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DYNAMICS FOR VORTEX CURVES OF THE GINZBURG-LANDAU EQUATIONS

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We study the asymptotic behaviour of solutions to the evolutionary Ginzburg- Landau equations in three dimensions. We show that the motion of the Ginzburg-Landau vortex curves is the flow by curvature.

1. INTRODUCTION

Let $Q = \Omega \times [0,l]$, $\Omega \subset R^2$, be a bounded smooth domain, $g : \Sigma = \partial\Omega \times [0,l] \to S^1$ a $C^{1,\alpha}$-map such that $\deg(g, \partial\Omega_z) = d > 0$ for all $0 \leqslant z \leqslant l$. Here $\Omega_z = \Omega \times \{z\}$. Let $a : Q \to R$ be a smooth function (say $C^3(\overline{Q})$) with positive lower bound.

We consider the following problem:

$$\frac{\partial u_\varepsilon}{\partial t} = \frac{1}{a(x)} \operatorname{div}\big(a(x)\nabla u\big) + \frac{1}{\varepsilon^2} u_\varepsilon\big(1 - |u_\varepsilon|^2\big) \quad \text{in } Q \times R_+, \tag{1.1}$$

$$u_\varepsilon(x,0) = u_\varepsilon^0(x), \quad x \in Q, \tag{1.2}$$

$$u_\varepsilon(x,t) = g(x), \quad x \in \Sigma,\ t \geqslant 0, \tag{1.3}$$

$$\frac{\partial u_\varepsilon}{\partial z} = 0 \quad \text{for } z = 0, l, \tag{1.4}$$

where $u_\varepsilon :\ Q \times R_+ \to R^2$. The system (1.1)–(1.4) can be viewed as a simplified evolutionary Ginzburg-Landau equation in the theory superconductivity of inhomonogence.

The aim of this article is to understand the dynamics of vortices, or zeros, of solutions u of (1.1)- (1.4). Its importance to the theory of superconductivity and applications are addressed in many earlier papers [**4, 9, 10, 11, 14**].

Let Γ_0 be a collection of d embedded C^2–curves in Q with $\partial\Gamma_0 \subset \Omega \times \{0,l\}$. Moreover, we assume Γ_0 intersects $\Omega \times \{0,l\}$ orthogonally along $\partial\Gamma_0$. Note that the last assumption is compatiable with the assumption $\dfrac{\partial u_\varepsilon^0}{\partial z} = 0$ for $z = 0, l$. (That is the natural compatibilty condition for problem (1.1)–(1.4). Similarly, we also assume that $u_\varepsilon^0 = g$ on Σ.)

For the initial data u_ε^0, we make the following assumptions:

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(H1) $\int_Q \rho^2(x)\Big[|\nabla u_\varepsilon^0|^2 + \big(|u_\varepsilon^0|^2 - 1\big)/(2\varepsilon^2)\Big]dx \leqslant K$ for all $0 < \varepsilon \leqslant 1$. Here $\rho(x) = \mathrm{dist}(x, \Gamma_0)$;

(H2) u_ε^0 converges as $\varepsilon \to 0^+$ in the C^0-norm away from Γ_0 to a map u^0 with its image in S^1;

(H3) Let Γ_0^i, $i = 1, \cdots, k$, be connected components of Γ_0, and let $\delta > 0$ be chosen so that the sets $\Gamma_0^i(\delta)$, $i = 1, \cdots, k$, are pairwise disjoint. Here $\Gamma_0^i(\delta) = \big\{x \in Q : \ \mathrm{dist}(x, \Gamma_0^i) \leqslant \delta\big\}$.

Let $T > 0$, and $\{\Gamma_t\}$, $0 \leqslant t \leqslant T$, be a family of embedded C^2 -curves inside Q with boundaries $\{\partial\Gamma_t\}$ contained in $\Omega \times \{0, l\}$, assume Γ_t intersects with $\Omega \times \{0, l\}$ orthogonally along $\partial\Gamma_t$, which are obtained from Γ_0 by the following equations in R^3:

$$(1.5) \qquad \begin{cases} \dfrac{dx(p,t)}{dt} = \vec{H}\,\big(x(p,t),t\big) - \pi\dfrac{\nabla a}{a}\big(x(p,t)\big), \\ x(p,0) = p \in \Gamma_0, \end{cases}$$

where π is the projection onto the normal space of Γ_t, and the curvature vector $\vec{H}$ of Γ_t is characterised by the property

$$\int_{\Gamma_t} \mathrm{div}^{\Gamma_t}\,\phi d\mathcal{H}^1 = -\int_{\Gamma_t} \vec{H}\cdot\phi d\mathcal{H}^1, \quad \forall\phi \in (\phi_1, \phi_2, \phi_3) \in C^1(R^3, R^3),$$

here $\mathrm{div}^{\Gamma_t}\phi = d_i^{\Gamma_t}\phi_i$ is the tangential divergence of ϕ [**12**]. In the case $a = 1$, equation (1.5) denotes the flow by mean curvature with codimension 2 in R^3.

Theorem 1.1. *Assume that $a \in C^3(\overline{Q})$ and $a_0 = \min_{\overline{Q}} a > 0$. Under assumptions (H1)-(H3) and for each t, $0 \leqslant t \leqslant T$, one has (by taking subsequences if necessary) that $u_\varepsilon(x,t) \rightharpoonup u_*(x,t)$ weakly in $H^1_{loc}(\overline{Q} \setminus \Gamma_t)$. Here $u_*(x,t)$ satisfies:*

$$(1.6) \qquad \partial_t u_* - \frac{1}{a}\mathrm{div}(a\nabla u_*) = |\nabla u_*|^2 u_* \ \text{ in } \ Q \setminus \Gamma_t.$$

Now we briefly describe some mathematical advances concerning this problem. In two space dimensions, $a = 1$, the dynamical law for vortices was formally derived in [**8, 14**]. The first rigorous mathematical proof of this dynamical law, which is of the form $\frac{d}{dt}x(t) = -\nabla w\big(x(t)\big)$, was given by Lin in [**4, 5**]. See also [**6**, Lecture 3]. In [**4, 5**], one allows vortices of degree ± 1 and assumes that they have the same sign. For vortices of degree ± 1 (possibly of different signs), the same type of dynamical law has recently been shown [**3**]. We refer to [**7**] for vortex dynamics under the Neumann boundary conditions for pinning conditions. In three space dimensions, $a = 1$, a similar dynamical law was also established in [**7**] for nearly parallel filaments. The short-time dynamical law for codimension 2 interfaces in higher dimensions was shown in [**7**]. In two space dimensions, $a \not\equiv 1$, the dynamical law was established in [**7**].

The rest of the paper is organised as follows. In Section 2, we collect some basic facts on the curve flow. In Section 3, we prove the weak convergence.

2. Mean curvature flow with codimension 2

Given a set $E \subset R^3$, we set

$$\eta_E(x) = \frac{1}{2}\big(\text{dist}(x, E)\big)^2.$$

The following results on the square distance function have been proved in [6]. Let γ be a smooth embedded curve in R^3; then η_γ is smooth in a suitable tubular neighbourhood Ω of γ. $-\Delta\nabla\eta_\gamma$ coincides, on γ, with the curvature vector $\vec{H}$ of γ.

Lemma 2.1. [2, Lemma 3.7] *Let $(\Gamma_t)_{t\in[0,T]}$ be a smooth flow. Then there exists $\sigma > 0$ such that the function*

$$\eta(x,t) := \frac{1}{2}\text{dist}^2(x, \Gamma_t)$$

is smooth in $\{(x,t) \in R^3 \times [0,T] : \eta \leqslant \sigma\}$. Moreover, the displacement of the flow is given by

$$\frac{dx(p,t)}{dt} = -\nabla\eta_t\big(x(p,t),t\big), \quad \forall t \in [0,T], \ p \in \Gamma_0.$$

In particular, $(\Gamma_t)_{t\in[0,T]}$ is a smooth curvature flow defined by (1.5) if and only if

$$\nabla\eta_t = \Delta\nabla\eta - \nabla^2\eta\frac{\nabla a}{a}, \quad \text{on } \Gamma_t.$$

Short time existence for curvature flow of smooth initial space curves is a consequence of a general theorem proved in [**1**, **13**].

Lemma 2.2. *Assume that γ_0 is a embedded C^2–curve in Q with $\partial\gamma_0 \in \Omega\times\{0,l\}$. Assume Γ_0 intersects $\Omega\times\{0,l\}$ orthogonally along $\partial\gamma_0$. Then there exist a positive number $t_0 > 0$ and a family of embedded C^2–curves inside Q with $\partial\Gamma_t \subset \Omega\times\{0,l\}$ such that the following system of equalities holds on γ_t:*

$$\frac{\partial\nabla\eta_\gamma}{\partial t}(t,p) - \Delta\nabla\eta_\gamma(t,p) + \nabla^2\eta\frac{\nabla a}{a}(p) = 0, \quad t \in [0,t_0], \ p \in \gamma_t,$$

and γ_t intersects with $\Omega\times\{0,l\}$ orthogonally along $\partial\gamma_t$.

3. The proof of Theorem 1.1

Lemma 3.1. (Uniformly estimate)

$$\int_0^T\int_{Q\setminus\Gamma_t(\delta)}\Big[|u_{\varepsilon t}|^2 + \frac{1}{2}a(x)\Big(|\nabla u_\varepsilon|^2 + \frac{1}{2\varepsilon^2}\big(1-|u_\varepsilon|^2\big)^2\Big)\Big]dxdt \leqslant C(\delta,T,\sigma), \tag{3.1}$$

where $\sigma > 0$ is such that the sets $\Gamma_t^i(4\sigma)$, $i = 1,2,\cdots,k$, are pairwise disjoint for all $0 \leqslant \Gamma$, $0 < \delta \leqslant \sigma$. Here $\Gamma_t^i(4\sigma) = \{x \in Q : \ \text{dist}\,(x,\Gamma_t^i) \leqslant 4\sigma\}$.

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PROOF: Let $\phi_\sigma : R_+ \to R_+$ be a smooth monotone function such that

$$\phi_\sigma(r) = \begin{cases} r^2 & \text{if} \quad r \leqslant \sigma \\ 4\sigma^2 & \text{if} \quad r \geqslant 2\sigma. \end{cases} \quad (\sigma > 0). \tag{3.2}$$

Define

$$\rho(x,t) = \operatorname{dist}(x, \Gamma_t). \tag{3.3}$$

Assume that

$$\min\Big\{|x-y| : \quad x \in \Gamma_t,\ y \in \sum,\ 0 \leqslant t \leqslant \Gamma\Big\} \geqslant 4\sigma. \tag{3.4}$$

Using integration by parts, one gets

$$\begin{aligned} \frac{d}{dt}\int_Q \frac{1}{2}\phi_\sigma\big(\rho(x,t)\big)a(x)\Big[|\nabla u|^2 + \frac{1}{2\varepsilon^2}\big(1-|u|^2\big)^2\Big] \\ = \int_Q \frac{1}{2}\Big(\frac{d}{dt}\phi_\sigma\Big)\cdot a(x)\Big[|\nabla u|^2 + \frac{1}{2\varepsilon^2}\big(1-|u|^2\big)^2\Big] \\ + \int_Q \phi_\sigma a\Big[\nabla u \cdot \nabla u_t + \frac{1}{2\varepsilon^2}\big(1-|u|^2\big)\cdot(-2uu_t)\Big] \\ =: I + II. \end{aligned} \tag{3.5}$$

We shall set $\phi_\sigma = \phi, \quad u_\varepsilon = u.$

$$\begin{aligned} II &= \int_Q \phi\Big[-\nabla(a\nabla u) - \frac{a(x)}{\varepsilon^2}\big(1-|u|^2\big)u\Big]u_t - \int_Q \nabla\phi\cdot a\cdot\nabla u\cdot u_t \\ &= -\int_Q \phi|u_t|^2 - \int_Q a\nabla\phi\nabla u\cdot u_t. \end{aligned} \tag{3.6}$$

Now we calculate the expression $a\nabla\phi\nabla u\cdot u_t$. We shall use the summation convention, and simplify notation.

$$\begin{aligned} a\nabla\phi\nabla u\cdot u_t &= \nabla\phi\nabla u\Big[\operatorname{div}(a\nabla u) + \frac{1}{\varepsilon^2}a\cdot\big(1-|u|^2\big)u\Big] \\ &= (au_j)_j u_i\phi_i + \frac{1}{\varepsilon^2}a\big(1-|u|^2\big)u\cdot u_i\phi_i \\ &= (au_iu_j)_j\phi_i - au_ju_{ij}\phi_i - \Big[\frac{1}{4\varepsilon^2}a\big(1-|u|^2\big)^2\Big]_i\phi_i + \frac{1}{4\varepsilon^2}\big(1-|u|^2\big)^2 a_i\phi_i \\ &= (au_iu_j)_j\phi_i - \frac{1}{2}\big(|u_j|^2\big)_i\phi_i - \Big[\frac{1}{4\varepsilon^2}a\big(1-|u|^2\big)^2\Big]_i\phi_i + \frac{1}{4\varepsilon^2}\big(1-|u|^2\big)^2 a_i\phi_i. \end{aligned} \tag{3.7}$$

Hence

$$\begin{aligned} \int_Q a\nabla\phi\nabla u\cdot u_t &= \int_Q -a\phi_{ij}u_iu_j + \frac{1}{2}\Delta\phi\cdot a|\nabla u|^2 + \frac{1}{2}\nabla a\cdot\nabla\phi\cdot|\nabla u|^2 \\ &\qquad + \Delta\phi\cdot\frac{1}{4\varepsilon^2}a\big(1-|u|^2\big)^2 + \frac{1}{4\varepsilon^2}\big(1-|u|^2\big)^2\cdot\nabla a\cdot\nabla\phi \\ &= -\int_Q a\phi_{ij}u_iu_j + \int_Q \Delta\phi\cdot\frac{1}{2}a\Big[|\nabla u|^2 + \frac{1}{2\varepsilon^2}\big(1-|u|^2\big)^2\Big] \\ &\qquad + \int_Q \frac{\nabla a}{a}\cdot\nabla\phi\cdot\frac{1}{2}a\Big[|\nabla u|^2 + \frac{1}{2\varepsilon^2}\big(1-|u|^2\big)^2\Big]. \end{aligned} \tag{3.8}$$

So, we have

$$\frac{d}{dt}\int_Q \phi\frac{1}{2}a\Big[|\nabla u|^2+\frac{1}{2\varepsilon^2}\big(1-|u|^2\big)^2\Big] \tag{3.9}$$
$$=\int_Q[\phi_t-\Delta\phi-\frac{\nabla a}{a}\cdot\nabla\phi]\frac{1}{2}a\Big[|\nabla u|^2+\frac{1}{2\varepsilon^2}\big(1-|u|^2\big)^2\Big]$$
$$+\int_Q a\phi_{ij}u_iu_j-\int_Q\phi|u_t|^2.$$

Next we observe that on the set $\{x\in Q:\ \ \rho(x,t)<\sigma\}$, $(\phi_{ij})\leqslant I$ in the sense that

$$\phi_{ij}\xi_i\xi_j\leqslant|\xi|^2 \qquad \text{for all}\ \ \xi\in R^3. \tag{3.10}$$

Also, on Γ_t, we have $\phi_t=0$, $\Delta\phi=0$. Since Γ_t is obtained from Γ_0 by curvature flow (1.5), by Lemma 2.1, we have

$$\nabla(\phi_t-\Delta\phi-\frac{\nabla a}{a}\cdot\nabla\phi)=0 \qquad \text{on}\ \ \Gamma_t. \tag{3.11}$$

Thus

$$\phi_t-\Delta\phi-\frac{\nabla a}{a}\cdot\nabla\phi\leqslant-2+C_0\cdot\rho^2(x,t) \tag{3.12}$$
$$=-2+C_1\phi.$$

Combining (3.9) and (3.12) with the fact that $(\phi_{ij})\leqslant I$, we have

$$\frac{d}{dt}\int_Q\frac{1}{2}\phi\cdot a\Big[|\nabla u|^2+\frac{1}{2\varepsilon^2}\big(1-|u|^2\big)^2\Big]\leqslant C\int_Q\phi\frac{1}{2}a\Big[|\nabla u|^2+\frac{1}{2\varepsilon^2}\big(1-|u|^2\big)^2\Big]. \tag{3.13}$$

Now we use Gronwall's inequality and the assumption (H1) to obtain

$$\sup_{0\leqslant t\leqslant T}\int_Q\phi_\sigma\big(\rho(x,t)\big)\frac{1}{2}a\Big[|\nabla u|^2+\frac{1}{2\varepsilon^2}\big(1-|u|^2\big)^2\Big]dx\leqslant C(\sigma,T,K). \tag{3.14}$$

The last inequality implies that

$$\int_{Q\backslash\Gamma_t(\delta)}\frac{1}{2}a\Big[|\nabla u|^2+\frac{1}{2\varepsilon^2}\big(1-|u|^2\big)^2\Big]dx\leqslant C(\delta,\sigma,T,K), \tag{3.15}$$

for all $0\leqslant t\leqslant T$ and $0<\varepsilon<<1$.

Next, for $0 \leqslant t_1 \leqslant t \leqslant t_2 \leqslant T$, we let $\eta(x)$ be a smooth cutoff function supported in $Q \setminus \bigcup_{t_1 \leqslant t \leqslant t_2} \Gamma_t$; then

$$\begin{aligned}
(3.16) \qquad & \frac{d}{dt}\int_Q \eta^2(x)\frac{1}{2}a\Big[|\nabla u|^2 + \frac{1}{2\varepsilon^2}\big(1-|u|^2\big)^2\Big]dx \\
&= \int_Q \eta^2 a\Big[\nabla u\cdot\nabla u_t - \frac{1}{\varepsilon^2}(1-|u|^2)u\cdot u_t\Big] \\
&= -\int_Q \eta^2\Big[\nabla(a\nabla u) + \frac{1}{\varepsilon^2}(1-|u|^2)u\Big]u_t - 2\int_Q a\eta\nabla\eta\cdot\nabla u\cdot u_t \\
&= -\int_Q \eta^2(x)|u_t|^2 - 2\int_Q a\eta\nabla\eta\nabla u\cdot u_t \\
&\leqslant -\frac{1}{2}\int_Q \eta^2(x)|u_t|^2 + C\int_Q |\nabla\eta|^2|\nabla u|^2.
\end{aligned}$$

From (3.15) and (3.16), we obtain that

$$\|u_\varepsilon\|_{H^1_{loc}(\overline{Q}\times[0,T]\setminus\bigcup_{0\leqslant t\leqslant T}\Gamma_t)} \leqslant C.$$

The proof of Lemma 3.1 is completed. ∎

Hence, by taking a subseqence if necessary, we have

$$u_\varepsilon \rightharpoonup u_* \qquad \text{weakly in } H^1_{loc}\Big(\overline{Q}\times[0,T]\setminus\bigcup_{0\leqslant t\leqslant T}\Gamma_t\Big).$$

It is easy to verify that u_* satisfies

$$\frac{\partial u_*}{\partial t} = \frac{1}{a}\operatorname{div}(a\nabla u_*) + u_*|\nabla u_*|^2 \quad \text{in } H^1_{loc}\Big(\overline{Q}\times[0,T]\setminus\bigcup_{0\leqslant t\leqslant T}\Gamma_t\Big).$$

The proof of Theorem 1.1 is completed.

References

[1] S. Altschuler, 'Singularities of the curve shortening flow for space curves', *J. Differential Geom.* **34** (1991), 491–514.

[2] L. Ambrosio and H.M. Soner, 'Level set approach to mean curvature flow in arbitrary codimension', *J. Differental Geom.* **43** (1996), 693–737.

[3] R. Jerrard and M. Soner, 'Dynamics of Ginzburg-Landau vortices', *Arch. Rational Mech. Anal.* **142** (1998), 99–125..

[4] F.H. Lin, 'Some dynamical properties of Ginzburg-Landau vortices', *Comm. Pure Appl. Math.* **49** (1996), 323–359.

[5] F.H. Lin, 'A remark on the previous paper: "Some dynamical properties of Ginzburg-Landau vortices"', *Comm. Pure Appl. Math.* **49** (1996), 361–364.

[6] F.H. Lin, 'Static and moving vortices in Ginzburg-Landau theories', in *Nonlinear partial differential equations in geometry and physics (Knoxville, TN, 1995)*, Progr. Nonlinear Differential Equations Appl. **29** (Birkhauser, Basel, 1997), pp. 71–111.

[7] F.H. Lin, 'Complex Ginzburg-Landau equations and dynamics of vortices, filaments, and codimension-2 submanifolds', *Comm. Pure Appl. Math.* **51** (1998), 385–441.

[8] J.C. Neu, 'Vortices in complex scalar fields', *Phys. D.* **43** (1990), 385–406.

[9] J.C. Neu, 'Vortex dynamics of the nonlinear wave equation', *Phys. D.* **43** (1990), 407–420.

[10] L. Peres and J. Rubinstein, 'Vortex dynamics in U(1) Ginzburg-Landau models', *Phys. D.* **64** (1993), 299–309.

[11] J. Rubinstein, 'On the equilibrium position of Ginzburg-Landau vortices', *Z. Angew. Math. Phys.* **46** (1995), 739–751.

[12] L. Simon, *Lectures on geometric measure theory* (Centre for Math. Analysis, Australian National University, A.C.T., 1984).

[13] A. Stahl, 'Regularity estimates for solutions to the mean curvature flow with a Neumann boundary condition', *Calc. Var. Partial Differential Equations* **4** (1996), 385–407.

[14] E. Weinan, 'Dynamics of vortices in Ginzburg-Landau theories with applications to supconductivity', *Phys. D.* **77** (1994), 383–404.

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