Canad. Math. Bull.Vol. **36** (4), 1993 pp. 419–425

SUBSOCLES SUPPORTING ISOTYPE AND BALANCED SUBGROUPS

PAUL HILL AND CHARLES MEGIBBEN

ABSTRACT. We identify a condition, which we refer to as *cohesiveness*, on a subgroup S of the socle $G[p] = \{x \in G : px = 0\}$ of an abelian p-group G which is necessary for S to be the socle of an isotype subgroup of G. It is shown, when S is countable, that this condition is both necessary and sufficient. A further restriction, definable in terms of the coset valuation on G/S, leads to the notion of S being *completely cohesive* in G. When S is countable, this latter condition is both necessary and sufficient for S to serve as the socle of a balanced subgroup of G. Also noteworthy is the fact that if H and K are, respectively, balanced and isotype subgroups of G with $H[p] = K[p]$, then K is necessarily balanced in G.

1. **Introduction.** All groups considered herein are abelian p-groups written additively. Recall that if G is such a group then the underlying vector space $G[p] = \{x \in G : px = 0\}$ is called the *socle* of G, and any subgroup S of $G[p]$ is known as a *subsocle*. If S is a subsocle of G and H is a subgroup for which $H[p] = S$, then we say that S *supports* H. This, along with terminology not explained, is in agreement with Fuchs [2].

Much attention has been given to the fundamental problem of determining under what conditions a subsocle S supports a pure subgroup H of G, and this has led to many results; see [5] and [6] for classical theorems and, for example, [1] and [3] for more recent ones. However, known results concerning the natural generalization of this problem to isotype subgroups are embarrassingly sparse even though isotype subgroups, introduced by Kulikov [4] in the early 1950's, play a major role in the structure theory of abelian groups. In this paper, we give necessary and sufficient conditions for a specified *countable* subsocle S of G to support an isotype subgroup, and we also find necessary and sufficient conditions in order that S support a balanced subgroup.

As in many of our earlier papers, we find it convenient to denote the height of an element x in the group G by $|x|$, or by $|x|_G$ when necessary for clarity. Likewise, if H is a subgroup of G, we denote the *coset valuation* of x relative to H by $\|x + H\|$. To summarize, $|x| = \alpha$ if $x \in p^\alpha G \setminus p^{\alpha+1}G$ (with $|x| = \infty$ if $x \in p^\alpha G$ for all ordinals α), and $\|x + H\| = \sup\{|x + h| + 1 : h \in H\}$. As one further convention, limit ordinals are assumed to be nonzero.

The authors were supported, respectively, by NSF grants DMS-90-96243 and DMS-89-00235.
Received by the editors March 9, 1992; revised July 8, 1992.
AMS subject classification: 20K10, 20K27.

https://doi.org/10.4153/CMB-1993-057-x Published online by Cambridge University Press

2. **Cohesive subsocles.** The results of this paper are based on the following new concept.

DEFINITION 2.1. Let S be a subsocle of the p-group G. Then the *lineage of G relative to S*, denoted by $\mathrm{Lin}(G,S)$, is defined as follows: If $|x| < \omega$ or $|x| = \infty$, then $x \in \mathrm{Lin}(G,S)$. If $|x| = \mu + n$ where μ is a limit ordinal and $n < \omega$, then $x \in \mathrm{Lin}(G,S)$ if and only if there is a $y \in G$ such that $\|y+S\| = \mu$ and $p^{n+1}y = x$.

Although $\mathrm{Lin}(G,S)$ in general is not a subgroup of G, it does enjoy the following important closure properties.

PROPOSITION 2.2. *Let S be a subsocle of the p-group G.*
(1) (Absorption) If $x \in \mathrm{Lin}(G,S)$ and $|z| > |x|$, then $x+z \in \mathrm{Lin}(G,S)$.
(2) (Weak Additivity) If $x_1, x_2 \in \mathrm{Lin}(G,S)$ and if $|x_1+x_2| = \min\{|x_1|,|x_2|\}$, then $x_1+x_2 \in \mathrm{Lin}(G,S)$.

PROOF. Clearly we need only deal with the situation where the elements have transfinite height. Suppose that $|x| = \mu+n$ where μ is a limit ordinal and $n<\omega$. Then $|z|>|x|$ implies that there exists a $w \in p^{\mu}G$ such that $p^{n+1}w = z$. If $\|y+S\| = \mu$ and $p^{n+1}y = x$, then it is readily seen that $\|(y+w)+S\| = \mu$ and $p^{n+1}(y+w) = x+z$, that is, $x+z \in \mathrm{Lin}(G,S)$. By the absorption property just established, the proof of (2) reduces to the case $|x_1+x_2| = |x_1| = |x_2| = \mu+n$ where μ is a limit ordinal and $n<\omega$. Since $x_1,x_2 \in \mathrm{Lin}(G,S)$, there exist elements y_1 and y_2 in G such that $\|y_1+S\| = \|y_2+S\| = \mu$, $p^{n+1}y_1 = x_1$ and $p^{n+1}y_2 = x_2$. Notice then that both py_1 and py_2 are necessarily in $p^{\mu}G$, and therefore $|p(y_1+y_2)| = \mu$ is a consequence of the fact that $|x_1+x_2| = \mu+n$. It follows that $\|y_1+y_2+S\| = \mu$ and hence $x_1+x_2 \in \mathrm{Lin}(G,S)$.

When applying weak additivity, it is helpful to keep in mind the following elementary observation: If $|a| \geq |a+b|$, then $|a+b| = \min\{|a|,|b|\}$. The restriction $|x_1+x_2| = \min\{|x_1|,|x_2|\}$ in Proposition 2.2 is necessary. For example, if G is a p-group containing elements x_1 and z with $|x_1| < |z| = \omega$, then both x_1 and $x_2 = z - x_1$ are in $\mathrm{Lin}(G,0) = p^{\infty}G \cup (G \setminus p^{\omega}G)$ while $x_1+x_2 = z \notin \mathrm{Lin}(G,0)$. The relevance of $\mathrm{Lin}(G,S)$ to the question of when S supports an isotype subgroup is evident from our next result.

PROPOSITION 2.3. *Let S be a subsocle of the p-group G. If H is an isotype subgroup of G with $H[p] \subseteq S$, then $H \subseteq \mathrm{Lin}(G,S)$.*

PROOF. It suffices to consider $x \in H$ with $|x|_G = |x|_H = \mu+n$ where μ is a limit ordinal and $n<\omega$. Fix a $y \in H$ with $|py|_G = |py|_H = \mu$ and $p^{n+1}y = x$. Then, for each $\alpha < \mu$, there exists an $h_\alpha \in p^{\alpha}H$ such that $ph_\alpha = py$. Hence, for each $\alpha<\mu$, $s_\alpha = h_\alpha - y \in H[p] \subseteq S$ and $|y+s_\alpha|_G = |h_\alpha|_G \geq \alpha$. Since $|y+s|_G < \mu$ for all $s \in S$, it follows that $\|y+S\| = \mu$ and therefore, by definition, $x \in \mathrm{Lin}(G,S)$.

DEFINITION 2.4. A subsocle S of G is said to be *cohesive* in G if $S \subseteq \mathrm{Lin}(G,S)$.

Although the definition of cohesiveness given in 2.4 is the most convenient one for our purposes, the following easily established paraphrase may provide the reader further insight into the meaning of the concept: A subsocle S of G is cohesive if and only if, for each limit ordinal μ and each $n < \omega$, $S \cap p^{\mu+n}G \subseteq p^{n+1}\bar{S}_\mu$ where $\bar{S}_\mu = \bigcap_{\alpha<\mu}(S+p^\alpha G)$. By Proposition 2.3, being cohesive in G is a necessary condition for S to support an isotype subgroup H of G. Since, however, all subsocles of separable p-groups are cohesive and an uncountable subsocle need not support a pure subgroup [7], cohesiveness does not suffice. Nonetheless, we prove in the next section that every *countable* cohesive subsocle does support an isotype subgroup. The proof of this fact requires one further preliminary result about cohesive subsocles.

PROPOSITION 2.5. *Suppose that the subsocle S is cohesive in G.*

(1) If $\|y+S\| = \mu$ is a limit ordinal, then $y \in \mathrm{Lin}(G,S)$.

(2) If $x \in \mathrm{Lin}(G,S)$ with $|x| = \beta$, then there exists a $y \in G$ such that

(a) $py = x$ and

(b) for each $\alpha < \beta$, there is an $s_\alpha \in S$ such that $y + s_\alpha \in \mathrm{Lin}(G,S)$ and $|y+s_\alpha| \geq \alpha$.

PROOF. (1) Choose an ordinal α such that $|y| < \alpha < \mu$. Since $\|y+S\| = \mu$, there is an $s \in S$ such that $|y-s| \geq \alpha$. Then $|y-s| > |y| = |s|$ and consequently $y = s + (y-s)$ is in $\mathrm{Lin}(G,S)$ by absorption (Proposition 2.2) and the fact that S is cohesive in G.

(2) It suffices to consider the situation $|x| = \beta = \mu + n$ where μ is a limit ordinal and $n < \omega$. Since $x \in \mathrm{Lin}(G,S)$, there is a $w \in G$ such that $\|w+S\| = \mu$ and $p^{n+1}w = x$. If $n \neq 0$, we let $y = p^n w$. Indeed, in this case, $y \in \mathrm{Lin}(G,S)$ by Definition 2.1 and we may take all the s_α's zero since $|y| = \beta - 1$. On the other hand, if $n = 0$, we need only let $y = w$. For then the desired s_α's exist because $\|y+S\| = \mu = \beta$ and (1) implies $y + s \in \mathrm{Lin}(G,S)$ for all $s \in S$.

3. **Support for isotype subgroups.**

We are now in position to prove one of our main results.

THEOREM 3.1. *A countable subsocle S of the p-primary group G supports an isotype subgroup of G if and only if S is cohesive in G.*

PROOF. We have already noted that cohesiveness is a necessary condition for S to support an isotype subgroup. Suppose then that S is countable and cohesive in G. Assume next that H is any finite subgroup of G that satisfies the following conditions.

(I) $H[p] \subseteq S$.

(II) $H \subseteq \mathrm{Lin}(G,S)$.

(III) If $h \in H$ and $|ph| \geq \alpha + 1$, then there exists an $s \in S$ such that $|h+s| \geq \alpha$ and $h + s \in \mathrm{Lin}(G,S)$.

Our aim is to show that we can expand any such H to an isotype subgroup supported by S.

CLAIM A. If $s \in S$, then $H' = \langle H, s\rangle$ continues to satisfy conditions (I)–(III). Clearly, H' satisfies (I). In order to demonstrate that H' satisfies (II), it suffices to show that $h+s \in \mathrm{Lin}(G,S)$ whenever $h \in H$ and $s \in S$. Since both H and S are contained in $\mathrm{Lin}(G,S)$, Proposition 2.2 reduces consideration to the case where $|h+s| > |h| = |s|$. Let $\alpha = |h+s|$

and observe that condition (III) implies that there exists an $s' \in S$ such that $|h+s'| \geq \alpha$ and $h+s' \in \mathrm{Lin}(G,S)$. But then $|h+s'| \geq |h+s| = |(h+s')+(s-s')|$ implies that $|h+s| = \min\{|h+s'|, |s-s'|\}$ (see remark following the proof of Proposition 2.2) and hence, by weak additivity and the cohesiveness of S, $h+s = (h+s')+(s-s')$ is in $\mathrm{Lin}(G,S)$. Finally, H' also satisfies (III) by what has just been proved. Indeed if $|p(h+s)| = |ph| \geq \alpha+1$ and if s' is chosen as above, then $s'-s$ is an element of S such that $|(h+s)+(s'-s)| \geq \alpha$ and $(h+s)+(s'-s) = h+s' \in \mathrm{Lin}(G,S)$.

CLAIM B. If H contains an element $x \notin pH$ with $|x| = \beta \neq 0$, then there is a $y \in G$ such that $py = x$ and $H' = \langle H, y\rangle$ continues to satisfy conditions (I)–(III). By finiteness, we may assume without loss of generality that $|x| \geq |x+ph|$ for all $h \in H$. Now choose y to satisfy the conditions in part (2) of Proposition 2.5 and let $H' = \langle H, y\rangle$. Since $x \notin pH$, it follows that $H'[p] = H[p]$ and consequently H' satisfies (I). In order to verify that H' satisfies condition (II), it suffices to prove that $h+y \in \mathrm{Lin}(G,S)$ for every $h \in H$. First observe that $|h+y| < \beta$ since $|p(h+y)| = |ph+x| \leq |x| = \beta$. Then, by the choice of y, there is an $s \in S$ such that $|y+s| \geq |h+y| = |(y+s)+(h-s)|$ and $y+s \in \mathrm{Lin}(G,S)$. Since also $h-s \in \mathrm{Lin}(G,S)$ by the proof of Claim A, it follows by weak additivity that $h+y = (y+s)+(h-s)$ is in $\mathrm{Lin}(G,S)$, as desired. Finally, we need to show that H' satisfies condition (III). To do this, it is enough to consider the situation $|p(h+y)| \geq \alpha+1$ where h is an arbitrary element of H. Notice that $\alpha < \beta$ since $|ph+x| \leq |x| = \beta$, and also $|ph| \geq \alpha+1$ since the contrary would contradict $|p(h+y)| \geq \alpha+1$. Then, by the fact that H satisfies (III), there is an $s \in S$ such that $|h+s| \geq \alpha$ and $h+s \in \mathrm{Lin}(G,S)$. By choice of y, there is also an $s' \in S$ with $|y+s'| \geq \alpha$ and $y+s' \in \mathrm{Lin}(G,S)$. If $|h+s|+1 < \beta$, we may furthermore choose s' so that $|y+s'| > |h+s|$; while if $|h+s|+1 \geq \beta$, we will have $|(h+s)+(y+s')| \leq |h+s|$ since $p(h+s+y+s') = ph+x$ has height at most β. In either case, $|(h+s)+(y+s')| = \min\{|h+s|, |y+s'|\}$ and weak additivity implies that $(h+y)+(s+s')$ is an element of $\mathrm{Lin}(G,S)$ having height at least α. This completes the proof of Claim B.

To finish the proof the theorem, we fix an enumeration $s_1, s_2, \ldots, s_n, \ldots$ of the countable set S and note that $x+pH \subseteq pH'$ when $H' = \langle H, y\rangle$ is as in Claim B. Thus repeated applications of Claim A and Claim B yield an ascending sequence of finite subgroups

$$0 = H_0 \subseteq H_1 \subseteq H_2 \subseteq \cdots \subseteq H_n \subseteq \cdots$$

that satisfy conditions (I)–(III) and for which the following also hold.

(IV) $s_n \in H_{n+1}$.

(V) $H_n \cap pG \subseteq pH_{n+1}$.

Let $H = \bigcup_{n<\omega} H_n$. Obviously, conditions (I) and (IV) imply that $H[p] = S$. In order to show that H is isotype in G, we prove that $H \cap p^\alpha G \subseteq p^\alpha H$ by induction on α. Since the induction automatically survives limits, it suffices to demonstrate that $H \cap p^{\alpha+1}G \subseteq p^{\alpha+1}H$ follows from the assumption that $H \cap p^\alpha G \subseteq p^\alpha H$. So let x be in $H \cap p^{\alpha+1}G$ and choose n so that $x \in H_n$. By condition (V), $x = py$ for some $y \in H_{n+1}$. Then $|py|_G \geq \alpha+1$ and condition (III) implies that there is an $s \in S$ such that $|y+s|_G \geq \alpha$. But then $h = y+s$ is an element of $H \cap p^\alpha G \subseteq p^\alpha H$ such that $ph = x$. Hence $x \in p^{\alpha+1}H$, as required.

https://doi.org/10.4153/CMB-1993-057-x Published online by Cambridge University Press

Our concept of cohesive subsocles suggest a number of new problems. Recall, for example, that regardless of cardinality a direct sum of cyclic p-groups is *pure-complete* [5] in the sense that every subsocle supports a pure subgroup. Thus, for suitably restricted classes of p-groups, the cardinality of a subsocle should be irrevelant. In particular, it seems reasonable to conjecture that simply presented p-groups [2, §83] are *isotype-complete* in the sense that all cohesive subsocles support isotype subgroups. Quite possibly the larger class of isotype subgroups of simply presented p-groups enjoys this property. We also note, in view of the notion of cohesiveness, that the definition of "isotype-complete" proposed in Problem 63 of [2] no longer seems viable since even countable p-groups fail to satisfy Fuchs' version of the concept. Indeed it is easily seen that the (countable) Prüfer group $G = H_{\omega+1}$ [2, p. 85] contains a noncohesive subsocle S which is nonetheless a cohesive subsocle of a larger p-group $K \simeq H_{\omega+1}$ containing G as an isotype subgroup.

4. **Support for balanced subgroups.** Recall that a subgroup of a p-primary group G is said to be *balanced* if H is both isotype and nice; that is, for each ordinal α, we have both $p^\alpha G \cap H = p^\alpha H$ and $p^\alpha(G/H) = (p^\alpha G + H)/H$. A straightforward induction establishes the following fundamental fact: An isotype subgroup H of G is balanced if and only if the coset valuation $\|x + H\|$ never assumes limit values for any x in G. Balanced subgroups play a significant role in the structure theory of primary groups, and the important class of simply presented groups consists precisely of the balanced projectives [2, Theorem 82.3].

In this section, we extend the results of the preceding section concerning isotype subgroups to corresponding results for balanced subgroups. More precisely, we find necessary and sufficient conditions for a *countable* subsocle S of G to support a balanced subgroup of G. In fact if the subsocle S supports an isotype subgroup of G, then we shall identify a further condition on S (independent of its cardinality) that insures that S supports a balanced subgroup of G. This new condition is already implicit in our next result.

PROPOSITION 4.1. *Suppose H is an isotype subgroup of the p-primary group G and let $S = H[p]$. Then H is a balanced subgroup of G if and only if, for each limit ordinal μ and each $y \in G$, $\|y + S\| = \mu$ implies that y is contained in the subgroup generated by H and $p^\mu G$.*

PROOF. First assume that H is balanced in G and that $\|y+S\| = \mu$ is a limit ordinal. Then, since $\|y + H\| \geq \|y + S\|$ and $\|y + H\|$ cannot be a limit ordinal, it follows that $\|y+H\| \geq \mu+1$. Therefore there is an $h \in H$ such that $|y+h| \geq \mu$ and hence $y \in H+p^\mu G$, as desired. Conversely, suppose $\{y \in G : \|y+S\| = \mu\} \subseteq H + p^\mu G$ for all limit ordinals μ and assume by way of contradiction that H fails to be balanced in G. This means that there must exist some $y \in G$ such that $\mu = \|y + H\|$ is a limit ordinal. Among all such pairs (y, μ), choose one for which $y + p^\mu G$ has minimal order. It must be the case that $0(y+p^\mu G) = p$. For otherwise, since $\|py+H\| \geq \|y+H\|$ and $0(py+p^\mu G) < 0(y+p^\mu G)$,

https://doi.org/10.4153/CMB-1993-057-x Published online by Cambridge University Press

it would follow that $||py + H|| \geq \mu + 1$; which, in view of the fact that $pG \cap H = pH$, yields a $y' = h + y$ such that $h \in H$, $py' \in p^{\mu}G$ and $||y' + H|| = ||y + H|| = \mu$. Thus, we have $||y + H|| = \mu$ and $py \in p^{\mu}G$ where μ is a limit ordinal. We claim, however, that these conditions imply that $||y + S|| = \mu$. Indeed, for each $\alpha < \mu$, there exists an $h_{\alpha} \in H$ such that $|y + h_{\alpha}| \geq \alpha$ where $ph_{\alpha} \in H \cap p^{\alpha+1}G$ because $py \in p^{\mu}G$. But H is isotype in G and consequently $ph_{\alpha} = ph'_{\alpha}$ for some $h'_{\alpha} \in p^{\alpha}H$, that is, $s_{\alpha} = h_{\alpha} - h'_{\alpha}$ is an element of $S = H[p]$ such that $|y + s_{\alpha}| \geq \alpha$. Since we have such an s_{α} for each $\alpha < \mu$, we see that $\mu \leq ||y + S|| \leq ||y + H|| = \mu$. Finally then, our hypotheses force us to conclude that $y \in H + p^{\mu}G$, which is contrary to the assumption that $||y + H|| = \mu$. This contradiction completes the proof that H is necessarily balanced in G.

DEFINITION 4.2. A cohesive subsocle S of the p-group G is said to be *completely cohesive* provided it satisfies the following condition: If $||y + S|| = \mu$ is a limit ordinal, then there exists an $x \in G$ such that (1) $x - y \in p^{\mu}G$, (2) $\langle x \rangle[p] \subseteq S$, and (3) $0(x) \leq 0(y)$.

As in the case of cohesiveness, we have chosen here a definition for completely cohesive subsocles that leads most quickly to our desired results. The reader, however, may find the following alternative formulation more attractive: A subsocle S of the p-group G is completely cohesive if and only if, for each limit ordinal μ and each $n < \omega$,

$$S \cap p^{\mu+n}G \subseteq p^{n+1}\bar{S}_{\mu} \quad \text{and} \quad \bar{S}_{\mu}[p^{n+1}] \subseteq p^{-n}S + p^{\mu}G$$

where $\bar{S}_{\mu} = \cap_{\alpha<\mu}(S + p^{\alpha}G)$ and $p^{-n}S = \{x \in G : p^n x \in S\}$.

THEOREM 4.3. *Suppose S is a subsocle of the p-primary group G that supports an isotype subgroup. Then a necessary and sufficient condition for S to support a balanced subgroup of G is that S be completely cohesive.*

PROOF. First consider the case where H is a balanced subgroup of G supported by S and let y be an arbitrary element of G such that $||y + S|| = \mu$ is a limit ordinal. By Proposition 4.1, there exist elements $x \in H$ and $z \in p^{\mu}G$ such that $y = x + z$. Thus $x - y \in p^{\mu}G$ and $\langle x \rangle[p] \subseteq H[p] = S$. If however $0(x) > 0(y) = p^n$, then $p^n x \in H \cap p^{\mu+n}G = p^{\mu+n}H$ and hence we have $p^n x = p^n h$ for some $h \in p^{\mu}H$. But then $x' = x - h$ is an element of H with $x' - y \in p^{\mu}G$ and $0(x') \leq p^n$. We conclude that S is completely cohesive.

Conversely, assume that S is a completely cohesive subsocle of G and let H be any isotype subgroup of G such that $H[p] = S$. We shall prove that H is necessarily balanced in G by demonstrating that the conditions of Proposition 4.1 are satisfied. Towards this end, suppose that $||y + S|| = \mu$ is a limit ordinal. Since S is completely cohesive, we can write $y = x + z$ where $\langle x \rangle[p] \subseteq S$, $z \in p^{\mu}G$ and $0(x) \leq 0(y)$. If $0(y) = p$, then x must be an element of $S = H[p]$ and we have the desired conclusion that $y \in H + p^{\mu}G$. Proceeding by induction on $0(y) = p^{n+1}$, we may assume that $g \in H + p^{\mu}G$ whenever g is an element of G with $||g + S|| = \mu$ and $0(g) \leq p^n$. In particular, if it should happen that $0(x) < 0(y)$, then $||x + S|| = ||y + S|| = \mu$ and our induction hypothesis implies that $x \in H + p^{\mu}G$. Hence $y = x + z$ is also in $H + p^{\mu}G$, as desired. Thus we may assume that

$p^n x$ is a nonzero element of S. Recalling that $\|y + S\| = \mu$ implies $py \in p^\mu G$, we see in fact that $p^n x \in S \cap p^{\mu+n-1}G$. Then, as in the proof of Proposition 2.3, we can exhibit an $h \in H$ such that $p^n h = p^n x$ and, for each $\alpha < \mu$, there exists an $s_\alpha \in S$ with $|h + s_\alpha| \geq \alpha$. Consequently $x' = x - h$ is an element of order at most p^n such that either $\|x' - S\| = \mu$ or else $|x' + s| \geq \mu$ for some $s \in S$. In the first instance, $x \in H + p^\mu G$ follows from our induction hypothesis; while in the second instance, $x \in H + p^\mu G$ follows immediately from the fact that $S \subseteq H$. We conclude therefore that H is a balanced subgroup of G.

COROLLARY 4.4. *If K is a balanced subgroup of the p-primary group G and if H is an isotype subgroup of G with $H[p] = K[p]$, then H is necessarily balanced in G.*

Combining Theorem 3.1 and Theorem 4.3, we obtain our promised characterization of those countable subsocles that support balanced subgroups.

THEOREM 4.5. *A countable subsocle S of a p-primary group G supports a balanced subgroup if and only if S is completely cohesive.*

REFERENCES

1. K. Benabdallah and G. Bouabdillah, *Straight and strongly straight modules over principal ideal domains*, Abelian Groups and Modules, CISM Courses and Lectures **287**, Springer-Verlag, (1984), 115–130.

2. L. Fuchs, *Infinite Abelian Groups, Vol. II*, Academic Press, New York, 1973.

3. J. Irwin and J. Swanek, *On purifiable subsocles of a primary abelian group*, Can. J. Math. **23**(1971), 48–57.

4. L. Kulikov, *Generalized primary groups*, Russian, Trudy Moskov. Mat. Obšč. **1**(1952), 277–326; **2**(1953), 85–167.

5. P. Hill and C. Megibben, *Minimal pure subgroups in primary groups*, Bull. Soc. Math. France **92**(1964), 251–257.

6. ———, *On primary groups with countable basic subgroups*, Trans. Amer. Math. Soc. **124**(1966), 49–59.

7. C. Megibben, *A note on a paper Bernard Charles (Étude sur les sous-groupes d'un groupe abélien)*, Bull. Soc. Math. France **91**(1963), 452–454.

Department of Algebra Combinatorics and Analysis
Auburn University
Auburn, Alabama 36849
U.S.A.

Department of Mathematics
Vanderbilt University
Nashville, Tennessee 37240
U.S.A.

https://doi.org/10.4153/CMB-1993-057-x Published online by Cambridge University Press