \mathbf{R}^+$ be strictly monotonic, C^1, and satisfy $\lfloor h(n+1) \rfloor > \lfloor h(n) \rfloor$ and $yh'(y) \sim Ch(y)$ as $y \uparrow \infty$, for some $C > 0$. Set $g = \sum t^{\lfloor h(n) \rfloor}$. If f is LLT, $\alpha \equiv \alpha(f) > 0$, and $\liminf_{t \uparrow 1} (1-t) f(t) h^{-1}((1-t)^{-1}) = \infty$, then fg is VRT. In particular, this applies if g is momentous and either $\alpha(f) + \alpha(g) > 1$ or $\liminf_{t \uparrow 1} (1-t)^{-1} fg(t)) = \infty$.

Proof As usual, set $a(n) = f(1 - 1/n)/n\Gamma(\alpha)$, and form $f_0 = \sum_1^\infty a(n) t^n$. First, we note that $((1-t)^{-1} g, t^N) = \lfloor h^{-1}(N) \rfloor$, and

$$\sum_1^{\delta N} a(n)(g, t^{N-n}) = \sum_{h^{-1}(\lceil N(1-\delta) \rceil)}^{h^{-1}(\lfloor N \rfloor)} a(N - h(n)).$$

If we assume (for now) that $\alpha < 1$, then there exist $c > 0$ and positive integer K such that $a(n) \geq a(n+1)$ for all $n \geq K$. Using the integral approximation (and keeping in mind the initial segment), we have

$$\begin{aligned}
&\sum_1^{\delta N} a(n)(g, t^{N-n}) \\
&\quad \leq \sum_{h^{-1}(N) \geq n \geq h^{-1}(N-k)} a(N - h(k)) + \frac{1}{\Gamma(\alpha)} \int_{h^{-1}(N(1-\delta)}^{h^{-1}(N-K)} \frac{f\left(1 - \frac{1}{N-h(x)}\right)}{N - h(x)}\, dx \\
&\quad = \sum_{h^{-1}(N) \geq n \geq h^{-1}(N-k)} a(N - h(k)) + \frac{1}{\Gamma(\alpha)} \int_K^{\delta N} \frac{f\left(1 - \frac{1}{s}\right)}{s \cdot h'(h^{-1}(N-s))}\, ds \\
&\quad \leq \sum_{h^{-1}(N) \geq n \geq h^{-1}(N-k)} a(N - h(k)) + \frac{1}{\Gamma(\alpha) h'(h^{-1}((1-\delta)N))} \int_K^{\delta N} \frac{f\left(1 - \frac{1}{s}\right)}{s}\, ds \\
&\quad \leq \sum_{h^{-1}(N) \geq n \geq h^{-1}(N-k)} a(N - h(k)) + \frac{1}{\Gamma(\alpha) h'(h^{-1}((1-\delta)N))} \sum_1^{\delta N} a(n) \\
&\quad \sim \sum_{h^{-1}(N) \geq n \geq h^{-1}(N-k)} a(N - h(k)) + \frac{1}{\Gamma(\alpha) h'(h^{-1}((1-\delta)N))} \cdot \frac{f\left(1 - \frac{1}{\delta N}\right)}{\Gamma(1+\alpha)}
\end{aligned}$$

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$$\sim \sum_{h^{-1}(N)\geq n\geq h^{-1}(N-k)} a(N-h(k)) + \frac{c}{h'(h^{-1}((1-\delta)N))}\cdot\delta^{\alpha} f\Big(1-\frac{1}{N}\Big)$$

$$\sim \sum_{h^{-1}(N)\geq n\geq h^{-1}(N-k)} a(N-h(k)) + \delta^{\alpha} c' \frac{f\big(1-\frac{1}{N}\big)}{N} h^{-1}(N)$$

$$\leq \sum_{1}^{K} a(n) + \delta^{\alpha} c'' \frac{f\left(1-\frac{1}{N}\right)h^{-1}(N)}{N}$$

The first equality (third line) is obtained via the substitution $s = N - h(x)$; $ds = -h'(x)dx$, and since $x = h^{-1}(N-s)$, we have $dx = -ds/h'(h^{-1}(N-s))$. Since h' and h^{-1} are increasing, the inequality in the fourth line results. The fifth follows from the converse of the integral test (where we have added in the first few terms just in case). The sixth and seventh come from f being momentous (and $\alpha > 0$).

The eighth line, the replacement of the complicated h term in the denominator by $h^{-1}(N)$, results from $h'(h^{-1}(N(1-\delta))) \sim CN(1-\delta)/h^{-1}(N(1-\delta)$ and the fact that h^{-1} is increasing.

Thus to verify the initial segment condition, it suffices that $1 = \boldsymbol{o}((f_0 g, t^N))$ and $f(1-1/N)h^{-1}(N) = \boldsymbol{O}(N(f_0 g, t^N))$. However,

$$(f_0 g, t^N) \geq \sum_{N/2}^{N} a(n)g(N-n) \geq a(N)\sum_{1}^{N/2}(g, t^N)$$
$$\sim c''' f(1-1/N)h^{-1}(N/2)/N \sim c'''' f(1-1/N)h^{-1}(N)/N.$$

Hence $f_0 g$ satisfies VRT. Since the initial segment condition is stronger than the $b(N) = \boldsymbol{o}(a * b(N))$ condition, it follows that fg is coefficientwise asymptotic to $f_0 g$, so the former is also VRT.

If $\alpha \geq 1$, only minor modifications are required, concerning asymptotic behaviour of ratios $a(n)/a(N)$ for $N/2 \leq n \leq N$. ■

We can weaken the hypotheses on h considerably. For instance, if we wish to deal with $g = f_E$ where E is the set of primes, set $h(x) = x\ln(x+1)$; then $h^{-1}(N) \sim N/\ln N$ and this is good enough. In this case, the sufficient condition is satisfied automatically when $\alpha(f) > 0$ (since $\alpha(fg) > 1$).

On the other hand, if $h(x) = x^2$, then $g = \sum t^{n^2}$ so g is momentous, $g \sim (1-t)^{-1/2}$, and $\alpha(g) = \frac{1}{2}$. The necessary condition from the result requires at least $\alpha(f) \geq \frac{1}{2}$ (and if $\alpha = \frac{1}{2}$, an additional condition on the growth), and we have seen before that there are LLT functions f with $\alpha(f)$ any desired value less than or equal to $\frac{1}{2}$ such that fg (and even $f_0 g$) is *not* VRT.

5 Elementary Perturbations

Here we show that if f is LLT and $g = \sum e(n)$ has absolutely summable coefficients (not necessarily nonnegative) and $g(1) > 0$ *and* $|e(n)| = \boldsymbol{O}(1/n)$, then fg has almost

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all of its coefficients nonnegative, and $(fg, t^N) \sim g(1)(f, t^N)$; in particular, fg is LLT. If the $\boldsymbol{O}(1/n)$ condition fails, the product fg need not be LLT.

Lemma 5.1 *Let $G = \sum_{n\geq 1} e(n)t^n$ where $e(n) \geq 0$, and suppose that g and h are real analytic with radius of convergence* 1 *and only nonnegative coefficients such that $|(g-h, t^N)| = \boldsymbol{o}((g, t^N))$. If $e(n) = \boldsymbol{o}((g, t^n))$ and g satisfies LRT, then $(hG, t^N) \sim (gG, t^N)$.*

Proof Given $\epsilon > 0$, there exists a polynomial p_ϵ with positive coefficients, say bounded above by $L \equiv L(\epsilon)$, and of degree $k \equiv k(\epsilon)$, and an integer K such that for all $n \geq K$, $-(p_\epsilon, t^n) - \epsilon(g, t^n) \leq (g-h, t^n) \leq (p_\epsilon, t^n) + \epsilon(g, t^n)$, whence

$$-(p_\epsilon G, t^n) - \epsilon(gG, t^n) \leq ((g-h)G, t^n) \leq (p_\epsilon G, t^n) + \epsilon(gG, t^n).$$

Now we could have chosen K sufficiently large that $(G, t^n) < \epsilon(Lk)^{-1}(g, t^n)$ for all $n \geq K - k$.

$$\begin{aligned}(p_\epsilon G, t^n) &\leq L\sum_{i=0}^{k}(G, t^{n-i}) \leq Lk\max\{(G, t^{n-i}) \mid i = 0, \dots, k\}\\ &\leq \left(\frac{\epsilon}{Lk}\right) Lk\max\{(g, t^{n-i}) \mid i = 0, \dots, k\}\\ &\leq 2\epsilon(g, t^n).\end{aligned}$$

The last line uses the LRT hypothesis. Also, $(gG, t^n) \geq (G, t) \cdot (g, t^{n-1}) \sim (G, t) \cdot (g, t^n)$, so the result follows. ∎

Lemma 5.2 *Suppose that $F = \sum c(n)t^n$ where $0 \leq c(n)$ and g has only nonnegative coefficients.*

(i) *If $c(n) = \boldsymbol{o}(1/n)$ and $(g, t^n) = \boldsymbol{o}(g(1-1/n))$, then $(gF', t^N) = \boldsymbol{o}(g(1-1/N))$;*
(ii) *if $c(n) = \boldsymbol{O}(1/n)$, then $(gF', t^N) = \boldsymbol{O}(g(1-1/N))$;*
(iii) *if $c(n) = \boldsymbol{o}(1/n)$, and $g'(1-1/N) = \boldsymbol{O}((g', t^N))$* and *$\rho(g) < \infty$, then $(gF', t^N) = \boldsymbol{o}((g'F, t^N)/N)$;*
(iv) *as* (iii)*, but with all little oh's converted to big Oh's.*

Remark The hypothesis $(g, t^n) = \boldsymbol{o}(g(1-1/n))$ is not a misprint for the much stronger $(g, t^n) = \boldsymbol{o}(g(1-1/n)/n)$, nor is it redundant—the function defined via $g = \sum 2^{2^k} t^{2^{2^k}}$ fails to satisfy it.

Proof (i) If $c(n) = \boldsymbol{o}(1/n)$, then $(F', t^n) = \boldsymbol{o}((1-t)^{-1}, t^n))$, hence its coefficients are bounded, say by K. For each $\epsilon > 0$, we can write $F' = p_\epsilon + h$, where $p_\epsilon \leq K\sum_0^{D(\epsilon)} t^n$ coefficientwise for some $D \equiv D(\epsilon)$, and $(h, t^n) \leq \epsilon$ for all n. Set $f_\epsilon = (K-\epsilon)\sum_0^{D(\epsilon)} t^n + \epsilon(1-t)^{-1}$, so that F' is coefficientwise less than or equal to f_ϵ. Then $(gF', t^N) \leq (K-1)\sum_{N-D(\epsilon)}^{N}(g, t^n) + \epsilon\sum_0^N (g, t^n)$. As $N \to \infty$, the first sum is bounded above by little oh of $D(\epsilon)g(1-1/N)$, and since the second summand is bounded above by $\epsilon g(1 - 1/N)$, the result follows by considering all sufficiently large N.

The obvious and easy argument yields (ii).

Without loss of generality, we may assume that $c(1) = 1$. To obtain (iii), we note that

$$\begin{aligned}(g'F, t^N) &= \sum_{n<N} c(N-n)(g', t^N) \geq c(1)(g', t^{N-1}) \\ &\geq c' \frac{g'(1 - 1/(N-1))}{N-1} \geq c''c' \frac{g'(1-1/N)}{N} \geq c'''g(1-1/N).\end{aligned}$$

Part (iv) is proved similarly. ∎

A consequence is that if $c(n) = \boldsymbol{o}(1/n)$ and g is (say) LLT, then for all positive integers k, $((gF)^{(k)}, t^N) \sim (g^{(k)}F, t^N)$. If $g^{(k)}$ has increasing coefficients and we assume that $\sum c(n) < \infty$, then obviously gF is momentous, and so is $g^{(k)}F$; since its coefficients are totally ordered, $g^{(k)}F$ is LLT. Since $((gF)^{(k)}, t^N) \sim (g^{(k)}F, t^N)$, this means that $(gF)^{(k)}$ is LLT, and therefore so is gF. Hence $(gF, t^N) \sim c(gF)'(1 - 1/N)/N^2 \sim cg'(1 - 1/N)F(1)/N^2$; in other words, $(gF, t^N) \sim F(1)(g, t^N)$. The fact that g LLT implies that gF is as well is a perturbation result, which does not always hold if the hypotheses are weakened. For example, if $F = \sum t^{2^{2^n}}/2^n$ and $\alpha(g) < 1$, then it is easy to check that $(gF, t^N) \neq \boldsymbol{O}(g'(1 - 1/N)/N^2)$.

We wish to refine this perturbation result. This leads to something parallel to Littlewood's Tauberian theorem, as we explain later.

Proposition 5.3 *Suppose that $F = \sum c(n)t^n$ where $c(n) \geq 0$, $\sum c(n) < \infty$, and $c(n) = \boldsymbol{O}(1/n)$. Let g be LLT with $\alpha(g) > 0$. Then gF is LLT and $(gF, t^N) \sim F(1)(g, t^N)$.*

Proof Pick $\beta < \alpha(f)$, and set $g_1 = (1 - t)^{-\beta}$, so that g_1 is LLT, and g_1' has increasing coefficients. Thus $g_1'F$ is momentous and has increasing coefficients, hence is LLT. Since $(g_1F', t^N) = \boldsymbol{O}(g_1'F, t^N)$, it follows immediately that $h_1' := (g_1F)'$ satisfies $(h_1', t^N) \approx h_1'(1 - 1/N)/N$, which is also $\approx h_1(1 - 1/N)$. Therefore, $(h_1, t^N) \approx h(1 - 1/N)/N$.

Now define $g_2 = c\sum g(1 - 1/N)/N^{1+\beta}t^n$; this is LLT by Lemma 4.4, and it is trivial that $(1 - t)^{-\beta}g_2(t) \sim g(t)$. Moreover, g_2 is VRT, and the condition on h_1 is enough to ensure that g_2h_1 is also VRT. Hence g_2h_1 is both VRT and momentous, whence LLT. Thus $(1 - t)^{-\beta}g_2F$ is LLT.

By Lemma 5.1 with $e(n) = \boldsymbol{O}(1/n) = \boldsymbol{o}(g(1 - 1/n)/n) = \boldsymbol{o}(g(1 - 1/n))$ and $h = g_2h_1$, we would obtain $(gF, t^N) \sim (hF, t^N)$ if we knew $(g - h, t^n) = \boldsymbol{o}((g, t^n))$. However, it follows from $h(t) \sim g(t)$ and both functions being LLT with $\alpha(h) = \alpha(g) > 0$ that $(h, t^n) \sim (g, t^n)$, which is of course the same as $(g - h, t^n) = \boldsymbol{o}((g, t^n))$. Hence $(gF, t^N) \sim (hF, t^N)$, so gF is LLT. ∎

The parallel with Littlewood's Tauberian theorem is the following. If instead we had little oh in the hypothesis, then the corresponding result is true, by the straightforward argument we used earlier. The little oh hypothesis is analogous to that in Tauber's eponymous theorem. Littlewood improved it to big Oh, and here we do more or less the same thing, by exploiting a factorization and that $(gF', t^N) = \boldsymbol{O}((g'F, t^N))$.

Corollary 5.4 *Suppose that F and G each have summable nonnegative coefficients, and moreover, $F(1) > G(1)$, $(F, t^N) = \boldsymbol{O}(1/N)$, and $(G, t^N) = \boldsymbol{O}(1/N)$. If f is LLT and $\alpha(f) > 0$, then $f \cdot (F - G)$ has at most finitely many negative coefficients, and $(f \cdot (F - G), t^N) \sim (F(1) - G(1))(f, t^N)$.*

Proof From $(fF, t^N) \sim F(1)(f, t^N)$ and $(fG, t^N) \sim G(1)(f, t^N)$, we have $((F - F(1)f, t^N) = \boldsymbol{o}(F(1)(f, t^N)) = \boldsymbol{o}((f, t^N))$, and similarly,

$$((G - G(1)f, t^N) = \boldsymbol{o}(G(1)(f, t^N)) = \boldsymbol{o}((f, t^N)).$$

Thus, $(f \cdot (F - G) - (F(1) - G(1))f, t^N) = \boldsymbol{o}((f, t^N))$. Thus $(f \cdot (F - G), t^N) \geq (F(1) - G(1))(f, t^N) - \boldsymbol{o}((f, t^N))$. The latter is eventually strictly positive, yielding the first statement, and the asymptotic result is now a consequence of the displayed formula. ■

Suppose that $F = \sum c(n)t^n$, $F_1 = \sum c_1(n)t^n$ where all the coefficients are nonnegative, $\sum c(n) = \sum c_1(n) = 1$, and $c_1(n)$ is close to $c(n)$ (in some sense). Can we say whether the coefficients of $(1 - F)^{-1}$ and $(1 - F_1)^{-1}$ are close? The following result says something in this direction, although the specific notion of closeness is often difficult to verify.

Lemma 5.5 *Suppose that $(1 - F)^{-1} \cdot (F_1 - F)$ has absolutely summable coefficients, $\sum nc(n) = \infty$, $c(n) > 0$ for all $n \geq 1$, $F' \cdot (1 - F)^{-1}$ has bounded coefficients, and $|c(n) - c_1(n)| = \boldsymbol{O}(c(n))$. Then $(1 - F_1)^{-1}(1 - F)$ has absolutely summable coefficients, which are additionally $\boldsymbol{O}(1/n)$.*

Proof Since $g := (1 - F)^{-1}(1 - F_1) = 1 + (1 - F)^{-1} \cdot (F_1 - F)$, it follows that g has absolutely summable coefficients. In particular, g is continuous on the closed disk and therefore $g(1) = \lim_{t\uparrow 1} g(t)$, and it easily follows from l'Hôpital's rule that $(g - 1)(t) \to 0$. Hence g has no zeroes on the closed unit disk, and therefore, its inverse has absolutely summable coefficients as well (by Wiener's preliminary Tauberian lemma).

Now we verify that the coefficients of g^{-1} are $\boldsymbol{O}(1/n)$; it is sufficient to show that $((g^{-1})', t^n)$ is bounded. We write $(g^{-1})'$ as $F_1'(1 - F_1)^{-2}(1 - F) - F'(1 - F_1)^{-1}$. Since a convolution of a bounded sequence with an absolutely summable one is bounded, it suffices to show that the two constituents of $(g^{-1})'$, multiplied by g^2 or g, have bounded coefficients.

The first term is $F_1'(1 - F_1)^{-2}(1 - F) \cdot g^2 = F_1' \cdot (1 - F)^{-1}$, the second is $F'(1 - F_1)^{-1} \cdot g = F'(1 - F)^{-1}$. The latter has bounded coefficients by hypothesis, hence so does $F'(1-F_1)^{-1}$. The former differs from $F'(1-F)^{-1}$ by $(F'-F_1')(1-F)^{-1}$, so it suffices to show this last has bounded coefficients. By hypothesis, $|(F_1'-F', t^n)| \leq K(F', t^n)$ for some $K > 0$, and it easily follows that the coefficients of $(F' - F_1')(1 - F)^{-1}$ are bounded in absolute value by a multiple of the corresponding coefficients of $F'(1 - F)^{-1}$. In particular, the coefficients are bounded, and thus so are those of $F_1'(1 - F_1)^{-2}(1 - F)$, and therefore, so are those of $(g^{-1})'$. Hence $(g^{-1}, t^n) = \boldsymbol{O}(1/(n-1)) = \boldsymbol{O}(1/n)$. ■

In some cases, we can show that the coefficients of $(1 - F_1)^{-1}(1 - F)$ are even $\boldsymbol{o}(1/n)$; it may be that this is true fairly generally.

Since $g^{-1}(1) = 1$, we turn to Corollary 5.4 to see whether we can conclude from Lemma 5.5 whether $(1-F_1)^{-1} = (1-F)^{-1} \cdot g^{-1}$ has coefficients asymptotic to those of $(1 - F)^{-1}$. Unfortunately, at the moment, Corollary 5.4 requires the additional assumption that F' be LLT, for which all perturbation results turn out to be moot.

6 Imitation Momentous Functions

For this section, $f\colon (0, 1) \to \mathbf{R}^+$ will be C^l for some $l \geq 1$, with the property that f and all of its derivatives (up to the l-th) are positive-valued and increasing. We define the following numbers associated with f.

For $k > 1$ (not necessarily an integer), set

$$\delta_k(f) = \liminf_{t\uparrow 1} \tfrac{f(t^k)}{f(t)} \quad \rho(f) = \limsup_{t\uparrow 1} \tfrac{f'(t)}{(1-t)^{-1}f(t)} \quad \sigma(f) = \liminf_{t\uparrow 1} \tfrac{f'(t)}{(1-t)^{-1}f(t)}$$

$$\delta^k(f) = \limsup_{t\uparrow 1} \tfrac{f(t^k)}{f(t)} \quad P(f) = \liminf_{t\uparrow 1} \tfrac{(1-t)^{-1}f(t)}{f'(t)} \quad \Sigma(f) = \limsup_{t\uparrow 1} \tfrac{(1-t)^{-1}f(t)}{f'(t)}$$

There are some obvious relations among these, *e.g.*, $0 \leq \delta_k(f) \leq \delta^k(f) \leq 1$, $\Sigma(f) = 1/\sigma(f)$, $P(f) = 1/\rho(f)$ (the "P" is capital rho), and so on. We want to deduce some consequences of $\sigma(f) > 0$ and $\rho(f) < 1$, among other things. A restatement of $\rho(f) < \infty$ is that $f'(t) = \boldsymbol{O}((1-t)^{-1}f(t))$ (as $t\uparrow 1$), and of $\sigma(f) > 0$, that $(1 - t)^{-1}f(t) = \boldsymbol{O}(f'(t))$.

Lemma 6.1 (Basic Lemma) *Let (t_n) be a strictly increasing sequence of positive real numbers converging to 1. Let $k > 1$, and let r, s_k, and R_k be nonnegative numbers. Consider the following properties:*

(i) $\dfrac{f'(t_n)}{(1-t_n)^{-1}f(t_n)} \to r;$

(ii) $\dfrac{f'(t_n^k)}{(1-t_n^k)^{-1}f(t_n^k)} \to R_k;$

(iii) $\dfrac{f(t_n^k)}{f(t_n)} \to s_k.$

Then $r \geq \frac{1-s_k}{k-1}$, and if $s_k > 0$, then $R_k \leq \frac{k}{s_k} \cdot \frac{1-s_k}{k-1}$.

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Proof Part (iii) implies that $s_k f(t_n) - f(t_n^k) = \boldsymbol{o}(f(t_n))$, so that $f(t_n) - f(t_n^k) = (1 - s_k + \boldsymbol{o}(1))f(t_n)$. By the mean value theorem, the left side is $(t_n - t_n^k)f'(t_n')$ for some t_n' with $t_n^k \le t_n' \le t_n$. As f' is increasing, $(t_n - t_n^k)f'(t_n) \ge (1 - s_k + \boldsymbol{o}(1))f(t_n)$. Thus

$$\frac{f'(t_n)}{(1-t_n)^{-1}f(t_n)} \cdot \frac{t_n - t_n^k}{1 - t_n} \ge 1 - s_k + \boldsymbol{o}(1).$$

As $n \to \infty$, the left side converges to $r(k-1)$, whence $r(k-1) \ge 1 - s_k$.

If $s_k > 0$, then $f(t_n)/f(t_n^k) \to 1/s_k$, and thus $f(t_n) - f(t_n^k)/s_k = \boldsymbol{o}(f(t_n^k))$. Hence $f(t_n) - f(t_n^k) = (s_k^{-1} - 1 + \boldsymbol{o}(1))f(t_n^k)$. Applying the mean value theorem and using monotonicity of f', we deduce $(t_n - t_n^k)f'(t_n^k) \le (s_k^{-1} - 1 + \boldsymbol{o}(1))f(t_n^k)$. Hence

$$\frac{f'(t_n^k)}{(1-t_n^k)^{-1}f(t_n^k)} \cdot \frac{t_n - t_n^k}{1 - t_n^k} \le 1 - s_k + \boldsymbol{o}(1).$$

As $n \to \infty$, the left side converges to $R_k \cdot (k-1)/k$. Hence $R_k \le k(1-s_k)/(k-1)s_k$, as desired. ∎

We deduce some consequences.

Corollary 6.2

(i) $\dfrac{1-\delta_k(f)}{k-1} \le \rho(f) \le \dfrac{k}{\delta_k}\dfrac{1-\delta_k(f)}{k-1}$.

(ii) $\dfrac{\delta^k}{k}\dfrac{k-1}{1-\delta^k(f)} \le \Sigma(f) \le \dfrac{k-1}{1-\delta^k(f)}$.

Proof (i) If $\rho(f) = \infty$ or $\delta_k(f) = 0$, the result is trivial. So we assume $\rho(f) < \infty$ and $\delta_k > 0$.

Let $\{t_n\}$ increase up to 1. Any convergent subsequence of $\{f(t^k)/f(t)\}$ converges to some number $s_j \ge \delta_k$. Since $\{f'(t)/(1-t)^{-1}(f(t))\}$ is bounded, some subsequence converges; hence we may assume (by reducing to a further subsequence) that $f(t_n^k)/f(t_n) \to s_k$ and $f'(t_n)/(1-t_n)^{-1}(f(t_n)) \to r$. Hence $r \ge (1-s_k)/(k-1)$, and since $\rho \ge r$, we deduce $\rho \ge (1-s_k)/(k-1)$. Since δ_k is defined as the lim inf of all the ratios, we may choose s_k to be arbitrarily close to δ_k. Hence $\rho \ge (1-\delta_k)/(k-1)$.

Similarly, let t_n be a sequence increasing to 1 such that the corresponding R_k is as close as we like to $\rho(f)$. By taking a subsequence, we may assume that $f(t_n^k)/f(t_n)$ converges, say to $s_k \ge \delta_k(f)$. Thus

$$R_k \le \frac{k}{k-1} \cdot \Big(\frac{1}{s_k} - 1\Big) \le \frac{k}{k-1} \cdot \Big(\frac{1}{\delta_k} - 1\Big) = \frac{k}{\delta_k} \cdot \frac{1-\delta_k}{k-1}.$$

Hence $\rho(f) \le \frac{k}{\delta_k} \cdot \frac{1-\delta_k}{k-1}$.

The proof of (ii) is parallel to that of (i). ∎

Of interest is the behaviour of the left and right terms in (i) and (ii) above as $k\downarrow 1$. Define $\mathcal{D}\colon [1,\infty) \to [0,1]$ via $\mathcal{D}(s) = \delta_s$. We see that $\mathcal{D}$ is monotone decreasing, and moreover, if $1 \le s \le s'$, then $\mathcal{D}(s') \ge \mathcal{D}(s) \cdot \mathcal{D}(s'/s)$. The latter follows from the factorization

$$\frac{f(t^{s'})}{f(t)} = \frac{f((t^{s'/s})^s)}{f(t^{s'/s})} \cdot \frac{f(t^{s'/s})}{f(t)}.$$

Obviously, $\delta_1 = 1$, so $\mathcal{D}$ is continuous at 1 if and only if $\lim_{k\downarrow 1} \delta_k = 1$. Assume $\rho \equiv \rho(f) < \infty$; from $\rho \cdot (k-1) \ge 1 - \delta_k$, we have $\delta_k \ge 1 - \rho \cdot (k-1)$, hence $\liminf_{k\downarrow 1} \delta_k \ge 1$, so $\lim \delta_k = 1$. We deduce that $\mathcal{D}$ is continuous at 1. However, more is true. We have that $\rho = \lim_{k\downarrow 1}(1-\delta_k)/(k-1)$ by (i), but the latter expression is the derivative (from the right) of $-\mathcal{D}$ at 1, *i.e.*, $-\mathcal{D}'(1^+)$. In particular, $\mathcal{D}'(1^+) = -\rho$.

Since $\mathcal{D}$ is continuous at 1, it follows that $\mathcal{D}$ is continuous on all of $[0,\infty)$ (regrettably, it does not seem possible to prove this for differentiability). Since $\mathcal{D}$ is monotone decreasing, it suffices to show that for $k_0 > 1$, $\inf_{1\le k<k_0} \mathcal{D}(k) \le \mathcal{D}(k_0)$. Since $\mathcal{D}(k_0) \ge \mathcal{D}(k) \cdot \mathcal{D}(k_0/k)$, as $k\uparrow k_0$, the right factor converges to $\mathcal{D}(1) = 1$, whence $\mathcal{D}(k_0) \ge \inf_{1\le k<k_0} \mathcal{D}(k)$, as desired.

A useful consequence is that if $\rho < 1$, then $k\delta_k > 1$ for all $k > 1$ sufficiently close to 1. Just note that the function $\mathcal{E}\colon s \mapsto s\mathcal{D}(s)$ satisfies $\mathcal{E}(s') \ge \mathcal{E}(s)\mathcal{E}(s'/s)$ (for $1 \le s \le s'$), and moreover, $\mathcal{E}$ is continuous and its derivative (from the right) at 1 is $\mathcal{D}(1) + \mathcal{D}'(1^+) = 1 - \rho > 0$. Hence $\mathcal{E}$ is strictly increasing at 1. If $\rho > 1$, then $\mathcal{E}$ is decreasing just to the right of 1.

If merely $\rho = 1$, there is no obvious conclusion, *e.g.*, if f is momentous with $\alpha(f) = 1$, then $\delta_k k = 1$ for all $k > 1$.

Immediate consequences of (i) and (ii) include the following.

Corollary 6.3 *Each of the following holds.*

(i) $\delta^k(f) = 0$ *for some* $k > 1 \iff \delta^k(f) = 0$ *for all* $k > 1 \iff \sigma(f) = \infty$.
(ii) $\delta^k(f) = 1$ *for some* $k > 1 \iff \delta^k(f) = 1$ *for all* $k > 1 \iff \sigma(f) = 0$.
(iii) $\delta_k(f) = 0$ *for some* $k > 1 \iff \delta_k(f) = 0$ *for all* $k > 1 \iff \rho(f) = \infty$.
(iv) $\delta_k(f) = 1$ *for some* $k > 1 \iff \delta_k(f) = 1$ *for all* $k > 1 \iff \rho(f) = 0$.

Proof The first equivalence in (iv) follows from monotonicity of $\mathcal{D}$, together with the property that $\mathcal{D}(s') \ge \mathcal{D}(s)\mathcal{D}(s'/s)$ for $1 \le s \le s'$. For (iii), $\delta_k = 0$ and monotonicity entails that $\delta_{k'} = 0$ if $k' > k$; if $1 < l < k$ and $\delta_k = 0$, but $\delta_l \ne 0$, then $\delta_{l/k} \le \delta_k/\delta_l = 0$. Set $l_0 = \sup\{l \mid \delta_l = 0\}$. Then $1 \le l < l_0$ implies $\delta_l \ne 0$; if $\delta_{l_0} = 0$, then $l_0 > 1$ and we obtain $\delta_{l_0/l} = 0$, a contradiction. Hence $\delta_{l_0} \ne 0$, but if $l_0 \ne 1$, the supermultiplicativity again yields a contradiction. The first equivalences of each of (i) and (ii) are proved in a parallel fashion; of course, the function $t \mapsto \delta^t$ is submultiplicative rather than super-multiplicative.

For the rest of (iii), we just let k decrease down to 1 in the left inequality of Corollary 6.2(i). For the rest of (iv), any value of $k > 1$ in the right side of Corollary 6.2(i) will yield the result. The rest of (i) and (ii) are proved in parallel fashion, exploiting Corollary 6.2(ii). ■

Obviously $\rho(f) \ge \sigma(f)$, so if $\rho(f) < \infty$, then $\sigma(f) < \infty$, and $\sigma(f) > 0$ implies $\rho(f) > 0$. Equally obviously, $\rho(f) = \sigma(f) < \infty$ if and only if f is weakly momentous, and in this case, $\alpha(f) = \rho(f)$.

Corollary 6.4 *If f satisfies $\rho(f) < \infty$ and $\sigma(f) > 0$, then $1 > \delta_k(f') > 0$ for all $k > 1$, and if additionally, $l \geq 2$, then $\rho(f') < \infty$ and $\sigma(f') \neq 0$.*

Proof Select k. For all $\epsilon > 0$ and for all t sufficiently close to 1, we have that $(\sigma - \epsilon)(1 - t^k)f(t^k) \leq f'(t^k) \leq (\rho + \epsilon)(1 - t^k)f(t^k)$, and since $t > t^k$, the same inequalities are true with t replacing t^k. Hence

$$\frac{(\sigma - \epsilon)(1 - t^k)f(t^k)}{(\rho + \epsilon)(1 - t)f(t)} \leq \frac{f'(t^k)}{f'(t)} \leq \frac{(\rho + \epsilon)(1 - t^k)f(t^k)}{(\sigma - \epsilon)(1 - t)f(t)}.$$

We conclude

$$\frac{\sigma}{\rho}\frac{\delta_k(f)}{k} \leq \delta_k(f') \leq \frac{\rho}{\sigma}\frac{\delta_k(f)}{k} \quad \text{and} \quad \frac{\sigma}{\rho}\frac{\delta^k(f)}{k} \leq \delta^k(f') \leq \frac{\rho}{\sigma}\frac{\delta^k(f)}{k}.$$

Next we observe that $\infty > \sigma(f) > 0$ implies $0 < \delta^k(f) < 1$ and similarly, $\delta_k > 0$, and $\delta_k/k \to 0$ as $k \to \infty$, so neither $\delta^k(f')$ nor $\delta_k(f')$ can be 1. If $l \geq 2$, then these conditions entail their counterparts in σ and ρ. ■

A particular consequence is that if f is C^∞ and all derivatives are increasing and positive, then $\rho(f) < \infty$ and $\sigma(f) > 0$ implies that the same is true of all derivatives, *i.e.*, $(1 - t)^{-1}f^{(l)}(t) = \boldsymbol{O}(f^{(l+1)}(t))$ and $f^{(l+1)}(t) = \boldsymbol{O}((1 - t)^{-1}f^{(l)}(t))$.

By analogy with our earlier notion of momentous, we say the Maclaurin series (or the function) $f = \sum a(n)t^n$ with only nonnegative coefficients and radius of convergence 1, is *imitation momentous* if $\rho(f) < \infty$. If $f = \sum t^{2^n}$, then we have seen that f is weakly momentous, but not momentous; however, it is imitation momentous (as is easy to verify). On the other hand, the exotic (very lacunary) $f = \sum t^{2^{2^n}}$ is weakly momentous but not even imitation momentous.

Here is a modest version of the LLT characterization, Theorem 2.7, weakening the momentous hypothesis to imitation momentous ($\rho(f) < \infty$). Unfortunately, the conclusion is considerably weaker too.

Lemma 6.5 *Suppose $f = \sum a(n)t^n$ is VRT, and f' is imitation momentous. Then $a(n) \approx f'(1 - 1/n)/n^2$.*

Proof On replacing f by f', we may assume that $\delta^k(f) < 1$, $\delta_k(f) > 0$ (the conclusion will be expressed in terms of $f(1-1/n)$ rather than $f'(1-1/n)$ to accommodate this assumption). Set $b(n) = na(n)/f(1 - 1/n)$ (or vice versa, that is, its inverse). It is sufficient to show that $b(n)$ is bounded above and below (away from zero, except for possibly finitely many zeroes). If not, without loss of generality (replacing b by its reciprocal if necessary), for all positive integers i, there exist integers $n(i)$ such that $b(n(i)) \geq i$; we may assume that $n(i)$ is strictly increasing in i.

Define

$$m(i) := \inf\{m \mid b(k) > i/2 \text{ for all } k \text{ such that } m \leq k \leq n(i)\},$$

$$M(i) := \sup\{m \mid b(k) > i/2 \text{ for all } k \text{ such that } n(i) \leq k \leq m\}.$$

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We must show that $M(i)$ is finite for all but finitely many i. If $M(i) = \infty$, we obtain that for all sufficiently large n, $na(n) > if(1 - 1/n)/2$. Hence for sufficiently large N, $\sum_1^N (f, t^k) \geq (i/3) \sum_1^N f(1 - 1/k)/k$ (select N sufficiently large that $n(i)$ is tiny in comparison). We deduce that for all sufficiently large N,

$$\sum_1^N (f, t^k) \geq \frac{i}{3e} \sum_{k=1}^N \sum_{j=1}^k \frac{a(j)}{k} \geq \frac{i}{3Ne} \sum_{k=1}^N \sum_{j=1}^k a(j) = \frac{i}{3Ne}((1-t)^{-2} f, t^N)$$

The left side is $((1-t)^{-1} f, t^N)$; however,

$$\begin{aligned} ((1-t)^{-2} f, t^N) &= \sum_1^N ((1-t)^{-1} f, t^n) \\ &\geq K(1-t)^{-1} f(t)|_{t=1-1/N} = KeNf(1 - 1/N) \end{aligned}$$

(for some small positive K depending only on f). And $((1-t)^{-1} f, t^N) = \sum_1^N (f, t^n) \leq ef(1 - 1/N)$. Therefore, for all sufficiently large N, $ef(1 - 1/N) \geq iKNf(1 - 1/N)/(3Ne)$, and thus $i \leq 3e^2/K$. Hence $M(i) = \infty$ for only finitely many i.

Now we show that $m(i) < n(i)/2$ for at most finitely many i. If $m(i) < n(i)/2$, then $na(n) > if(1 - 1/n)$ for $n(i)/2 \leq n \leq n(i)$, and thus (in parallel with the preceding argument),

$$\frac{n(i)}{2} \sum_1^N a(n) \geq \sum_{n(i)/2}^{n(i)} na(n) > \frac{in(i)}{2} f\Big(1 - \frac{2}{n(i)}\Big).$$

Therefore,

$$f\Big(1 - \frac{2}{n(i)}\Big) \leq \frac{\sum_1^{n(i)} a(n)}{i} \leq K \frac{f(1 - 1/n(i))}{i},$$

whence

$$\frac{f(1 - 2/n(i))}{f(1 - 1/n(i))} \leq \frac{K}{i}.$$

If the last line holds for infinitely many i, then $\delta_2(f) = 0$, a contradiction.

Thus we can assume that for all i (deleting an initial segment, if necessary), $m(i) \geq n(i)/2$ and $M(i) < \infty$. We observe that

$$\frac{b(n(i))}{b(m(i) - 1)}, \frac{b(n(i))}{b(M(i) + 1)} > \frac{i}{i/2} = 2.$$

Hence neither of the sequences $\{n(i) - m(i)\}$ nor $\{M(i) - n(i)\}$ can be $\boldsymbol{o}(n(i))$ by the VRT hypothesis. By taking a subsequence, we may assume that $(n(i) - m(i))/n(i) \geq$

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$\kappa n(i)$ for some $\frac{1}{2} > \kappa > 0$ and all i. Then

$$\begin{aligned}\sum_{(1-\kappa)n(i)}^{n(i)} a(n) &\geq i \sum_{(1-\kappa)n(i)}^{n(i)} \frac{f(1-1/n)}{n} \\ &\geq \frac{i}{n(i)} \sum_{(1-\kappa)n(i)}^{n(i)} f\Big(1-\frac{1}{n}\Big) \geq \frac{i}{n(i)} Kn(i) f\Big(1-\frac{1}{(1-\kappa)n(i)}\Big) \\ &\geq iK\kappa f(1-\frac{2}{n(i)}) \geq iK'\kappa f(1-1/n(i)).\end{aligned}$$

However, the left side is bounded above by $\sum_1^{n(i)} a(n) \leq ef(1-1/n(i))$. If $i > e/K'k$, we obtain a contradiction. ■

7 Imitation Partial Sums

Assume for this section that $f = \sum a(n)t^n$ has only nonnegative coefficients, has radius of convergence 1, and that $\sum a(n)$ diverges. We want to use the results of Section 6 to find a relation between $f(1-1/N)$ and $\sum_1^N a(n)$ for large N. An obvious one, requiring no additional assumptions at all, is that

$$f(1-1/N) \geq e^{-1-1/N} \sum_1^N a(n).$$

This follows from substituting $t = 1 - 1/N$ into the power series. A particular consequence is that $\sum_1^N a(n) = \boldsymbol{O}(f(1-1/N))$. Much more subtle are conditions to guarantee $f(1-1/N) = \boldsymbol{O}(\sum_1^N a(n))$.

We show that if $1 > \delta_k(f) > 0$ (equivalently, $f(t) = \boldsymbol{O}((1-t)f'(t))$ and $f'(t) = \boldsymbol{O}((1-t)^{-1}f(t))$), then the latter does hold, and thus it holds for all derivatives of f as well. We work with $k = 2$. A very easy application of the mean value theorem yields that $\liminf_{N\to\infty} f(1-1/N)/f(1-1/2N) = \delta_2(f)$. Given ϵ, for all sufficiently large N, $f(1-1/N) > (\delta_2 - \epsilon)f(1-1/2N)$. Now expand f.

For M a large integer, $f(1-1/M) = \sum a(n)(1-1/M)^n$ will be broken up into smaller pieces as follows. Let K be a nonnegative integer, and set $S_K(M) = \sum_{KM+1}^{(K+1)M} a(n)$. For $K \geq 1$, $(1-1/M)^{KM} \sim e^{-K}$ (the left side is slightly smaller). We have

$$(1-\epsilon(M))e^{-1}\Big(S_0(M) + \sum_{K\geq 1} e^{-K}S_K(M)\Big) \leq f(1-1/M) \leq S_0(M) + \sum_{K\geq 1} e^{-K}S_K(M),$$

where $\epsilon(M)$ is small to begin with and goes to zero as $M \to \infty$. We deduce

$$S_0(N) + \sum_{K\geq 1} e^{-K}S_K(N) \geq (\delta_2 - \epsilon)(1-\epsilon(2N))e^{-1}\Big(S_0(2N) + \sum_{K\geq 1} e^{-K}S_K(2N)\Big).$$

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Since $S_K(2N) = S_{2K}(N) + S_{2K+1}(N)$, we obtain

$$\Big(1 - \frac{\delta_2 - \epsilon(2N)}{e}\Big) S_0(N) \geq \sum_{K\geq 1}\big((\delta_2 - \epsilon(2N))e^{-\lfloor K/2\rfloor -1} - e^{-K}\big) S_K(N).$$

The coefficient of S_K appearing in the last formula exceeds e^{-K} if and only if $\exp(K - \lfloor K/2\rfloor) \geq 2e/(\delta_2 - \epsilon(N))$. Sufficient for this is $K \geq 2 + 2\ln 2 - \ln(\delta_2 - \epsilon(2N))$ (notice that if $\delta_2 \geq 1/2$, *i.e.*, $2\delta_2 \geq 1$ (qv earlier), then we obtain that $K \geq 4$ is sufficient). In particular, there exists $K' > 0$ (independent of N) such that

$$\sum_{K\geq K'} e^{-K} S_K(N) \leq \Big(1 - \frac{\delta_2 - \epsilon(2N)}{e}\Big) S_0(N) + \sum_{1\leq K<K'} e^{-K} S_K(N).$$

Thus $f(1 - 1/N) \leq (2 - \delta_2 + \epsilon(2N))S_0(N) + 2\sum_{1\leq K<K'} e^{-K} S_K(N)$.

Now pick $L \geq K'$. For $\epsilon' > 0$ and for all N sufficiently large, $f(1 - 1/LN) \leq f(1 - 1/N)/(\delta_L - \epsilon')$. To see this, note that $(1 - 1/LN)^L \leq (1 - 1/N) + \boldsymbol{O}(1/N^2)$, so

$$0 \leq f((1 - 1/LN)^L) - f(1 - 1/N) \leq ((1 - 1/LN)^L - (1 - 1/N))f'(1 - 1/N'),$$

where $(1 - 1/N) \leq (1 - 1/N') \leq (1 - 1/LN)^L$ and of course, neither L nor N' need be an integer. This is bounded above by $\boldsymbol{O}(1/N^2)f'((1 - 1/LN)^N)$. This is $\boldsymbol{O}(f(1 - 1/LN)^L)/N$, hence $f(1 - 1/N)/f((1 - 1/LN)^L) \to 1$. We deduce

$$\begin{aligned} f\Big(1 - \frac{1}{LN}\Big) &\leq \frac{f\big(1 - \frac{1}{N}\big)}{\delta_L - \epsilon'} \\ &\leq \frac{2\big(S_0(N) + \sum_{1\leq K<K'} e^{-K} S_K(N)\big)}{\delta_L - \epsilon'} \\ &= \frac{2S_0(K'N)}{\delta_L - \epsilon'}. \end{aligned}$$

Since K' does not depend on N and the last inequality is true for all sufficiently large L (not necessarily an integer), it follows that $f(1 - 1/m) = \boldsymbol{O}(\sum_1^m a(n))$.

We thus have the following.

Proposition 7.1 *Let $f = \sum a(n)t^n$ (with $a(n) \geq 0$) satisfy $0 < \sigma(f)$ and $\rho(f) < \infty$. Then $f(1 - 1/N) = \mathbf{O}(\sum_1^N a(n))$.*

The converse fails — any weakly momentous function with $\alpha(f) = 0$ satisfies $f(1 - 1/N) \sim \sum_1^N a(n)$, but $f'(t) = \boldsymbol{o}((1 - t)^{-1} f(t))$, so $\rho(f) = 0$.

With a minor modification, we can prove a strengthening of the result. Instead of finding minimal K, so that the coefficient of $S_K(N)$ is at least e^{-K}, we find K'' so that

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$(\delta_2-\epsilon)\exp(-K/2-1) \geq (K+2)e^{-K}$ for all $K \geq K''$. It suffices to choose K'' so that $K'' > 2\ln(K''+2)/(\delta_2-\epsilon)+1$, in other words K'' exists. Now consider

$$\begin{aligned} tf'\Big(1-\frac{1}{N}\Big) &= \sum a(n)n\Big(1-\frac{1}{n}\Big)^N \\ &\leq NS_0(N) + \sum_{K\geq 1} e^{-K}(K+1)NS_K(N) \\ &\leq N\big(S_0(N) + \sum_{K\geq 1} e^{-K}(K+1)S_K(N)\big). \end{aligned}$$

Selecting $L \geq K''$ and proceeding as before, we deduce that $f'(1-1/N) = \boldsymbol{O}(N\sum_1^N a(n))$; this provides an alternative proof to the earlier result, Corollary 6.4. Notice that this condition by itself guarantees that $f'(t) = \boldsymbol{O}((1-t)^{-1}f(t))$

8 Boundary Behaviour

This section deals with behaviour of the functions on the unit circle, near the singularity at 1, and consequences for the coefficients in the Maclaurin expansions. The open unit disk is denoted D and the closed unit disk is denoted $\overline{D}$.

Lemma 8.1 *Suppose that $0 \leq a < 1$ and suppose $H\colon [a,1] \to \mathbf{R}^+$ is continuous and in addition, is differentiable on $(a,1)$. If for some $\eta \geq 0$, $H'(x) \geq \eta(1-x)^{-1}$ for all x sufficiently close to 1, then $\eta = 0$.*

Proof We may assume the inequality holds on $[a,1)$. As H' is positive, H is increasing; define J via $J(t) = \int_a^t H'(x)\,dx$. Then $H = J + H(a)$, and $J(t) \geq \eta\int_a^t \frac{dx}{1-x} = \eta\ln(a/(1-t))$. Thus, if $\eta > 0$, then $J(t) \to \infty$, contradicting $H(1) < \infty$. ∎

Lemma 8.2 *Suppose that $0 < \rho(f) = r < \infty$. Then for all $\epsilon > 0$, $f(1-\theta) = \boldsymbol{o}(\theta^{-r-\epsilon})$ as $\theta\downarrow 0$.*

Proof Select $\epsilon > 0$ and define g by means of $g(t) = (1-t)^{-r-\epsilon}$. For t sufficiently close to 1,

$$\frac{f'(t)}{g'(t)} \leq \frac{(r+\epsilon/2)(1-t)^{-1}f(t)}{(1-t)^{-1-r-\epsilon}(r+\epsilon)} = \Big(1-\frac{\epsilon/2}{r+\epsilon}\Big)\frac{f(t)}{g(t)}.$$

Hence $f/g \geq f'/g' + \epsilon' f/g$ (for t sufficiently close to 1). Now $f/g - f'/g' = (fg'-f'g)/gg' = (g/f)'(f^2/gg')$. In particular, $(g/f)' > 0$, so g/f is increasing on $[a,1)$ for some $a < 1$. Moreover, $(g/f)'f^2/gg' > \epsilon' f/g$ yields $(g/f)' > \epsilon' g'/f = (r+\epsilon)\epsilon'(1-t)^{-1}g/f$.

Since g/f is increasing on $(a,1)$, f/g is decreasing. So if $\limsup_{t\uparrow 1} f/g(t) := R > 0$, then $f(t)/g(t) \to R$, so $g(t)/f(t) \to 1/R$. There thus exists b with $1 > b > a$ such that for all t in $(b,1)$, $(g/f)'(t) > (r+\epsilon)\epsilon'(1-t)^{-1}/2R$. With $H = g/f$, Lemma 8.1 yields a contradiction. Thus $f(t)/g(t) \to 0$, that is, $f(1-\theta) = \boldsymbol{o}(\theta^{-(r+\epsilon)})$. ∎

It is also true that $\theta^{-s} = \boldsymbol{o}(f(1-\theta))$ for all $s > \sigma(f)$, but there is no occasion to use this.

Lemma 8.3 *Suppose $f = \sum a(n)t^n$ with $a(n) \geq 0$ and define $\Delta a(n) = a(n) - a(n-1)$. If the following are true,*

(a) $\sum_{n>N} |\Delta a(n)| = \boldsymbol{O}(f(1-1/N)/N)$,
(b) $a(n) \to 0$ *as* $n \to \infty$,

then the following hold

(i) $a(n) = \boldsymbol{O}(f(1-1/n)/n)$,
(ii) f *is continuous on* $\overline{D} \setminus \{1\}$,
(iii) $|f(e^{i\theta})| = \boldsymbol{O}(f(1-\theta))$ *as* $\theta \downarrow 0$.

Proof Let K be the constant in the big Oh term in (a).

(i) For any positive integers n and m, $a(n) = a(n+m) - \sum_{j=1}^{m} \Delta a(n+j)$, so $a(n) \leq a(n+m) + \sum_1^m |\Delta a(n+j)| \leq a(n+m) + Kf(1-1/n)/n$. Hence $a(n) - Kf(1-1/n)/n \leq a(n+m)$; this is true for all m and $a(n+m) \to 0$ (by (b)), so $a(n) \leq Kf(1-1/n)/n$.

(ii) Obviously (a) implies that $\sum |\Delta a(n)|$ converges, so that $(1-t)f = \sum \Delta a(n)t^n$ has absolutely summable coefficients, and thus is continuous on the closed unit disk. Hence f is continuous on $\overline{D} \setminus \{1\}$.

(iii) Let $k = \lfloor 1/\theta \rfloor$. Then $\sum_{n>k} |\Delta a(n)| \leq Kf(1-1/k)/k \leq K\theta f(1-\theta)$. Now

$$(1-t)f = (1-t)\sum_{n=0}^{k} a(n)t^n + \sum_{n>k} \Delta a(n)t^n + a(k)t^k.$$

So

$$\begin{aligned} |(1-t)f|_{t=e^{i\theta}} &\leq |1-e^{i\theta}| \sum_{n=0}^{k} a(n) + \sum_{n>k} |\Delta a(n)| + |a(k)| \\ &\leq \theta\Big(ef\Big(1-\frac{1}{k}\Big) + Kf\Big(1-\frac{1}{k}\Big) + Kf\Big(1-\frac{1}{k}\Big)\Big) \\ &\leq K'\theta f(1-\theta). \end{aligned}$$

As $|1-e^{i\theta}| = 2\sin\theta/2 = \theta + \boldsymbol{O}(\theta^3)$, we deduce $|f(e^{i\theta})| = \boldsymbol{O}(f(1-\theta))$. ■

The following is elementary, but very useful.

Lemma 8.4 *Suppose that $A(n)$ is a sequence of positive numbers and there exists $\eta > 0$ such that $A(n) - A(n+1) \geq (1+\eta)A(n+1)/n$. Then there is $C(N)\downarrow 1$*

$$\sum_{n>N} A(n) < \frac{NA(N)C(N)}{\eta}.$$

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Proof We have $A(n+1) \leq A(n)(1-(1+\eta)/(n+1))$, so $A(N+1+k) \leq \prod_{i=1}^{k}(1-(1+\eta)/(N+1+i))$. Taking logarithms, and with $C(N)$ approximately $\exp((1+\eta)^2 \sum 1/2(N+1+i)^2)$, we deduce $\prod_{i=1}^{k}(1-(1+\eta)/(N+1+i)) \leq C(N)((N+1)/(N+1+k))^{1+\eta}$. Therefore,

$$\begin{aligned}\sum_{n>N} A(n) &\leq C(N)A(N+1)\sum_{k=1}^{\infty}\Big(\frac{N+1}{N+k+1}\Big)^{1+\eta}\\ &\leq C(N)A(N+1)(N+1)^{1+\eta}\int_N^{\infty}\frac{dx}{x^{1+\eta}}\\ &= \frac{C(N)A(N+1)(N+1)^{1+\eta}N^{-\eta}}{\eta} \leq \frac{NA(N)C'(N)}{\eta},\end{aligned}$$

for $C'(N)$ slightly larger than $C(N)$. ∎

Corollary 8.5 *Suppose $f = \sum a(n)t^n$ satisfies $\rho(f) < 1$. Then*

$$\sum_{n>N}\frac{f(1-\frac{1}{n})}{n^2} = \boldsymbol{O}\Big(\frac{f\big(1-\frac{1}{N}\big)}{N}\Big).$$

Proof For $\epsilon < 1-\rho(f)$, for all sufficiently large n, $f'(1-1/n) \leq (\rho(f)+\epsilon)\, nf(1-1/n)$. Set $A(n) = f(1-1/n)/n^2$. Then for all sufficiently large n,

$$\begin{aligned}&A(n)-A(n+1)\\ &\quad= \frac{(2n+1)f(1-1/n) - n^2\big(f(1-1/(n+1)-f(1-1/n)\big)}{n^2(n+1)^2}\\ &\quad= \frac{(2n+1)f(1-1/n) - \frac{n^2}{n(n+1)}f'(1-1/n')}{n^2(n+1)^2}\\ &\quad\geq \frac{(2n+1)f(1-1/n) - \frac{n}{n+1}f'(1-1/(n+1))}{n^2(n+1)^2}\\ &\quad\geq \frac{(2n+1)f(1-1/n) - \frac{n}{n+1}(n+1)(\rho(f)+\epsilon)f(1-1/(n+1)}{n^2(n+1)^2}\\ &\quad= \frac{(2n+1-n(\rho(f)+\epsilon))f(1-1/n) - n(f(1-1/n)-f(1-1/(n+1))}{n^2(n+1)^2}\\ &\quad\geq (1+\eta)\frac{f(1-1/n)}{n^2(n+1)} + \boldsymbol{o}\Big(\frac{f(1-1/n)}{n^2(n+1)}\Big)\end{aligned}$$

The n' appearing in the third line of the display is a real number between n and $n+1$ arising from the mean value theorem, and η (bottom line) is positive and slightly smaller than $1-\rho(f)-\epsilon$ which exceeds zero. This yields $A(n)-A(n+1) \geq (1+\eta')A(n)/n > (1+\eta)A(n+1)/n$, so Lemma 8.4 applies. ∎

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Lemma 8.6 *Suppose that $\rho(f) < 1$ and $|\Delta a(n)| = \boldsymbol{O}(a(n)/n)$. Then f is VRT, f is continuous on $\overline{D} \setminus \{1\}$, and $|f(e^{i\theta})| = \boldsymbol{O}(f(1-\theta))$ as $\theta\downarrow 0$.*

Proof The condition on Δa is sufficient for f to be VRT, by Proposition 2.6. Now VRT and $\rho(f) < \infty$ together imply (more than) the existence of $K > 0$ such that $a(n) \leq K f(1-1/n)/n$. Therefore

$$\sum_{n>N} |\Delta a(n)| \leq K \sum_{n>N} \frac{f\left(1-\frac{1}{n}\right)}{n^2} \leq K'' \frac{f\left(1-\frac{1}{N}\right)}{N}.$$

Finally, VRT and $\rho(f) < 1$ entail that $a(n) \to 0$ (*e.g.*, $f(1-1/n)/n \to 0$). The result follows from Lemma 8.3. ■

Corollary 8.7 *Suppose that $F = \sum_{n=1}^{\infty} c(n)t^n$ where $c(n) \geq 0$, $\sum c(n) = 1$, and the following properties hold:*

(i) *F' is momentous with $1 > \alpha \equiv \alpha(F') > 0$;*
(ii) *$|\Delta c(n)| = \boldsymbol{O}(c(n)/n)$;*

Then $|F'(e^{i\theta})| \leq \boldsymbol{O}(F'(1-\theta))$ (as $\theta\downarrow 0$).

Proof Set $f = F'/c(1)$. We claim that $|\Delta(n)| = \boldsymbol{O}(a(n)/n)$. We observe that $a(n) = (n+1)c(n+1)$, so that

$$\begin{aligned} |\Delta a(n)| &= |(n+1)c(n+1) - nc(n)| \\ &= |(n+1)\Delta c(n+1) + c(n)| \\ &\leq \boldsymbol{O}(c(n)) \\ &= \boldsymbol{O}\Big(\frac{a(n)}{n}\Big). \end{aligned}$$

By the previous result, $|F'(e^{i\theta})| \leq \boldsymbol{O}(F'(1-\theta))$. ■

Quite a bit easier is an upper bound for $|(1-F)^{-1}(e^{i\theta})|$ (we already know that $1-F$ is nonzero on the closed unit disk, except at 1, so the expression makes sense). Lower bounds for these functions are also available, but never required.

Lemma 8.8 *Suppose that $F = \sum c(n)t^n$ with $c(n) \geq 0$, $\sum c(n) = 1$, and F' is weakly momentous with $\alpha(F') > 0$. Then $|(1-F)^{-1}(e^{i\theta})| = \boldsymbol{O}((1-F)^{-1}(1-\theta))$.*

Proof We observe that $\mathrm{Re}(1-F)(e^{i\theta}) = 1 - \sum c(n)\cos n\theta$. Since $\sum c(n) = 1$, the expression simplifies to $\sum c(n)(1-\cos n\theta) = 2\sum c(n)\sin^2(n\theta/2)$. For values of n in the interval $(\pi/2\theta, 3\pi/2\theta)$, $\sin^2(n\theta/2)$ is at least one-half. Hence $\mathrm{Re}(1-F)(e^{i\theta}) \geq \sum_{\pi/2\theta}^{3\pi/2\theta} c(n)/2$.

Set $K = (1-t)^{-1}(1-F)$. This is weakly momentous (as easily follows from F' being weakly momentous), and its coefficients are decreasing; hence K is LLT. Moreover, $\alpha(K) = \alpha(F')$. Let $u = [\pi/2\theta + 1]$ and $v = [3\pi/2\theta]$, so that $\sum_{\pi/2\theta}^{3\pi/2\theta} c(n) \geq \sum_u^v c(n)$, and the latter is just $(K, t^u) - (K, t^v)$. Since $v \sim 3u$, $(K, t^v)/(K, t^u) \to 3^{-\alpha(K)}$ as $\theta \to 0$. Hence $(K, t^u) - (K, t^v) \sim (K, t^u)(1 - 3^{-\alpha(K)})$.

Finally, since K is LLT and $\alpha(K) > 0$, $(K, t^u) \sim c' K(1 - 1/u)/u$, and the latter is $c'(1 - F)(1 - 1/u)$, which is asymptotically $c'(1 - F)(1 - 2\theta/\pi) \sim c''(1 - F)(1 - \theta)$. Thus $|(1-F)(e^{i\theta})| \geq \mathrm{Re}(1-F)(e^{i\theta}) \geq c''(1-F)(1-\theta)$. This yields $|(1-F)^{-1}(e^{i\theta})| = \boldsymbol{O}((1 - F)^{-1}(1 - \theta))$. ■

Proposition 8.9 *Suppose that $\rho(f) < 1$, f and f' extend continuously to $\overline{D} \setminus \{1\}$, and additionally*

$$|f(e^{i\theta})| = \boldsymbol{O}(f(1 - \theta)) \quad \textit{and} \quad |f'(e^{i\theta})| = \boldsymbol{O}(f'(1 - \theta)) \quad (\theta \downarrow 0).$$

Then $(f, t^N) = \boldsymbol{O}(f(1 - 1/N)/N)$.

Proof By Lemma 8.2, $|f(1 - \theta)| = \boldsymbol{o}(\theta^{-1+\eta})$ for some $\eta > 0$, and it follows from this and the hypothesis that f belongs to $\mathrm{H}^1(D)$. Hence $(f, t^N)\pi = \int_0^\pi f(e^{i\theta})e^{-Ni\theta}\, d\theta$ (that is, the contour can be taken over the unit circle with 1 deleted). We chop the integral into pieces. For each $k = 0, 1, \dots \lceil N/2\pi \rceil - 1$, define

$$I_k = \int_{2\pi k/N}^{2\pi(k+1)/N} f(e^{i\theta})e^{-Ni\theta}\, d\theta.$$

Let K be a constant larger than those of the hypothesized big Oh terms, and let r be any number exceeding $\rho(f)$ but less than 1. For all sufficiently small θ, $f'(1-\theta) \leq r\theta f(1 - \theta)$.

First, we bound I_0.

$$|I_0| \leq \limsup_{\epsilon \downarrow 0} \int_\epsilon^{2\pi/N} |f(e^{i\theta})|\, d\theta \leq K \limsup_{\epsilon \downarrow 0} \int_\epsilon^{2\pi/N} f(1 - \theta)\, d\theta.$$

Evaluate the latter integral by parts, setting $u = f(1 - \theta)$ and $dv = d\theta$. We obtain

$$\begin{aligned} \int_\epsilon^{2\pi/N} f(1 - \theta)\, d\theta &= \theta f(1 - \theta)\Big|_\epsilon^{2\pi/N} + \int_\epsilon^{2\pi/N} \theta f'(1 - \theta)\, d\theta \\ &\leq \frac{2\pi f\left(1 - \frac{2\pi}{N}\right)}{N} - \epsilon f(1 - \epsilon) + r \int_\epsilon^{2\pi/N} f(1 - \theta)\, d\theta. \end{aligned}$$

Therefore,

$$(1 - r) \int_\epsilon^{2\pi/N} f(1 - \theta)\, d\theta \leq \frac{2\pi f\left(1 - \frac{2\pi}{N}\right)}{N},$$

so

$$\int_\epsilon^{2\pi/N} f(1 - \theta)\, d\theta \leq \frac{2\pi f\left(1 - \frac{2\pi}{N}\right)}{(1 - r)N},$$

whence,

$$|I_0| \le K_0 \frac{f\left(1 - \frac{2\pi}{N}\right)}{N}.$$

Next, we bound the sum of a large number of I_k. Set $\theta_k = 2\pi(k + \frac{1}{2})/N$ (the midpoints of the intervals over which the integration is taking place). Since

$$\int_{2\pi k/N}^{2\pi(k+1)/N} e^{-Ni\theta}\, d\theta = 0,$$

we have that for all $1 \le k \le \lceil N/2\pi \rceil - 1$,

$$\begin{aligned}
|I_k| &\le \Big| \int_{2\pi k/N}^{2\pi(k+1)/N} \big(f(e^{i\theta}) - f(e^{i\theta_k}) \big)\, e^{-Ni\theta}\, d\theta \Big| \\
&\le \frac{2\pi}{N} \max_{\theta \in [2\pi k/N, 2\pi(k+1)/N]} |f(e^{i\theta} - f(e^{i\theta_k}))| \\
&\le \max_{\theta \in [2\pi k/N, 2\pi(k+1)/N]} |f'(e^{i\theta})| \cdot \frac{2\pi^2}{N^2} \\
&\le K \frac{f'(1 - 2\pi k/N)}{N^2}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\sum_{1}^{\lfloor N/2\pi \rfloor} |I_k| &\le \frac{K}{N^2} \sum_{1}^{\lfloor N/2\pi \rfloor} f'(1 - 2\pi k/N) \\
&\le \frac{K}{N^2} \Big(\int_1^{N/2\pi} f'\Big(1 - \frac{2\pi x}{N}\Big)\, dx + f'\Big(1 - \frac{2\pi}{N}\Big) \Big) \\
&\le \frac{K}{N^2} \Big(-\frac{N}{2\pi} \int_{1-2\pi/N}^{0} f'(u)\, du + f'\Big(1 - \frac{2\pi}{N}\Big) \Big) \\
&\le \frac{K'}{N} f\Big(1 - \frac{2\pi}{N}\Big).
\end{aligned}$$

The remaining bit is easily incorporated into this estimate, so $\sum |I_k| = \boldsymbol{O}(f(1 - 2\pi/N)/N)$, and since $\delta_{2\pi}(f) < 1$, this is $\boldsymbol{O}(f(1 - 1/N)/N)$. ■

Now we show if F' has decreasing coefficients and is momentous, then $(1 - F)^{-1}$ satisfies similar boundary properties. The argument exploits the bijection between the set of $F = \sum c(n)t^n$ such that $c(n) \ge 0$, $\sum c(n) = 1$ and $f = \sum a(n)t^n$ such that $a(n) \ge a(n+1) \ge 0$ with $a(0) = 1$. The bijection is given by $F \mapsto (1 - t)^{-1}(1 - F)$ and $f \mapsto 1 - (1 - t)f$.

Lemma 8.10 *Suppose $G = \sum c(n)t^n$ where $c(n) \ge 0$ and $\sum c(n) = 1$; also suppose that G' is weakly momentous. Define $f = (1 - t)^{-1}(1 - G)$. Then*

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(i) $|(1-G)(e^{i\theta})| = \boldsymbol{O}((1-G)(1-\theta))$ *for* $\theta\downarrow 0$;
(ii) $|f(e^{i\theta})| = \boldsymbol{O}(f(1-\theta))$ *for* $\theta\downarrow 0$.

Proof We have that $f = \sum t^n \sum_{j>n} c(j)$. In particular, the coefficients of f are nonnegative, decreasing to zero, and the constant term is 1. Set $a(n) = \sum_{j>n} c(j)$.

Since G' is weakly momentous, it follows that $(1-G)^{-1}$ is weakly momentous, and it also follows that f is weakly momentous. Since the coefficients of f are decreasing, f is LLT (this is Feller's result, Proposition 2.10), and in particular, $a(N) \sim cf'(1-1/N)/N^2$ for some nonzero constant c.

Select (small) $\theta > 0$ and set $N = [1/\theta]$. In particular, $|N\theta - 1| \leq \theta/2$. We have the following.

$$\Big|\sum_{n>N} c(n)(1-e^{in\theta})\Big| \leq 2\sum_{n>N} c(n) = 2a(N);$$

$$\begin{aligned}\Big|\sum_{n\leq N} c(n)(1-e^{in\theta})\Big| &= |1-e^{i\theta}|\cdot\Big|\sum_1^N c(n)\sum_{j\leq N-1} e^{ij\theta}\Big|\\ &\leq 2\sin\theta/2\cdot\sum_1^N nc(n)\\ &\leq 3e\theta G'(1-1/N) \leq 4e\theta G'(1-\theta),\end{aligned}$$

if θ is sufficiently small.

Now $G'(t)\cdot(1-G)^{-1}(t) = \boldsymbol{O}((1-t)^{-1})$, so $G'(1-\theta) = \boldsymbol{O}(\theta^{-1}(1-G)(1-\theta))$ and thus $\theta G'(1-\theta) = \boldsymbol{O}((1-G)(1-\theta))$. Next, $a(N) = \boldsymbol{O}(f'(1-1/N)/N^2) = \boldsymbol{O}(f'(1-\theta)\theta^2)$. Since f is momentous, $f'(1-\theta) = \boldsymbol{O}(f(1-\theta)/\theta)$ (little oh if $\alpha(f)=0$). Hence $a(N) = \boldsymbol{O}(\theta f(1-\theta)) = \boldsymbol{O}((1-G)(1-\theta))$.

Combining these, we obtain

$$\begin{aligned}\big|(1-G)(e^{i\theta})\big| &= \big|1-\sum c(n)e^{in\theta}\big| = \Big|\sum c(n)(1-e^{in\theta})\Big|\\ &\leq \Big|\sum_{n\leq N} c(n)(1-e^{in\theta})\Big| + \Big|\sum_{n>N} c(n)(1-e^{in\theta})\Big|\\ &= \boldsymbol{O}((1-G)(1-\theta)) + \boldsymbol{O}((1-G)(1-\theta)) = \boldsymbol{O}((1-G)(1-\theta)).\end{aligned}$$

This yields (i), and (ii) is an obvious consequence. ■

Corollary 8.11 *Suppose that f has decreasing, nonnegative coefficients and is weakly momentous with $\alpha(f) < 1$. Then $|f(e^{i\theta})| = \boldsymbol{O}(f(1-\theta))$ as $\theta\downarrow 0$. In particular, if $F = \sum c(n)t^n$ where $c(n) \geq 0$, $\sum c(n) = 1$, $(nc(n))$ is decreasing, and F' is weakly momentous with $\alpha(F') < 1$, then $|F'(e^{i\theta})| = \boldsymbol{O}(F'(1-\theta))$.*

Proof Since $\alpha(f) < 1$ and the coefficients are monotone, $(f, t^n) \to 0$. Hence $(1-t)f(t) \to 0$ as $t\uparrow 1$. Set $G = 1-(1-t)f$, and it is easy to check that the hypotheses of Lemma 8.10 apply. ■

9 Absolute Summability

Some of the conditions discussed in Section 8 are enough to show that the coefficients of $(1-t)(1-F)^{-1}$ are absolutely summable (see [D, p. 49]).

Proposition 9.1 *Suppose that $F = \sum c(n)t^n$ satisfies the following for some $\beta > 0$.*

(i) *F' is continous on $\overline{D} \setminus \{1\}$ and $|F'(e^{i\theta})| = \boldsymbol{O}(F'(1-\theta))$ (as $\theta\downarrow 0$);*
(ii) *The map $\theta \mapsto \theta^{\beta-1}(1-F)^{-1}(1-\theta)$ belongs to $\mathrm{L}^1(0,\frac{1}{2})$.*

Then $(1-t)^{\beta}(1-F)^{-1}$ has absolutely summable coefficients.

Proof Set $g = (1-t)^{\beta}(1-F)^{-1}$. Sufficient for g to have absolutely summable coefficients is that g' belong to $\mathrm{L}^1(\mathbf{T})$ [D, p. 49]. From Lemma 8.8, $|(1-F)^{-1}(e^{i\theta})| = \boldsymbol{O}((1-F)^{1-\theta})$. We calculate $g' = (1-t)^{\beta-1}(1-F)^{-1}((1-t)F'(1-F)^{-1}-\beta)$ and thus

$$\begin{aligned}|g'(e^{i\theta})| &\leq \boldsymbol{O}(|1-e^{i\theta}|^{\beta-1}|(1-F)^{-1}(e^{i\theta})|(\theta|1-ei\theta|^{-1}+\beta))\\ &\leq \boldsymbol{O}(\theta^{\beta-1}(1-F)^{-1}(1-\theta)).\end{aligned}$$

■

Via Lemma 8.2, we obtain the two corollaries below; more are possible, *e.g.*, in the context of Lemma 8.11. It can be improved somewhat, replacing $(1-t)^{-\beta}$ by suitable LLT functions.

Corollary 9.2 *If the coefficients of F satisfy $|\Delta c(n)| = \boldsymbol{O}(c(n)/n)$ and $0 < \delta_k(F') < 1$, then $(1-t)(1-F)^{-1}$ has absolutely summable coefficients.*

It is easy to see that this can be improved to absolute summability of the coefficients of $(1-t)^s(1-F)^{-1}$ for suitable values of $s < 1$.

Corollary 9.3 *Suppose that $\delta_k(F') < 1$, there exists $\eta > 0$ such that $(\ln n)^{1+\eta} = \boldsymbol{O}(n^2c(n))$, and $|F'(e^{i\theta})| = \boldsymbol{O}(F'(1-\theta))$. Then $(1-t)(1-F)^{-1}$ has absolutely summable coefficients.*

10 Results on LLT

We have a relatively easy, almost definitive, characterization of LLT for $(1-F)^{-1}$. The following lemma is completely elementary.

Lemma 10.1 *Suppose that (Y_n) is an unbounded increasing sequence of nonnegative real numbers, and suppose that (Z_n) is a sequence of real numbers with $|Z_n| = \boldsymbol{o}(Y_n)$. Then there exists an increasing sequence of nonnegative real numbers, (X_n) such that $|Z_n| \leq X_n$ and $X_n = \boldsymbol{o}(Y_n)$.*

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Proof Define $X_1 = |Z_1|$ and inductively,

$$X_{n+1} = \begin{cases} X_n & \text{if } |X_n| > |Z_{n+1}|, \\ |Z_{n+1}| & \text{if } |X_n| \le |Z_{n+1}|. \end{cases}$$

Alternatively, $X_n = \max\{|Z_i| \mid i = 1, 2, \dots, n\}$ leads to the same definition. Obviously (X_n) is increasing and $X_n \ge |Z_n|$ for all n. Suppose there exist $n(i)$ with $n(i) < n(i+1)$ and $\delta > 0$ such that $X_{n(i)}/Y_{n(i)} > \delta$ for all i. If $(X_{n(i)})$ were bounded above, we would obtain an immediate contradiction from $Y_n \uparrow \infty$. Hence, by deleting suitable terms, we may assume that $X_{n(i)} < X_{n(i+1)}$ for all i. For each i, let $m(i)$ be the smallest index such that $|Z_{m(i)}| = \max\{|Z_j| \mid j = 1, 2, \dots, n(i)\}$. In particular, $m(i) \to \infty$, so that $Z_{m(i)}/Y_{m(i)} \to 0$. Thus

$$\frac{X_{n(i)}}{Y_{n(i)}} = \frac{|Z_{m(i)}|}{Y_{n(i)}} \le \frac{|Z_{m(i)}|}{Y_{m(i)}} \to 0. \qquad \blacksquare$$

The following is a type of perturbation result.

Proposition 10.2 *Suppose that f is LLT and $\alpha \equiv \alpha(f) > 1$.*

(i) *There exist LLT g and a function h such that almost all coefficients of g are increasing, $f = g + h$, and $|(h, t^n)| = \boldsymbol{o}((g, t^n))$.*
(ii) *If p is weakly momentous, then fp is LLT and $(fp, t^n) \sim (gp, t^n)$.*

Proof Define $g = \sum_1^\infty t^n f(1 - 1/n)/n\Gamma(\alpha)$; then g is LLT, and $|(f - g, t^n)| = \boldsymbol{o}((f, t^n)) = \boldsymbol{o}((g, t^n))$. Set $h = f - g$. From $\alpha > 1$, the mean value theorem, and the fact that f is momentous, it follows that for all sufficiently large n, $(g, t^n) < (g, t^{n+1})$. This yields (i).

(ii) Suppose the coefficients of g are increasing for all $n \ge n_0$. Since $\alpha(g) > 1$, the coefficients must be unbounded (above). By absorbing the first n_0 terms into h, we may assume that the coefficients of g are increasing. Thus we can write $g = (1-t)^{-1}G$ where G has only nonnegative coefficients. Necessarily, G is weakly momentous (note that $G(1) = \infty$ follows from $\alpha(g) = \alpha > 1$) and thus $\alpha(G)$ is defined, and therefore $\alpha(G) = \alpha(g) - 1 > 0$. Hence G is momentous.

Set $Y_n = (g, t^n)$ and $Z_n = (h, t^n)$. By Lemma 10.1, there exists an increasing sequence of nonnegative numbers X_n such that $|Z_n| \le X_n = \boldsymbol{o}((g, t^n))$. Hence if q is defined as the power series $\sum X_n t^n$, then q has radius of convergence at least that of g, which is 1. Since the coefficients of q are increasing, $(1 - t)q$ has nonnegative coefficients, so we can write $q = (1 - t)^{-1}Q$ where Q has nonnegative coefficients and has radius of convergence (at least) 1.

Since Qp has only nonnegative coefficients, $\sum_0^N (Qp, t^n) \le e^{1+1/N} Qp(1 - 1/N)$ (evaluate the power series for Qp at $1 - 1/N$ and simply observe that $(1 - 1/N)^j \ge e^{-1-1/N}$ for all $j \le N$; this is a result about series with no negative coefficients, and does not require the momentous property—an important point is that it is not clear whether X_n can be chosen so that q is momentous). Thus

$$(qp, t^N) = \sum_0^N (Qp, t^n) \le e^{1+1/N} Qp(1 - 1/N) = e^{1+1/N} \frac{qp(1 - 1/N)}{N}.$$

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Hence for all sufficiently large N, $|(hp, t^N)| \leq (qp, t^N) \leq 3qp(1 - 1/N)/N$.

On the other hand, $gp = (1 - t)^{-1}Gp$ has increasing coefficients and is weakly momentous; hence gp is LLT, and since $\beta \equiv \alpha(gp) > 0$,

$$(gp, t^N) \sim \frac{gp(1 - 1/N)}{N\Gamma(\beta)}.$$

Since $X_n = \boldsymbol{o}((g, t^n))$, it follows that $q(t) = \boldsymbol{o}(g(t))$ (as $t \uparrow 1$) and thus

$$\begin{aligned} |(hp, t^N)| &\leq \frac{3qp(1 - 1/N)}{N} \\ &= \boldsymbol{o}(\frac{gp(1 - 1/N)}{N}) = \boldsymbol{o}((gp, t^N)). \end{aligned}$$

Therefore $(fp, t^N) = (gp, t^N) + (hp, t^N) \sim (gp, t^N)$. Since gp is LLT, so is fp. ■

In the following, one of the boundary cases, $\alpha(F') = 1$, is permitted. However, the proof fails if $\alpha(F') = 0$.

Corollary 10.3 *Suppose that $F = \sum c(n)t^n$ where $c(n) \geq 0$, $\sum c(n) = 1$, F' is LLT, and $\alpha \equiv \alpha(F') > 0$. Then $(1 - F)^{-1}$ is LLT; in particular, if $0 < \alpha < 1$, then $((1 - F)^{-1}, t^n) \sim (1 - \alpha)(\sin \pi\alpha)/(\pi n^2 c(n))$.*

Proof Let k be any integer exceeding $1/\alpha$, and form $((1 - F)^{-1})^{(k)}$, that is, the k-th derivative of $(1-F)^{-1}$. When we expand this in terms of products of derivatives of F' with powers of $(1 - F)^{-1}$, we find that this is a positive integer combination of terms of the form $\prod(F^{(j(i))})^{l(i)}$ multiplied with a power of $(1 - F)^{-1}$, where $k = \sum j(i)l(i)$ ($j(i) > 0$). It is easy to see that the α value of each term $\prod(F^{(j(i))})^{l(i)}$ is at least $k\alpha > 1$ (the worst case, that is, the term giving the least alpha value, arises when $j(1) = 1$ and $l(1) = k$; all others have an F'' or higher derivative, so their α value exceeds 1 as well). By Proposition 10.2, with f being the product of derivatives and p the corresponding power of $(1 - F)^{-1}$, each term in the expansion is thus LLT. A sum of LLT functions is LLT, so $((1 - F)^{-1})^{(k)}$ is LLT, and thus, so is $(1 - F)^{-1}$. The final estimate is a consequence of Lemma 3.5. ■

Now we want to show the analogue of Corollary 10.3 under appropriate assumptions involving ρ and σ, rather than assuming F' is momentous (which is implicit in Corollary 10.3).

Lemma 10.4 *Suppose f is robust and satisfies $\sigma(f) > 0$ and $\rho(f) < \infty$. Then $(f, t^n) \approx f(1 - 1/n)/n$.*

Proof By Proposition 7.1 and the comment at the beginning of Section 7, $f(1 - 1/N) \approx \sum_0^N a(N)$. With the argument paralleling that at the beginning of that section, $f(1 - 1/N) < (\delta^2(f) + \epsilon)f(1 - 1/2N)$ for all sufficiently large N. It follows that $\sum_0^N a(n) \approx \sum_{N/2}^N a(n)$, so that $f(1-1/N) \approx \sum_{N/2}^N a(n)$. As f is robust, there exists $1 > s > 0$ such that $s < a(n)/a(N) < 1/s$ for all n with $N/2 \leq n \leq N$. It follows easily that $f(1 - 1/N)/N \approx a(N)$. ■

All of the following are routine consequences of the definitions (for which, see Sections 6 and 7).

Lemma 10.5 *Suppose that f, g, and h have only nonnegative coefficients, and radii of convergence at least* 1.

(i) *If $\rho(f), \rho(g) < \infty$, then $\rho(fg)$ is finite and bounded above by $\rho(f) + \rho(g)$, with equality if f or g is momentous.*
(ii) *$\sigma(fg) \geq \sigma(f) + \sigma(g)$, with equality if f or g is momentous.*
(iii) *If $\rho((1-t)^{-1}h) < \infty$, then either $\rho(h)$ is finite and equals $\rho((1-t)^{-1}h) - 1$, or the coefficients of h are absolutely summable.*

Lemma 10.6 *If f is robust, $\sigma(f) > 1$, and $\rho(f) < \infty$, then there exists g having only nonnegative and increasing coefficients such that $(f, t^n) \approx (g, t^n)$.*

Proof Since $\sigma(f) > 1$, it easily follows that $(1-t)f\uparrow\infty$ for t sufficiently close to 1. Hence $(f(1-1/n)/n)_{n \geq n_0}$ is increasing for sufficiently large n_0. Thus $\sum t^n f(1-1/n)/n$ has almost all its coefficients increasing, and by altering a finite number of coefficients, we can define g with coefficients positive and increasing, for which $(g, t^n) \sim f(1-1/n)/n$. By Lemma 10.4, $(f, t^n) \approx (g, t^n)$. ■

Proposition 10.7 *Suppose that f is robust, $\sigma(f) > 1$, and $\rho(f) < \infty$. If $\rho(p) < \infty$, then fp is robust.*

Proof Find g as in Lemma 10.6. Then gp has increasing coefficients and $\rho(fp) = \rho(gp) \leq \rho(g) + \rho(p) < \infty$. We may write $gp = (1-t)^{-1}h$ where h has only nonnegative coefficients. By Lemma 10.5, $\rho(h) < \infty$. Also, $\sigma((1-t)^{-1}) + \sigma(h) = \sigma(gp) > 1$, so $\sigma(h) > 0$.

Now $(gp, t^N) = \sum_0^N (h, t^n) \leq 3h(1-1/N) = 3gp(1-1/N)/N$ (for $N > 5$). By Lemma 3.5, $h(1-1/N) = \boldsymbol{O}(\sum_0^N (h, t^n)) = \boldsymbol{O}((gp, t^N))$. Hence $(gp, t^N) \approx gp(1-1/N)/N$ and it easily follows that gp is robust.

Finally, $(f, t^n) \approx (g, t^n)$ implies $(fp, t^n) \approx (gp, t^n)$, so fp is thus also robust. ■

Corollary 10.8 *Suppose that $F = \sum c(n)t^n$ where $c(n) \geq 0$, $\sum c(n) = 1$, $\sigma(F') > 0$, and $\rho(F') < \infty$.*

(i) *If F' is robust, then so is $(1-F)^{-1}$.*
(ii) *If F' satisfies VRT, then so does $(1-F)^{-1}$.*

Proof By Lemma 10.5(ii), there exists k such that $\sigma((F')^k) > 1$, and $\sigma(F'') > 1$ in any case, and the same is true for all higher derivatives. As in the proof of Corollary 10.3, we can apply Proposition 10.7 to each of the products appearing in the decomposition of $((1-F)^{-1})^{(k)}$, so that the latter is a sum of robust functions, hence is robust itself. Therefore, $(1-F)^{-1}$ is robust. This yields (i). For (ii), Lemma 6.5 entails that $(1-F)^{-1}$ is robust, and thus for both (ii) and (iii), $(1-F)^{-2}$ is robust. Therefore $((1-F)^{-1})' = F' \cdot (1-F)^{-2}$ satisfies VRT, by Proposition 4.11; so $(1-F)^{-1}$ satisfies VRT. ■

11 Sufficient Conditions for Imitation Momentous

Here, $F = \sum c(n)t^n$ with $c(n) \geq 0$, $\sum c(n) = 1$ and $F'(1) = \infty$. We investigate under what circumstances $0 < \rho(F'), \rho((1-F)^{-1}) < \infty$ and related properties; the conditions are in terms of the $c(n)$.

Lemma 11.1 *For F as defined here, we have the following.*

(i) *For all N, $((1-t)^{-1}tF', t^N) \leq ((1-t)^{-2}(1-F), t^N)$ and for all t in $(0,1)$, $tF' \cdot (1-F)^{-1}(t) \leq (1-t)^{-1}$.*
(ii) *Sufficient for $((1-t)^{-1}(1-F), t^N) = \boldsymbol{O}((tF', t^N))$ is $\sum_{n>N} c(n) = \boldsymbol{O}(Nc(N))$.*
(iii) *Sufficient for $(1-t)^{-1} = \boldsymbol{O}(F' \cdot (1-F)^{-1}(t))$ is $N \sum_{n>N} c(n) = \boldsymbol{O}(\sum_{n \leq N} nc(n))$.*

Proof (i) We have $((1-t)^{-1}tF', t^N) = \sum_{n \leq N} nc(n)$ and $((1-t)^{-1}(1-F), t^N) = \sum_{n>N} c(n)$. Therefore

$$\begin{aligned}(11.1) \qquad \big((1-t)^{-2}(1-F), t^N\big) &= \sum_{n=0}^{N} \sum_{j>n} c(j) \\ &= \sum_{n=1}^{N} nc(n) + (N+1) \sum_{n>N} c(n).\end{aligned}$$

This yields the first inequality, and we deduce from it that $(1-t)^{-1}tF'(t) = \boldsymbol{O}((1-t)^{-2}(1-F)(t))$, and the second inequality follows from this. The computation of the coefficients also yields (ii) and (iii). ■

In particular, we deduce without any extra conditions on the coefficients of F that $F' \cdot (1-F)^{-1}(t) = \boldsymbol{O}((1-t)^{-1})$. The other way around holds if the coefficients of F satisfy relatively modest conditions.

Lemma 11.2 *Set $h = (1-F)^{-1}$.*

(i) *$\rho(h) < \infty$.*
(ii) *$\sigma(h) > 0$ if and only $(1-t)^{-1}(1-F) = \boldsymbol{O}(F')$.*

Proof (i) By definition, $\rho(h) < \infty$ if and only if $h' = \boldsymbol{O}((1-t)^{-1}h)$, and of course, $h' = F' \cdot h^2$; but $F' \cdot h = \boldsymbol{O}((1-t)^{-1})$ from Lemma 11.1.

(ii) Similarly, $\sigma(h) > 0$ if and only if $(1-t)^{-1}h = \boldsymbol{O}(h')$, and the result follows immediately. ■

Before investigating when the corresponding results hold for F' in place of $h = (1-F)^{-1}$, we consider the conditions on the coefficients.

Lemma 11.3 *Suppose that $c(n) \geq 0$, $\sum c(n) = 1$, and $\sum nc(n) = \infty$. Suppose that $(C(n))$ is another sequence of nonnegative reals such that $c(n) \approx C(n)$. Define $h \colon \mathbf{N} \to \mathbf{R}^+$ via $h(n) = n(n+1)C(n)$.*

(i) *If $\limsup_{n\to\infty} \frac{n(h(n)-h(n-1))}{h(n)} < 1$, then $\sum_{n>N} c(n) = \boldsymbol{O}(Nc(N))$;*

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(ii) *if* $\liminf_{n\to\infty} \frac{n(h(n)-h(n-1))}{h(n)} > -\infty$ *and* $\limsup_{n\to\infty} \frac{n(h(n)-h(n-1))}{h(n)} < 1$, *then* $\sum_{n>N} c(n) \approx Nc(N)$.

Proof We reduce immediately to $c(n) = C(n)$. The discrete form of integration by parts is simply

$$\sum_{n>N} c(n) = \sum_{n>N} \frac{h(n)}{n(n+1)} = \frac{h(N)}{N} + \sum_{n>N} \frac{h(n)-h(n-1)}{n}. \tag{11.2}$$

(i) Let r be the lim sup. Given $\epsilon < 1-r$, for all sufficiently large n, $(n+1)(h(n+1) - h(n))/h(n+1) < r+\epsilon$, *i.e.*,

$$\frac{h(n+1)-h(n)}{n} < \frac{(r+\epsilon)h(n)}{n(n+1)} = (r+\epsilon)c(n).$$

Summing over $n > N$ and adding $h(N)/N$, we have $(r+\epsilon)\sum_{n>N} c(n) + h(N)/N > \sum_{n>N} c(n)$, and therefore

$$(1-r-\epsilon)\sum_{n>N} c(n) < \frac{h(N)}{N} = (N+1)c(N) = \boldsymbol{O}(Nc(N)),$$

and since $r+\epsilon < 1$, the conclusion of (i) follows.

(ii) There exists s (necessarily less than or equal to r and therefore less than 1, but s could be negative), such that for all sufficiently large n, $n(h(n)-h(n-1))/h(n) > s$, that is, $(h(n)-h(n-1))/n > sh(n)/n^2$. Putting this into (i), we obtain $h(N)/N \leq (1-s)\sum_{n>N} c(n)$, yielding the other inequality for (ii). ■

The conditions in Lemma 11.3 can be re-expressed as $\Delta h(n) < (1-\eta)h(n)/n$ (for some $\eta > 0$ and all sufficiently large n) and (in the presence of this), $|\Delta h(n)| = \boldsymbol{O}(h(n)/n)$. This contrasts somewhat with the condition of Lemma 11.1, which is $\Delta A(n) \leq -(1+\eta)A(n)/n$.

We now wish to exclude the possibility that $\delta_k(F') = 1$.

Lemma 11.4 *For* $f = \sum a(n)t^n$ *with* $a(n) \geq 0$ *and* $\sum a(n) = \infty$, *then* $\delta_k(f) = 1$ *for some* $k > 1$ *implies that* f *is weakly momentous and* $\alpha(f) = 0$.

Proof By Corollary 6.3, $\delta_k(f) = 1$ for all $k > 1$, so by definition, f is weakly momentous and $\alpha(f) = -\ln_2 1 = 0$. ■

Lemma 11.5 *Suppose that* $\delta_k(F') = 1$ *for some* $k > 1$.

(i) $(1-F)^{-1}$ *is momentous and* $\alpha((1-F)^{-1}) = 1$.

(ii) $F' \cdot (1-F)^{-1} \sim (1-t)^{-1}$.

Proof By Lemma 11.4, $\delta_k(F') = 1$ for all $k > 1$. Set $G(t) = (1-F)(t^k)$. Then $G'(t)/(1-F)'(t) = (kt^{k-1}) \cdot (F'(t^k)/F'(t))$. The second factor converges to 1, so $\lim_{t\uparrow 1} G'(t)/(1-F)'(t)$ exists and equals k. Since $(1-F)(1) = 0$, we can apply l'Hôpital's rule, and deduce $\lim_{t\uparrow 1}(1-F)(t^k)/(1-F)(t)$ exists and equals k. It easily follows that $\lim_{t\uparrow 1}(1-F)^{-1}(t^k)/(1-F)^{-1}(t)$ exists and equals $1/k$, whence $(1-F)^{-1}$ is weakly momentous. Moreover, $\alpha((1-F)^{-1}) = -\ln_2 1/2 = 1 > 0$, so $(1-F)^{-1}$ is momentous.

Since $\alpha = 1$, we have $((1-F)^{-1})' \sim (1-t)^{-1}(1-F)^{-1}$ and (ii) follows. ∎

Lemma 11.6 *If $\sum_{n\le N} nc(n) = \boldsymbol{O}(N\sum_{n>N} c(n))$, then*

$$\limsup_{t\uparrow 1} \frac{F'(t)(1-F)^{-1}(t)}{(1-t)^{-1}} < 1;$$

in particular, $\rho(F'), \delta_k(F') < 1$.

Proof We have $\sum_{n\le N} nc(n) \le KN\sum_{n>N} c(n)$ for some $K > 0$. By (11.1),

$$((1-t)^{-2}(1-F), t^N) \ge \Big(1+\frac{1}{K}\Big)\sum_{n\le N} nc(n) = \frac{K+1}{K}((1-t)^{-1}tF', t^N).$$

Hence on $(0,1)$, $(1-t)^{-2}(1-F)(t) \ge (1-t)^{-1}tF'(t)K/(K+1)$. For t sufficiently close to 1, we thus have $F'(t) \le (1-t)^{-1}(1-F)(t)/(1+1/2K)$. Therefore, $F' \cdot (1-F)^{-1}(t) \le (1-t)^{-1}/(1+1/2K)$ for such t.

By the preceding, $\delta_k(F') = 1$ entails $F' \cdot (1-F)^{-1} \sim (1-t)^{-1}$, which yields a contradiction here. ∎

A Appendix

Here we elaborate on part of the criteria in Theorem 4.6, namely conditions on a Maclaurin series f guaranteeing that f^k has all or almost all of its coefficients increasing for some integer k.

Proposition A.1 *Suppose that $P = \sum a(n)t^n$ is LLT and $\alpha(P) > 0$, and Q is a Maclaurin series with radius of convergence at least 1, such that for some $\gamma > 0$, $|(Q, t^n)| = \boldsymbol{O}((P, t^n)/n^\gamma)$ and $(P+Q, t^n) > 0$ for all n. If some power of P has increasing coefficients, then there exists a power of $P+Q$ that has increasing coefficients.*

Remark As observed in [Ha] (immediately following Proposition 4.6), the n^γ condition cannot be improved to its limiting case, *i.e.*, $\boldsymbol{O}((P, t^n)/(\ln n))$ or similar (an example is given there; consider $(1-t)^{-1} + \sum t^{2n}/(\ln(n+1))^M$ for any $M > 0$; no power has almost all its coefficients increasing).

Proof We will apply [Ha, Proposition 4.6]. By shrinking γ if necessary, we may suppose that $\gamma < \alpha$. Set $h = \sum P(1-1/n)(n^{1+\gamma}\Gamma(\alpha))^{-1}t^n$. Since $a(n) \sim$

$P(1-1/n)/n\Gamma(\alpha)$, we have that $(h,t^n) \sim a(n)/n^{\gamma}$, whence there exists $K > 0$ such that $|(Q,t^n)| \leq K(h,t^n)$, and so for all integers m, we have that $-K^m(h^m,t^n) \leq (Q^m,t^n) \leq K^m(h^m,t^n)$ for all integers m. (This relation is denoted $Q^m \prec h^m$ in [Ha].) Set $j = (1-t)^{\gamma}P$. Since $\alpha(P) > \gamma$, it easily follows from Lemma 4.4 that $h_0 := \sum j(1-1/n)t^n/n$ is LLT. Obviously $h_0 \sim \Gamma(\alpha)h$, so that h is LLT, and $\alpha(h) = \alpha(P) - \gamma > 0$. Therefore h^m is LLT for any positive integer m, and $(h^m,t^n) \sim c_m h^m(1-1/n)/n$ (where c_m is a constant depending on m). We can rewrite this as $c_m(h(1-1/n)/n)^m n^{m-1}$. Since $h(1-1/n)/n \sim c_0 P(1-1/n)/n^{1+\gamma}$, we have that $(h^m,t^n) \sim c'_m(P(1-1/n)/n)^m n^{m-1-m\gamma}$.

Now $(P^{m-1},t^n) \sim c''_m(P(1-1/n)/n)^{m-1}n^{m-2}$. Hence, if $P(1-1/n) = \boldsymbol{O}(n^{\gamma m-1})$, we will obtain $Q^m \prec P^{m-1}$. Of course, $P(1-1/n) = \boldsymbol{o}(n^{\alpha(P)+\epsilon})$ for all $\epsilon > 0$ (Lemma 8.2), so we select m such that $m\gamma - 1 > \alpha(P)$.

Now the hypotheses of [Ha, Proposition 4.6] apply, so that if f is a convergent Maclaurin series with at most polynomial growth on its coefficients, then whenever $P^k f$ has no negative coefficients, it follows that $(P+Q)^l f$ has no negative coefficients for some integer l. Setting $f = 1-t$, we obtain the desired conclusion. ∎

Proposition A.1 is a perturbation theorem. What is needed now is a more explicit criterion for some power to have its coefficients increasing. Here is a severe—but easy to verify—criterion, in terms of the coefficients. A power series f has *almost all of its coefficients increasing* if all but finitely many coefficients of $(1-t)f$ are nonnegative.

Lemma A.2 *Suppose f has only nonnegative coefficients with $f(1) = \infty$, and there exists $s > 0$ such that for (almost) all n, $((1-t)f',t^n) \geq s(f,t^n)$. Then for any integer k such that $k \geq 1/s$, the power series f^k has (almost) all coefficients increasing.*

Proof We may suppose that $f(0) \neq 0$. First, suppose the inequality holds for all $n = 0,1,2,\dots$. We calculate $((1-t)f^k)' = k(1-t)f'f^{k-1} - f^k$. Hence

$$\begin{aligned}(((1-t)f^k)',t^N) &= (f^{k-1}(k(1-t)f' - f),t^N)\\ &\geq (f^{k-1}\cdot \sum (ks-1)(f,t^n)t^n,t^N) \geq 0.\end{aligned}$$

Thus $((1-t)f^k)'$ has all of its coefficients nonnegative; of course, the constant coefficient of $(1-t)f^k$ is positive, so all coefficients of the latter are nonnegative.

Now assume that the inequality holds only for all $N \geq N_0$; we prove the corresponding result for the coefficients of f^k. Write $(1-t)f' = p + sf + q$ where p is a polynomial of degree at most N_0 and q has only nonnegative coefficients. There exists K such that $|(pf^{k-1},t^n)| \leq K(f^{k-1},t^n)$, and the latter is $\boldsymbol{o}((f^k,t^n))$. It follows that $((1-t)f'f^{k-1},t^N) \geq 0$ for all sufficiently large N. ∎

Write $f = \sum a(n)t^n$, so that the condition in Lemma A.2 boils down to $(n+1)a(n+1) - na(n) \geq sa(n)$, that is, $a(n+1)/a(n) \geq (n+s)/(n+1) = 1-(1-s)/(n+1)$. This can also be written as $a(n+1)-a(n) \geq -(1-s)a(n)/(n+1)$, related to a condition on the drop, d_f, discussed in Section 2.

There is a weaker but more complicated hypothesis that will work to yield the same conclusion—that $(f')^2$ have increasing coefficients and $((1-t)f'', t^n) \geq s(f^2, t^N)$—obtained by expanding $((1-t)f^k)''$. A sufficient condition for $(f')^2$ is obtainable form Lemma A.2, simply by replacing f by f' and setting $s = 1/2$; in terms of the coefficients of f, this boils down to $a(n+1)/a(n) \geq (n-1/2)/(n+1)$. To obtain $((1-t)f'', t^n) \geq s(f^2, t^N)$, we assume f is LLT, and only get a result with $>$ (rather than $\geq$). However, this slight improvement may also be obtainable from Proposition A.1 and Lemma A.2. No new additional sufficient conditions appear to come from expanding $((1-t)f^k)'''$.

We can combine the previous two results, and obtain a slightly more general criterion, which is a quantitative version of Proposition 2.1.

Proposition A.3 *Suppose that $f = \sum a(n)t^n$ with $a(n) \geq 0$ and $f(1) = \infty$. Suppose there exist $0 < r < 1$ and $\alpha > 0$ such that*

$$\Big| a(N) - \frac{\alpha \sum_{m=0}^{N-1} a(m)}{N} \Big| = \boldsymbol{O}(\frac{a(N)}{N^r}).$$

Then f is LLT with $\alpha(f) = \alpha$ and there exists k such that f^k has almost all of its coefficients increasing.

Proof Since $a(N) = \boldsymbol{o}(\sum_0^{n-1} a(m))$, we have that $(f', t^n) \sim \alpha \cdot ((1-t)^{-1} f, t^n)$, hence $f'(t) \sim \alpha \cdot (1-t)^{-1} f(t)$, whence f is momentous. That f is LLT is now obvious. We will write $f = g + h$ where $((1-t)g, t^N) \geq s(g, t^N)$ almost everywhere for some value of s and $|(h, t^N)| = \boldsymbol{O}((g, t^N)/N^r)$, so that Proposition A.1 and Lemma A.2 together yield that f^k has almost all coefficients increasing.

We define $g = \sum b(n)t^n$ so that $(g', t^N) = \alpha \sum_0^N a(n)$ (so $g' = \alpha \cdot (1-t)^{-1} f$); that is, $(g, t^N) = \alpha \sum_0^{N-1} a(n)/N$. So $(1-t)g' = \alpha f$. Since f is LLT, $(f, t^N) \sim \alpha \sum_0^N a(n)/N$, and since $a(N) = \boldsymbol{o}(\sum_0^N a(n))$, we have that $(f, t^N) \sim (g, t^N)$. Hence for any $s < \alpha$, for all sufficiently large N, $((1-t)g', t^N) > s(g, t^N)$. Thus $g^{\lceil \epsilon + 1/\alpha \rceil}$ has almost all of its coefficients nonnegative.

Now consider $h := f - g$ in order to apply Proposition A.1 with $g = P$ and $h = Q$. By hypothesis, $|(h, t^N)| = \boldsymbol{O}(a(N)/N^r) = \boldsymbol{O}((g, t^N)/N^r)$. ■

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