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SEMIGROUPS OF COMPOSITION OPERATORS ON LOCAL DIRICHLET SPACES

GEORGIOS STYLOGIANNIS

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Abstract

We study the strong continuity of semigroups of composition operators on local Dirichlet spaces.

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1. Introduction

1.1. The Dirichlet space $D(\mu)$. Let $\mathbb{D}$ be the open unit disk in the complex plane $\mathbb{C}$ and $\mathbb{T}$ be its boundary. For $\lambda \in \mathbb{T}$, let P_λ denote the Poisson kernel at λ:

$$P_\lambda(z) = \frac{1-|z|^2}{|\lambda - z|^2}, \quad z \in \mathbb{D}.$$

For a nonnegative finite Borel measure μ on $\mathbb{T}$, define the harmonic function

$$P_\mu(z) = \int_{\mathbb{T}} \frac{1-|z|^2}{|\lambda - z|^2}\, d\mu(\lambda).$$

Thus, when $\mu = \delta_\lambda$, the Dirac point mass at λ, we have $P_{\delta_\lambda}(z) = P_\lambda(z)$.

The Dirichlet space $D(\mu)$ consists of those analytic functions f that belong to the Hardy space H^2 such that

$$D_\mu(f) = \int_{\mathbb{D}} |f'(z)|^2 P_\mu(z)\, dA(z) < \infty,$$

where dA denotes the normalised two-dimensional Lebesgue measure. If $\mu = 0$, then it is convenient to identify $D(\mu)$ with the Hardy space H^2. The space $D(\mu)$ is a Hilbert space with norm

$$\|f\|^2_{D(\mu)} = \|f\|^2_{H^2} + D_\mu(f)$$

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and inner product

$$\langle f, g\rangle_{D(\mu)} = \langle f, g\rangle_{H^2} + \int_{\mathbb{D}} f(z)\overline{g(z)}P_\mu(z)\, dA(z), \quad f, g \in D(\mu).$$

In the special case when $\mu = \delta_\lambda$, the space $D(\mu)$ is called the local Dirichlet space $D(\delta_\lambda)$. We write $D_\lambda(f)$ instead of $D_\mu(f)$. It follows that if $D_\lambda(f)$ is finite, then the oricyclic limit of f at λ exists and is denoted by $f(\lambda)$, that is, $f(z) \to f(\lambda)$ as $z \to \lambda$ in any oricyclic approach region

$$Q_k(\lambda) = \{z \in \mathbb{D} : |z - \lambda|^2 < k(1 - |z|^2)\}, \quad k > 0.$$

Moreover, every function $f \in D(\delta_\lambda)$ can be written as $f(z) = f(\lambda) + (z - \lambda)g(z)$ for some $g \in H^2$. It can be proved that

$$D_\lambda(f) = \frac{1}{2\pi} \int_0^{2\pi} \left| \frac{f(e^{it}) - f(\lambda)}{e^{it} - \lambda} \right|^2 dt$$

and thus $D_\lambda(f) = \|g\|^2_{H^2}$. In addition, the following identity holds:

$$D_\mu(f) = \int_{\mathbb{T}} D_\lambda(f)\, d\mu(\lambda).$$

For additional information about the Dirichlet spaces $D(\mu)$, see [11] and [10].

1.2. Composition operators on $D(\mu)$. Let $\varphi : \mathbb{D} \to \mathbb{D}$ be analytic. For f analytic in the unit disk $\mathbb{D}$, the composition operator C_φ is given by

$$C_\varphi(f)(z) = f(\varphi(z)), \quad z \in \mathbb{D}.$$

By the subordination principle, composition operators act continuously on the Hardy spaces H^p for $1 \le p < \infty$ [7]. Recently, Sarason and Silva [12] studied composition operators on the Dirichlet space $D(\mu)$. They used counting functions along with Bergman embedding theorems to describe when C_φ is bounded or compact on $D(\mu)$. Among other things, they showed that if C_φ is bounded on $D(\delta_\lambda)$, then $\varphi(\lambda)$ exists in the oricyclic sense and it is either in $\mathbb{D}$ or it is equal to λ. In the latter case, that is, $\varphi(\lambda) = \lambda$, the operator C_φ is bounded on $D(\delta_\lambda)$ if and only if the angular derivative $\varphi'(\lambda)$ exists [12, Theorem 2].

1.3. Semigroups of composition operators. A semigroup of analytic functions is a family $\{\varphi_t : t \ge 0\}$ of analytic self maps of the unit disk $\mathbb{D}$ satisfying the following conditions:

(1) φ_0 is the identity map of $\mathbb{D}$;
(2) $\varphi_s \circ \varphi_t = \varphi_{s+t}$ for $s, t \ge 0$;
(3) the map $(t, z) \to \varphi_t(z)$ is jointly continuous on $[0, \infty) \times \mathbb{D}$.

For each semigroup the following basic properties hold [1]. The limit

$$\lim_{t \to 0^+} \frac{\partial \varphi_t(z)}{\partial t} = G(z)$$

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exists uniformly on compact subsets of $\mathbb{D}$. The analytic function G satisfies the identities

$$G(\varphi_t(z)) = \frac{\partial \varphi_t(z)}{\partial t} = G(z)\frac{\partial \varphi_t(z)}{\partial z}, \quad z \in \mathbb{D},\ t \geq 0.$$

Moreover, $G(z)$ has the unique representation

$$G(z) = (\overline{b}z - 1)(z - b)F(z), \quad z \in \mathbb{D},$$

where $b \in \overline{\mathbb{D}}$ and $F(z)$ is analytic in $\mathbb{D}$ with nonnegative real part. The function G is called the infinitesimal generator of $\{\varphi_t : t \geq 0\}$. The point b is the common Denjoy–Wolff point of all the functions φ_t. This point plays an important role in the dynamical behaviour of the semigroup (see, for example, [2, 3, 5]).

Each semigroup $\{\varphi_t\}$ gives rise to a semigroup $\{C_t\}$ of composition operators

$$C_t(f)(z) = f(\varphi_t(z)), \quad f \text{ analytic in } \mathbb{D}.$$

If X is a Banach space of analytic functions on $\mathbb{D}$, we say that the semigroup $\{C_t\}$ is strongly continuous on X if

$$\lim_{t\to 0^+} \|C_t f - f\|_X = 0 \quad \text{for every } f \in X.$$

Furthermore, if the semigroup $\{C_t\}$ is strongly continuous on X, then the linear operator Γ defined by

$$\mathcal{D}(\Gamma) = \Big\{ f \in X : \lim_{t\to 0^+} \frac{C_t f - f}{t} \text{ exists} \Big\}$$

and

$$\Gamma(f) = \lim_{t\to 0^+} \frac{C_t f - f}{t}$$

for $f \in \mathcal{D}(\Gamma)$ is the infinitesimal generator of the semigroup $\{C_t\}$ with domain $\mathcal{D}(\Gamma)$.

The properties of these semigroups $\{C_t\}$ on various spaces of analytic functions have been studied over the last few decades. For example, the strong continuity of $\{C_t\}$, their spectral properties and the corresponding resolvent operators were the subject of several papers such as [1, 13]. Here we are interested in the strong continuity of the semigroup $\{C_t\}$ on the Dirichlet space $D(\mu)$.

2. Composition operator semigroups on $D(\mu)$

In this section we will characterise the strong continuity of a composition operator semigroup $\{C_t\}$ on $D(\mu)$ in terms of the growth of the norms $\|C_t\|$ for $t \in [0, 1]$.

Lemma 2.1. *Point evaluation functionals are continuous on $D(\mu)$.*

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PROOF. Let $f = \sum_{n=0}^{\infty} a_n z^n \in D(\mu)$. By the Cauchy–Schwarz inequality,

$$\begin{aligned} |f(z)|^2 &\leq \Big(\sum_{n=0}^{\infty} |a_n||z|^n\Big)^2 \leq \Big(\sum_{n=0}^{\infty} |a_n|^2\Big)\Big(\sum_{n=0}^{\infty} |z|^{2n}\Big) \\ &= \|f\|_{H^2}^2 \frac{1}{1-|z|^2} \\ &\leq \|f\|_{D(\mu)}^2 \frac{1}{1-|z|^2}. \end{aligned}$$

□

By the Riesz representation theorem, for every $w \in \mathbb{D}$, there is a $K_w \in D(\mu)$ such that for every $f \in D(\mu)$,

$$f(w) = \langle f, K_w \rangle_{D(\mu)}.$$

The function K_w is called the reproducing kernel at w. Thus, $D(\mu)$ is a reproducing kernel Hilbert space.

THEOREM 2.2. *Let $\{\varphi_t\}$ be a semigroup of analytic functions and $\{C_t\}$ be the induced operator semigroup consisting of bounded operators on $D(\mu)$. Then $\{C_t\}$ is strongly continuous on $D(\mu)$ if and only if*

$$\sup\{\|C_t\| : 0 \leq t \leq 1\} < \infty.$$

PROOF. If $\{C_t\}$ is strongly continuous on $D(\mu)$, then, from the general theory of semigroups [9, Theorem 2.2, page 4], it follows that $\sup\{\|C_t\| : 0 \leq t \leq 1\} < \infty$.

Conversely, suppose that $\sup\{\|C_t\| : 0 \leq t \leq 1\} < \infty$. Since $C_t(f)(z) \to f(z)$ pointwise as $t \to 0^+$, by [6, Corollary 1.3, page 3], it follows that $C_t f \to f$ as $t \to 0^+$ weakly for every $f \in D(\mu)$. Thus, by [9, Theorem 1.4, page 44], $\{C_t\}$ is strongly continuous on $D(\mu)$. □

3. Composition operator semigroups on $D(\delta_\lambda)$

In this section we consider composition operator semigroups $\{C_t\}$ acting on $D(\delta_\zeta)$. Following the work of Sarason and Silva [12] on composition operators, we will distinguish two cases. The first case is when ζ is a fixed point for some member (and hence for every member [4]) of the family $\{\varphi_t : t \in (0, \infty)\}$ and the second case is when $|\varphi_t(\zeta)| < 1$ for every $t > 0$.

3.1. Case $\varphi_t(\zeta) = \zeta$ for every $t \geq 0$. The following theorem provides us with a characterisation of the strong continuity of $\{C_t\}$ in terms of the generator G.

THEOREM 3.1. *Let $\zeta \in \mathbb{T}$, $\{\varphi_t\}$ be a semigroup of analytic functions with generator G and $\{C_t\}$ be the induced composition operator semigroup. If the semigroup $\{\varphi_t\}$ has the point ζ as a common boundary fixed point, the following are equivalent:*

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(i) *$\{C_t\}$ is strongly continuous on $D(\delta_\zeta)$;*
(ii) *the angular limit*

$$\alpha := \angle \lim_{z\to\zeta} \frac{G(z)}{z-\zeta} \tag{3.1}$$

exists finitely;
(iii) *C_t is bounded on $D(\delta_\zeta)$ for every $t > 0$.*

Proof. (ii) $\Rightarrow$ (i). Assume first that the angular limit in (3.1) is finite. By [5, Theorem 1], $\alpha \in \mathbb{R}$ and, for every $t \geq 0$, the angular derivative $\varphi_t'(\zeta)$ exists with $\varphi_t'(\zeta) = e^{\alpha t}$. Thus, C_t is bounded for every $t > 0$ [12, Theorem 2]. Therefore,

$$\sup\{\varphi_t'(\zeta) : 0 \leq t \leq 1\} = \sup\Big\{\sup\Big\{\frac{P_\zeta(z)}{P_\zeta(\varphi_t(z))} : z \in \mathbb{D}\Big\} : 0 \leq t \leq 1\Big\} < \infty.$$

Recall that by the Julia–Carathéodory theorem,

$$\varphi_t'(\zeta) = \sup\Big\{\frac{P_\zeta(z)}{P_\zeta(\varphi_t(z))} : z \in \mathbb{D}\Big\}$$

(see [8]). Thus, there is an absolute constant $K > 0$ such that

$$P_\zeta(z) \leq K P_\zeta(\varphi_t(z))$$

for every $z \in \mathbb{D}$ and every $0 \leq t \leq 1$. If we show that $\sup\{\|C_t\| : 0 \leq t \leq 1\} < \infty$, then, by Theorem 2.2, the semigroup $\{C_t\}$ will be strongly continuous on $D(\delta_\zeta)$. Indeed,

$$\|C_t f\|^2_{D(\delta_\zeta)} = \|C_t f\|^2_{H^2} + D_\zeta(C_t f).$$

We estimate both terms on the right-hand side of the above equality. Since C_φ is bounded on H^2, by [6, Corollary 3.7],

$$\|C_t f\|^2_{H^2} \leq \frac{1 + |\varphi_t(0)|}{1 - |\varphi_t(0)|}\|f\|^2_{H^2} \leq \frac{1 + |\varphi_t(0)|}{1 - |\varphi_t(0)|}\|f\|^2_{D(\delta_\zeta)}.$$

For the second term,

$$\begin{aligned} D_\zeta(C_t f) &= \int_{\mathbb{D}} |f'(\varphi_t(z))|^2 |\varphi_t'(z)|^2 P_\zeta(z)\, dA(z) \\ &\leq K \int_{\mathbb{D}} |f'(\varphi_t(z))|^2 |\varphi_t'(z)|^2 P_\zeta(\varphi_t(z))\, dA(z) \\ &= K \int_{\varphi_t(\mathbb{D})} |f'(z)|^2 P_\zeta(z)\, dA(z) \\ &\leq K \int_{\mathbb{D}} |f'(z)|^2 P_\zeta(z)\, dA(z) \leq K\|f\|^2_{D(\delta_\zeta)}. \end{aligned}$$

Combining the above estimates yields

$$\|C_t\|^2 \leq K + \frac{1 + |\varphi_t(0)|}{1 - |\varphi_t(0)|}$$

and it follows that $\sup\{\|C_t\| : 0 \leq t \leq 1\} < \infty$, since the set $\{\varphi_t(0) : 0 \leq t \leq 1\}$ is a compact subset of $\mathbb{D}$.

(i) $\Rightarrow$ (iii). This is obvious.

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(iii) ⇒ (ii). Since C_t is bounded for every $t > 0$, the angular derivative $\varphi_t'(\zeta)$ exists for every $t > 0$ [12, Theorem 2]. This means that $\zeta \in \mathbb{T}$ is a nonsuperrepulsive fixed point for the semigroup $\{\varphi_t\}$ [5, Lemmas 1 and 3] and thus by [5, Theorem 1]

$$\angle \lim_{z \to \zeta} \frac{G(z)}{z - \zeta}$$

exists finitely. □

Example 3.2. Let

$$\varphi_t(z) = 1 - (1 - z)^{e^{-t}}$$

with generator $G(z) = -(1 - z) \log(1/(1 - z))$. The point 1 is the common boundary fixed point for the semigroup. The image $\varphi_t(\mathbb{D})$ is an angular region inside $\mathbb{D}$ whose angle vertex is at 1. The angular limit $\angle \lim_{z \to 1} G(z)/(z - 1)$ does not exist. Thus, C_t is not bounded on $D(\delta_1)$ for every $t > 0$ and therefore $\{C_t\}$ is not strongly continuous on $D(\delta_1)$.

Example 3.3. Let

$$\varphi_t(z) = \frac{e^{-t} z}{(e^{-t} - 1)z + 1}$$

with generator $G(z) = -z(1 - z)$. The point 1 is the common boundary fixed point for the semigroup. The image $\varphi_t(\mathbb{D})$ is a disk tangent to the unit circle at 1 whose diameter shrinks to $1/2$ as $t \to \infty$. The angular limit $\angle \lim_{z \to 1} G(z)/(z - 1)$ exists and therefore $\{C_t\}$ is strongly continuous on $D(\delta_1)$.

3.2. Case $|\varphi_t(\zeta)| < 1$ for every $t > 0$. In this case we were unable to fully describe strong continuity of $\{C_t\}$ in terms of the generator G. The main theorem of this section is a partial result in this direction.

Lemma 3.4. *Let $\zeta \in \mathbb{T}$ and $\{\varphi_t\}$ be a semigroup of analytic functions with generator G. If*

$$K := \sup_{z \in \mathbb{D}} \operatorname{Re} G'(z) < \infty,$$

then, for every $t \geq 0$,

(α) $\varphi_t' \in H^\infty$ *with* $\|\varphi_t'\|_{H^\infty} \leq \exp\{Kt\}$ *and*

(β) $(\varphi_t(z) - \varphi_t(\zeta))/(z - \zeta) \in H^\infty$ *with* $\|(\varphi_t(z) - \varphi_t(\zeta))/(z - \zeta)\|_{H^\infty} \leq \exp\{Kt\}$.

Proof. Since $\varphi_t'(z) = \exp\{\int_0^t G'(\varphi_s(z))\, ds\}$ (see [1]),

$$|\varphi_t'(z)| = \exp\left\{\int_0^t \operatorname{Re} G'(\varphi_s(z))\, ds\right\} \leq \exp\{Kt\}$$

and this finishes the first part of the lemma. For the second half, observe that φ_t belongs to the disk algebra for every $t \geq 0$, so that

$$\frac{\varphi_t(z) - \varphi_t(\zeta)}{z - \zeta} = \frac{1}{z - \zeta} \int_\zeta^z \varphi_t'(u)\, du,$$

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where the integration is performed along the segment joining z and ζ. Thus,

$$\left|\frac{\varphi_t(z)-\varphi_t(\zeta)}{z-\zeta}\right| = \left|\frac{1}{z-\zeta}\int_\zeta^z \varphi_t'(u)\,du\right| = \left|\int_0^1 \varphi_t'((1-\lambda)\zeta+\lambda z)\,d\lambda\right|$$
$$\leq \int_0^1 |\varphi_t'((1-\lambda)\zeta+\lambda z)|\,d\lambda \leq \exp\{Kt\}$$

and the proof of the lemma is complete. □

Lemma 3.5. *Let $\lambda \in \mathbb{D}$ and $g \in H^2$. Then*

$$\left\|\frac{g(z)-g(\lambda)}{z-\lambda}\right\|_{H^2}^2 \leq \frac{1}{(1-|\lambda|)^2}\|g\|_{H^2}^2.$$

Proof. For λ and g as above,

$$\left\|\frac{g(z)-g(\lambda)}{z-\lambda}\right\|_{H^2}^2 = \frac{1}{2\pi}\int_0^{2\pi}\frac{|g(e^{it})|^2}{|1-\overline{\lambda}e^{it}|^2}\,dt - \frac{|g(\lambda)|^2}{1-|\lambda|^2} \leq \frac{1}{(1-|\lambda|)^2}\|g\|_{H^2}^2,$$

as we wanted to show. □

Lemma 3.6. *Let $g \in H^2$ and $\{\varphi_t\}$ be a semigroup of analytic functions with generator G which satisfies*

$$K := \sup_{z\in\mathbb{D}} \operatorname{Re} G'(z) < \infty.$$

Suppose that $\zeta \in \mathbb{T}$ is such that $|\varphi_t(\zeta)| < 1$ for every $t > 0$. Then

$$(1-|\varphi_t(\zeta)|)^2 D_\zeta(g\circ\varphi_t) \leq \exp\{2Kt\}\left(\frac{1+|\varphi_t(0)|}{1-|\varphi_t(0)|}\right)\|g\|_{H^2}^2.$$

Proof. Let $g \in H^2$. Using Lemma 3.4,

$$\begin{aligned}
D_\zeta(g\circ\varphi_t) &= \left\|\frac{g(\varphi_t(z))-g(\varphi_t(\zeta))}{z-\zeta}\right\|_{H^2}^2 \\
&= \left\|\frac{\varphi_t(z)-\varphi_t(\zeta)}{z-\zeta}\,\frac{g(\varphi_t(z))-g(\varphi_t(\zeta))}{\varphi_t(z)-\varphi_t(\zeta)}\right\|_{H^2}^2 \\
&\leq \exp\{2Kt\}\left\|\frac{g(\varphi_t(z))-g(\varphi_t(\zeta))}{\varphi_t(z)-\varphi_t(\zeta)}\right\|_{H^2}^2 \\
&\leq \exp\{2Kt\}\left(\frac{1+|\varphi_t(0)|}{1-|\varphi_t(0)|}\right)\left\|\frac{g(z)-g(\varphi_t(\zeta))}{z-\varphi_t(\zeta)}\right\|_{H^2}^2 \\
&\leq \exp\{2Kt\}\left(\frac{1+|\varphi_t(0)|}{1-|\varphi_t(0)|}\right)\frac{1}{(1-|\varphi_t(\zeta)|)^2}\|g\|_{H^2}^2,
\end{aligned}$$

where in the last inequality we used Lemma 3.5. □

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THEOREM 3.7. *Let $\{\varphi_t\}$ be a semigroup of analytic functions with generator G which satisfies*

$$K := \sup_{z\in\mathbb{D}} \operatorname{Re} G'(z) < \infty. \tag{3.2}$$

Suppose that $\zeta \in \mathbb{T}$ is such that $|\varphi_t(\zeta)| < 1$ for every $t > 0$ and

$$|\zeta - \varphi_t(\zeta)| \le k(1 - |\varphi_t(\zeta)|) \tag{3.3}$$

for $0 < t \le 1$ and some $k > 0$. Then the semigroup $\{C_t\}$ is strongly continuous on $D(\delta_\zeta)$.

PROOF. By Theorem 2.2, it is enough to show that $\sup\{\|C_t\| : 0 \le t \le 1\} < \infty$. Let $f \in D(\delta_\zeta)$ and write $f(z) = f(\zeta) + (z - \zeta)g(z)$ for some $g \in H^2$. Then

$$f(\varphi_t(z)) = f(\zeta) + (\varphi_t(z) - \zeta)g(\varphi_t(z)).$$

Straightforward calculations show that

$$\frac{f(\varphi_t(z)) - f(\varphi_t(\zeta))}{z-\zeta} = \frac{\varphi_t(z) - \varphi_t(\zeta)}{z-\zeta} g(\varphi_t(z)) + (\varphi_t(\zeta) - \zeta)\frac{g(\varphi_t(z)) - g(\varphi_t(\zeta))}{z-\zeta}.$$

Thus,

$$\begin{aligned}
\|C_t f\|^2_{D(\delta_\zeta)} &= \left\| \frac{f(\varphi_t(z)) - f(\varphi_t(\zeta))}{z-\zeta} \right\|^2_{H^2} \\
&= \left\| \frac{\varphi_t(z) - \varphi_t(\zeta)}{z-\zeta} g(\varphi_t(z)) + (\varphi_t(\zeta) - \zeta)\frac{g(\varphi_t(z)) - g(\varphi_t(\zeta))}{z-\zeta} \right\|^2_{H^2} \\
&\le 2\left\| \frac{\varphi_t(z) - \varphi_t(\zeta)}{z-\zeta} g(\varphi_t(z)) \right\|^2_{H^2} + 2|\varphi_t(\zeta) - \zeta|^2 \left\| \frac{g(\varphi_t(z)) - g(\varphi_t(\zeta))}{z-\zeta} \right\|^2_{H^2}.
\end{aligned}$$

For the first term, we apply Lemma 3.4 to give

$$\begin{aligned}
\left\| \frac{\varphi_t(z) - \varphi_t(\zeta)}{z-\zeta} g(\varphi_t(z)) \right\|^2_{H^2} &\le \exp\{2Kt\} \|g \circ \varphi_t\|^2_{H^2} \\
&\le \exp\{2Kt\} \Big(\frac{1 + |\varphi_t(0)|}{1 - |\varphi_t(0)|} \Big) \|g\|^2_{H^2} \\
&\le \exp\{2Kt\} \Big(\frac{1 + |\varphi_t(0)|}{1 - |\varphi_t(0)|} \Big) \|f\|^2_{D(\delta_\zeta)}.
\end{aligned}$$

For the second term, Lemma 3.6 implies that

$$\begin{aligned}
|\varphi_t(\zeta) - \zeta|^2 \left\| \frac{g(\varphi_t(z)) - g(\varphi_t(\zeta))}{z-\zeta} \right\|^2_{H^2} &\le k^2(1 - |\varphi_t(\zeta)|)^2 \left\| \frac{g(\varphi_t(z)) - g(\varphi_t(\zeta))}{z-\zeta} \right\|^2_{H^2} \\
&\le k^2 \exp\{2Kt\} \Big(\frac{1 + |\varphi_t(0)|}{1 - |\varphi_t(0)|} \Big) \|g\|^2_{H^2} \\
&\le k^2 \exp\{2Kt\} \Big(\frac{1 + |\varphi_t(0)|}{1 - |\varphi_t(0)|} \Big) \|f\|^2_{D(\delta_\zeta)}.
\end{aligned}$$

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Summing up,

$$\|C_t f\|^2_{D(\delta_\zeta)} \le 2(k^2+1)\exp\{2Kt\}\Big(\frac{1+|\varphi_t(0)|}{1-|\varphi_t(0)|}\Big)\|f\|^2_{D(\delta_\zeta)}$$

and, since the set $\{\varphi_t(0) : 0 \le t \le 1\}$ is a compact subset of $\mathbb{D}$, we see that $\sup\{\|C_t\| : 0 \le t \le 1\} < \infty$. This completes the proof of the theorem. □

REMARK 3.8.

(a) The condition

$$|\zeta - \varphi_t(\zeta)| \le k(1 - |\varphi_t(\zeta)|)$$

for $0 < t \le 1$ and some $k > 0$ says that for these values of the parameter t the point $\varphi_t(\zeta)$ lies inside some nontangential approach region of the point ζ.

(b) From the proof of Theorem 3.7, C_t is also bounded on $D(\delta_\zeta)$ for every $t > 0$.

(c) In all the proofs, the interval $[0, 1]$ for the parameter t can be replaced by any interval of the form $[0, \delta]$ with $\delta > 0$.

REMARK 3.9. Conditions (3.2) and (3.3) are independent. This can be seen by the following examples. Let

$$\varphi_t(z) = 1 - (1 - z)^{e^{-t}}$$

with generator

$$G(z) = -(1 - z)\log\frac{1}{1 - z}.$$

Obviously, $\sup_{z\in\mathbb{D}} \operatorname{Re} G'(z) = \infty$ and $|\varphi_t(-1)| < 1$ for every $t > 0$, although we see that $|-1 - \varphi_t(-1)| = 1 - |\varphi_t(-1)|$ for t close to zero. Thus, (3.3) does not imply (3.2). To see that (3.2) does not imply (3.3), consider the semigroup

$$\varphi_t(z) = \frac{(1 + i)e^{-(1+i)t}z}{1 + i - (1 - e^{-(1+i)t})z}$$

with generator

$$G(z) = -z(1 + i - z).$$

By straightforward calculations, $|\varphi_t(1)| < 1$ for every $t > 0$ and

$$\lim_{t\to 0^+} \frac{1 - \varphi_t(1)}{|1 - \varphi_t(1)|} = i.$$

Thus, $\varphi_t(1)$ approaches 1 as $t \to 0^+$ tangentially, although $\sup_{z\in\mathbb{D}} \operatorname{Re} G'(z) < \infty$.

REMARK 3.10. The next two examples illustrate Theorem 3.7. First, let

$$\varphi_t(z) = e^{-t}z$$

be the dilation semigroup with generator $G(z) = -z$. Obviously, $\operatorname{Re} G'(z) < \infty$ and $|\zeta - \varphi_t(\zeta)| \le k(1 - |\varphi_t(\zeta)|)$ for every $k \ge 1$. Thus, $\{C_t\}$ is strongly continuous on $D(\delta_\zeta)$ for every $\zeta \in \mathbb{T}$.

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Second, consider

$$\varphi_t(z) = e^{-t}z + 1 - e^{-t}$$

with generator $G(z) = 1 - z$. The image $\varphi_t(\mathbb{D})$ is a small disk tangent to the unit circle at 1, whose diameter goes to 0 as $t \to \infty$. It follows that $\operatorname{Re} G'(z) = -1$ and $|\varphi_t(\zeta)| < 1$ for every $\zeta \in \mathbb{T} \setminus \{1\}$ and every $t > 0$. Moreover, $\varphi_t(\zeta)$ approaches ζ nontangentially as $t \to 0^+$ for every $\zeta \in \mathbb{T} \setminus \{1\}$ and thus $\{C_t\}$ is strongly continuous on $D(\delta_\zeta)$.

The following theorem provides a description of the infinitesimal generator Γ of $\{C_t\}$ in terms of G.

Theorem 3.11. *Let $\{\varphi_t\}$ be a semigroup of analytic functions with generator G and $\{C_t\}$ be the induced composition operator semigroup. If $\{C_t\}$ is strongly continuous on $D(\mu)$, then the infinitesimal generator Γ of $\{C_t\}$ on $D(\mu)$ has domain*

$$\mathcal{D}(\Gamma) = \{f \in D(\mu) : Gf' \in D(\mu)\}$$

and, for every $f \in \mathcal{D}(\Gamma)$,

$$\Gamma(z) = G(z)f'(z), \quad z \in \mathbb{D}.$$

Proof. By definition, the domain of Γ is

$$\mathcal{D}(\Gamma) = \left\{f \in D(\mu) : \lim_{t\to 0^+} \frac{C_t f - f}{t} \text{ exists in } D(\mu)\right\}.$$

Let $\mathcal{D} = \{f \in D(\mu) : Gf' \in D(\mu)\}$. We will show that if $f \in \mathcal{D}(\Gamma)$, then $Gf' \in D(\mu)$. Indeed, if $f \in \mathcal{D}(\Gamma)$, then $\Gamma(f) \in D(\mu)$ and

$$\lim_{t\to 0^+} \left\| \frac{C_t f - f}{t} - \Gamma(f) \right\|_{D(\mu)} = 0.$$

Convergence in the norm of $D(\mu)$ implies uniform convergence on compact subsets of $\mathbb{D}$ and therefore pointwise convergence. So, for every $z \in \mathbb{D}$,

$$\begin{aligned}\Gamma(f)(z) &= \lim_{t\to 0^+} \frac{f(\varphi_t(z)) - f(z)}{t} = \lim_{t\to 0^+} \frac{f(\varphi_t(z)) - f(\varphi_0(z))}{t} \\ &= \frac{\partial[f(\varphi_t(z))]}{\partial t}\bigg|_{t=0} = G(z)f'(z).\end{aligned}$$

Therefore, $G(z)f'(z) = \Gamma(f)(z) \in D(\mu)$ and $\mathcal{D}(\Gamma) \subseteq \mathcal{D}$. On the other hand, for λ in the resolvent set $\rho(\Gamma)$ of Γ, it is easy to see that

$$\mathcal{D} = \{f \in D(\mu) : Gf' \in D(\mu)\} = \{f \in D(\mu) : Gf' - \lambda f \in D(\mu)\} = R(\lambda, \Gamma)(D(\mu)),$$

where $R(\lambda, \Gamma)$ is the resolvent of Γ at the point λ. But $R(\lambda, \Gamma)(D(\mu)) \subseteq \mathcal{D}(\Gamma)$ and the proof is finished. □

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GEORGIOS STYLOGIANNIS, Loukianou 12, Karditsa 43132, Greece
e-mail: stylog@math.auth.gr, g.stylog@gmail.com