CRITICALITY DEPENDENCE ON DATA AND PARAMETERS FOR A PROBLEM IN COMBUSTION THEORY

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Abstract

A central problem in the theory of combustion, consisting of a nonlinear parabolic equation together with initial and boundary conditions, is considered. The influence of the initial and boundary data are examined. In the main part of the study, a two-step linearization is developed such that the interesting features of the original problem are given by the solution of a non-linear ordinary differential equation. Approximate solutions are obtained and upper and lower solutions are used to assess the validity of the approximations. Whenever possible, results are compared with those obtained previously and there is good agreement in all cases.

I. Introduction

A central problem in combustion theory is the initial and boundary value problem (IBVP) consisting of the nonlinear parabolic equation

$$\frac{\partial \theta}{\partial t} = \nabla^2 \theta + \delta f(\theta); \qquad \mathbf{x} \in D, \, t > 0, \tag{1-1}$$

where $f(\theta) = \exp[\alpha\theta/(\alpha + \theta)]$, and the conditions

$$\theta(\mathbf{x}, 0) = h(\mathbf{x}) \geqslant 0, \qquad \mathbf{x} \in D, \tag{1-2}$$

and

$$\theta(\mathbf{x}, t) = g(\mathbf{x}, t); \qquad \mathbf{x} \text{ on } \partial D, \, t > 0. \tag{1-3}$$

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Here, θ is the temperature, x and t are the non-dimensionalized spatial and time variables, δ is a parameter incorporating the chemical properties of the combustible material, the temperature of assembly, as well as its geometrical dimensions, and α is a parameter related to the activation energy of the material occupying the region D. Typically, α is considerably greater than unity.

A number of investigators have studied this problem. The derivation of (1–1) and reference to the early work can be found in [4]. A usual simplification is the so-called "expanding the exponent", namely, the nonlinear term in (1–1) is replaced by $\exp(\theta)$. The result is that the time-independent version of (1–1) is reduced to the Liouville equation, which is an important equation used to describe a variety of physical phenomena, among them fluid mechanics and plasma physics. Exact solutions can be found for the Liouville equation for simple geometries, and the early analytical work on combustion theory is in this category. It is found that when δ exceeds a certain value, δ_{cr}, which is dependent on the geometry, the Liouville equation has no solution, which is interpreted to mean that ignition results. While the simpler equation gives useful information on δ_{cr}, it does have the disadvantage of losing track of the influence of α on δ_{cr}. More importantly, replacing the term $\exp[\alpha\theta/(\alpha + \theta)]$, which is bounded by e^α regardless of the value of θ, by the term $\exp(\theta)$, which grows exponentially with θ, changes the character of the nonlinear term. The IBVP (1–1), (1–2), (1–3), has for every L a unique solution bounded for all $0 < t < L$, [1], for all values of δ. Such a result is not obtainable for the simplified equation without further restrictions.

In 1961, Parks [6] integrated numerically equations (1–1), (1–2), and (1–3), with $g(x, t) = h(x) = 0$, and obtained the critical values δ_{cr}, for a range of values of α, for a number of simple geometrical shapes. In 1975, Sattinger [7] considered the stability of the multiple time-independent solutions corresponding to (1–1), (1–2), (1–3) with $h(x) = g(x, t) = 0$, using a comparison theorem for parabolic equations. An attempt was also made to construct an approximate solution, when $\delta < \delta_{cr}$ and θ remains small, based on the asymptotic consideration $\alpha \gg 1$. A number of studies on the critical value δ_{cr}, and its dependence on α, have been done by Fenaughty, Lacey and Wake [3], and Kordylewski [5], among others. These studies are based on the time-independent equation corresponding to (1–1), and the dependence of the number of solutions on α was examined.

The construction of upper and lower solutions for (1–1), (1–2), (1–3), and the influence of the initial and boundary data, was studied in a number of papers by Tam, [8], [9], [10], Tam and Kiang [14], Tam and Chapman [13]. These are also for simple geometric configurations. Comparison theorems as well as iteration schemes based on the integral equation derived from (1–1) are used.

In this paper, we present a procedure for the construction of upper and lower solutions. The only restriction on the domain D is that the eigenvalue problem $\nabla^2\varphi = -\lambda^2\varphi$; $x \in D$; $\varphi = 0$ on ∂D shall have a discrete spectrum of eigenvalues.

In II, we reconsider the construction of upper and lower solutions with homogeneous data. The importance of the first eigenfunction is emphasized. In III, we consider the influence of the initial data, with a view of constructing approximate, as well as upper and lower solutions. A two-step linearization of the given problem is introduced. In IV, we give a similar treatment for the influence of the boundary conditions. In V, we consider the effect of the Biot number on the critical parameter. Whenever possible, we compare the present results with those obtained previously. There is good agreement in all cases.

Throughout this paper, we deal with a number of differential or transcendental equations whose solutions depend on the parameters α and δ. They all share the following feature: When α is sufficiently large and δ is greater than a critical value, say δ_{cr}, which is particular to the specific equation under consideration, the equation has a unique solution whose maximum value is of order $O(e^\alpha)$. When δ is less than δ_{cr}, but greater than an extinction value, say δ_e, the equation has three solutions, the largest of order $O(e^\alpha)$ and the smallest of order $O(1)$. When δ is less than δ_e, there is again a unique solution of order $O(1)$. We refer to the large solution as "super-critical" and the small solution as "sub-critical". These terms and the symbols δ_{cr} and δ_e are used generically. When we wish to refer to δ_{cr} or δ_e of a particular equation, we make them specific by referring to the equation either by number or by context.

II. Upper and lower solutions with homogeneous data

In two previous papers [8], [10], the steady state upper solutions for the three simple geometries, viz, slab, circular cylinder, and sphere, were obtained in the manner to be described. Here, the result is presented for a general geometry, and the steady state is that corresponding to homogeneous boundary data. The construction for the lower solutions in [8], [10] was also based on the individual cases. Here, it is for a general geometry and given in terms of the first eigenfunction. The significance of the first eigenfunction was recognized in [9], but has not been exploited in the construction process.

IIa. The steady state

To start with, we take $\partial/\partial t = h(x) = g(x, t) = 0$, and we focus our attention on the influence of δ. Let λ_1^2 be the smallest eigenvalue with corresponding normalized eigenfunctions $\varphi_1(x) > 0$ of the problem

$$\nabla^2\varphi = -\lambda^2\varphi, \qquad x \in D, \tag{2-1}$$

$$\varphi(x) = 0, \qquad x \text{ on } \partial D. \tag{2-2}$$

Let k be a numerical constant whose magnitude is to be determined. We have:

LEMMA 1. *Let λ_1^2 and φ_1 be as defined in (2–1), (2–2) with φ_{1m} denoting the maximum value of φ_1 on D; and let k be the solution (the smallest if there is more than one solution) of*

$$\lambda_1^2 k \varphi_{1m} = \delta \exp\left[\alpha k \varphi_{1m}/(\alpha + k\varphi_{1m})\right].$$

Then $u = k\varphi_1$ is a lower solution of

$$\nabla^2\theta + \delta f(\theta) = 0, \qquad \mathbf{x} \in D, \tag{2–3}$$

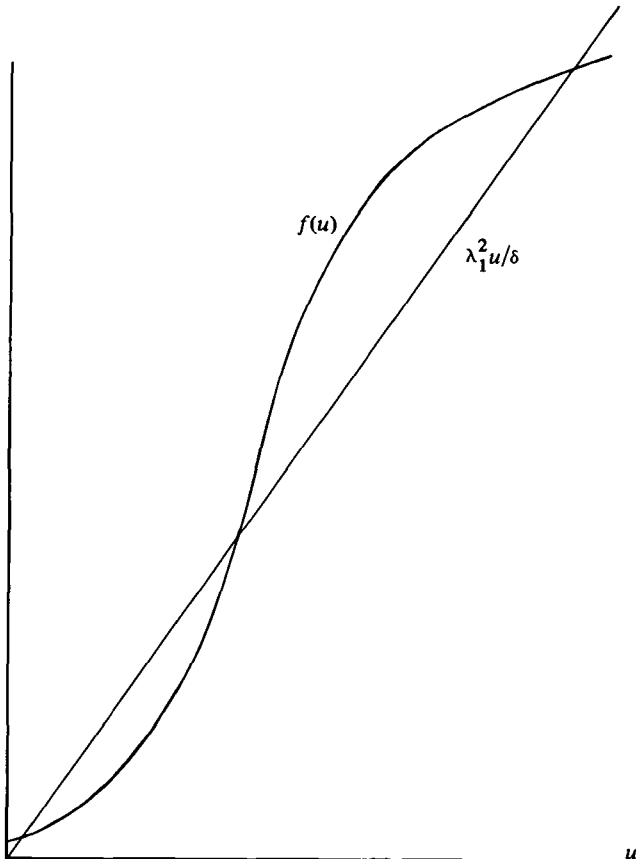$$\theta(\mathbf{x}) = 0, \qquad \mathbf{x} \in \partial D. \tag{2–4}$$

FIGURE I. $\lambda_1^2 u/\delta$ and $f(u)$ vs. u.

PROOF. We write $P\theta \equiv \partial\theta/\partial t - \nabla^2\theta - \delta f(\theta)$. Hence we have $Pu = \lambda_1^2 u - \delta f(u)$. Plotting $\lambda_1^2 u/\delta$ and $f(u)$ versus u in Figure I, it is clear that for $u \leqslant k\varphi_{1m}$, we have $f(u) \geqslant \lambda_1^2 u/\delta$, with the equality sign holding for $u = k\varphi_{1m}$. Since $0 < \varphi_1 \leqslant \varphi_{1m}$, we have $P(k\varphi_1) \leqslant 0$, implying that $k\varphi_1$ is a lower solution. Q.E.D.

LEMMA 2. *Let χ be the solution of*

$$\nabla^2\chi = -1, \qquad x \in D,$$

$$\chi(x) = 0, \qquad x \text{ on } \partial D,$$

and χ_m denote the maximum of χ in D. Let K be the solution (the smallest if there is more than one solution) of

$$K = \delta f(K\chi_m). \tag{2-5}$$

Then $U = K\chi$ is an upper solution of (2-3), (2-4).

PROOF. Since $f(\theta)$ is an increasing function of θ, we have

$$PU = K - \delta f(K\chi) \geqslant K - \delta f(K\chi_m). \tag{2-6}$$

Thus if K is determined from (2-5), we have $PU \geqslant 0$, implying that U is an upper solution. Q.E.D.

Now, from Lemma 1, we can obtain an upper bound for δ_{cr}, for equations (2-3), (2-4). It is clear from Figure I that if δ is increased from zero until $\lambda_1^2 u\delta^{-1} - f(u)$ has a double zero at $k\varphi_{1m}$, a critical value $\bar{\delta}_{cr}$ is reached. Further increase in δ will lead to a value of k which is exponentially large, that is, the lower solution for (2-3) and (2-4) becomes super-critical. Thus, a necessary condition for the solution of (2-3) and (2-4) to remain subcritical is that its lower solution $k\varphi_1$ must not be exponentially large, implying that we must have $\delta_{cr} < \bar{\delta}_{cr}$. The computation of $\bar{\delta}_{cr}$ is straightforward. We have

$$\bar{\delta}_{cr} = \lambda_1^2 u^* \exp[-\alpha u^*/(\alpha + u^*)]$$

where

$$u^* = \frac{1}{2}\left\{\alpha(\alpha - 2) - \sqrt{\alpha^3(\alpha - 4)}\right\}.$$

In a similar manner, we can also obtain from the upper solution a lower bound for δ_{cr}, say $\tilde{\delta}_{cr}$. We have

$$\tilde{\delta}_{cr} = K_1 \exp[-\alpha\chi_m K_1/(\alpha + \chi_m K_1)]$$

where

$$K_1 = u^*/\chi_m.$$

Thus, we have $\tilde{\delta}_{cr} = \bar{\delta}_{cr}/(\chi_m\lambda_1^2)$. We observe that these bounds cease to have any meaning if α is decreased until (2–6) has only one solution regardless of the value of δ.

IIb. The time-dependent case

With homogeneous initial and boundary data, the time-dependent case can be dealt with in an entirely analogous manner.

LEMMA 1′. *Let φ_1 be defined as before and $k(t)$ be a suitably chosen (as detailed in proof) increasing function of t, tending to k_m as $t \to \infty$. Then $u = k\varphi_1$ is a lower solution of* (1–1), (1–2), (1–3), *with $h(\mathbf{x}) = g(\mathbf{x}, t) = 0$.*

PROOF. Let φ_{1m} be the maximum value of $\varphi_1(x)$. We first consider $\delta \leqslant \bar{\delta}_{cr}$. We have

$$Pu = k'\varphi_1 + \lambda_1^2 k\varphi_1 - \delta f(k\varphi_1).$$

If $k(t)$ is determined from

$$k'(t) = \left\{ \delta f(k\varphi_{1m}) - \lambda_1^2 k\varphi_{1m} \right\}/\varphi_{1m}$$

satisfying $k(0) = 0$, it is easily seen that $k(t)$ is a monotonic increasing function, reaching the equilibrium value k_m given by

$$\delta f(k_m\varphi_{1m}) = \lambda_1^2 k_m\varphi_{1m}. \tag{2–7}$$

Thus, we have

$$Pu = \frac{\varphi_1}{\varphi_{1m}} \left\{ \delta f(k\varphi_{1m}) - \lambda_1^2 k\varphi_{1m} \right\} - \left\{ \delta f(k\varphi_1) - \lambda_1^2 k\varphi_1 \right\}.$$

For $0 < k < k_m$, quantities in both brackets are positive. Thus, we have

$$Pu \leqslant \left(\delta f(k\varphi_{1m}) - \lambda_1^2 k\varphi_{1m} \right) - \left(\delta f(k\varphi_1) - \lambda_1^2 k\varphi_1 \right). \tag{2–8}$$

From Figure I, it is clear that $\lambda_1 k\varphi_1 - \delta f(k\varphi_1) < \lambda_1 k\varphi_{1m} - \delta f(k\varphi_{1m})$, so that $Pu \leqslant 0$, implying that $u = k\varphi_1$ is a lower solution. For $\delta > \bar{\delta}_{cr}$, the inequality (2–8) no longer holds. In this case, we determined $k(t)$ from the equation

$$k'(t) = \frac{\varepsilon}{\varphi_{1m}} \left\{ \delta f(k\varphi_{1m}) - \lambda_1^2 k\varphi_{1m} \right\}$$

where $\varepsilon < 0$, and so

$$Pu = \varepsilon \left\{ \delta f(k\varphi_{1m}) - \lambda_1^2 k\varphi_{1m} \right\} + \left\{ \lambda_1^2 k\varphi_1 - \delta f(k\varphi_1) \right\}. \tag{2–9}$$

We observe that the quantity in the second bracket on the right is negative for $k\varphi_1 < k_m\varphi_{1m}$. Thus, if ε is sufficiently small, the right side of (2–9) remains negative for $k\varphi_{1m} < k_m\varphi_{1m}$, and so we have $Pu \leqslant 0$, again implying that $u = k\varphi$ is a lower solution. In this case, the solution of (2–7) is $O(e^\alpha)$.   Q.E.D.

As for the upper solution, we observe that for $g(\mathbf{x}, t) = h(\mathbf{x}) = 0$, the upper solution for the time-independent case also serves an upper solution for the time-dependent case. If we want to have an explicit time-dependent upper solution, we can use the upper solution for the time-dependent case to linearize the given differential equation. The procedure is straightforward, and will not be pursued.

III. The influence of the initial data

For δ in a certain range, and for α sufficiently large, the steady state equation admits more than one solution. The intial and/or boundary data then determine which steady state is realized. The influence of the initial data was investigated in [9], [10]. The approach there was to re-write the partial differential equation as an integral equation and to use an iteration scheme based on an asymptotic analysis of the large time behaviour of the solution. The initial data were used in the first iterate.

In the present paper, we approach the problem from the standpoint of constructing upper and lower solutions. We begin by considering an associated ordinary differential equation which we believe to contain the essence of the given non-linear problem. We then show that upper and lower solutions to the given problem can be constructed by using suitable multiples of the solution of the ordinary differential equation to linearize the given problem. A similar idea has been used by the author in the study of swirling-flow boundary layers in fluid mechanics [12]. Lastly, we compare the current result with those in [9], [10].

IIIa. The associated ordinary differential equation
Let $H(\mathbf{x}, t)$ denote the solution of the IBVP

$$H_t - \nabla^2 H = 0,$$

$$H(\mathbf{x}, 0) = h(\mathbf{x}) \geqslant 0, \qquad H(\mathbf{x}, t) = 0, \qquad \mathbf{x} \in \partial D.$$

We have

$$H(\mathbf{x}, t) = G(\mathbf{x}, \xi, t) \cdot h(\mathbf{x}) \equiv \int_D G(\mathbf{x}, \xi, t) h(\xi) \, dV, \qquad (3\text{–}1)$$

where

$$G(\mathbf{x}, \xi, t) = \sum_{n=1}^{\infty} e^{-\lambda_n^2 t} \varphi_n(\mathbf{x}) \varphi_n(\xi) \qquad (3\text{–}2)$$

and $\{\lambda_n^2\}$ and $\{\varphi_n(\mathbf{x})\}$ are respectively the eigenvalues and normalized eigenfunctions of (2-1) and (2-2). Let $T(t)$ denote the solution of the IVP

$$\frac{dT(t)}{dt} = -\lambda_1^2 T(t) + \delta \int_D \varphi_1(\boldsymbol{\xi}) \exp\left[\frac{\alpha(T(t)\varphi_1(\boldsymbol{\xi}) + H(\boldsymbol{\xi}, t))}{\alpha + T(t)\varphi\varphi_1(\boldsymbol{\xi}) + H(\boldsymbol{\xi}, t)}\right] dV, \qquad (3\text{-}3)$$

$$T(0) = 0. \qquad (3\text{-}4)$$

Equivalently, T can be expressed as the solution of the integral equation

$$T = \delta e^{-\lambda_1^2 t} \int_0^t e^{\lambda_1^2 \tau} \varphi_1(\boldsymbol{\xi}) \cdot \exp\left[\frac{\alpha(T(\tau)\varphi_1(\boldsymbol{\xi}) + H(\boldsymbol{\xi}, \tau))}{\alpha + T(\tau)\varphi_1(\boldsymbol{\xi}) + H(\boldsymbol{\xi}, \tau)}\right] d\tau. \qquad (3\text{-}5)$$

We observe that, since all eigenfunction components of $H(\mathbf{x}, t)$ decrease exponentially to zero, the equilibrium values of T are the solutions of the equation

$$\frac{\lambda_1^2 T}{\delta} = \int_D \varphi_1(\boldsymbol{\xi}) \exp\left[\frac{\alpha T \varphi_1(\boldsymbol{\xi})}{\alpha + T\varphi_1(\boldsymbol{\xi})}\right] dV. \qquad (3\text{-}6)$$
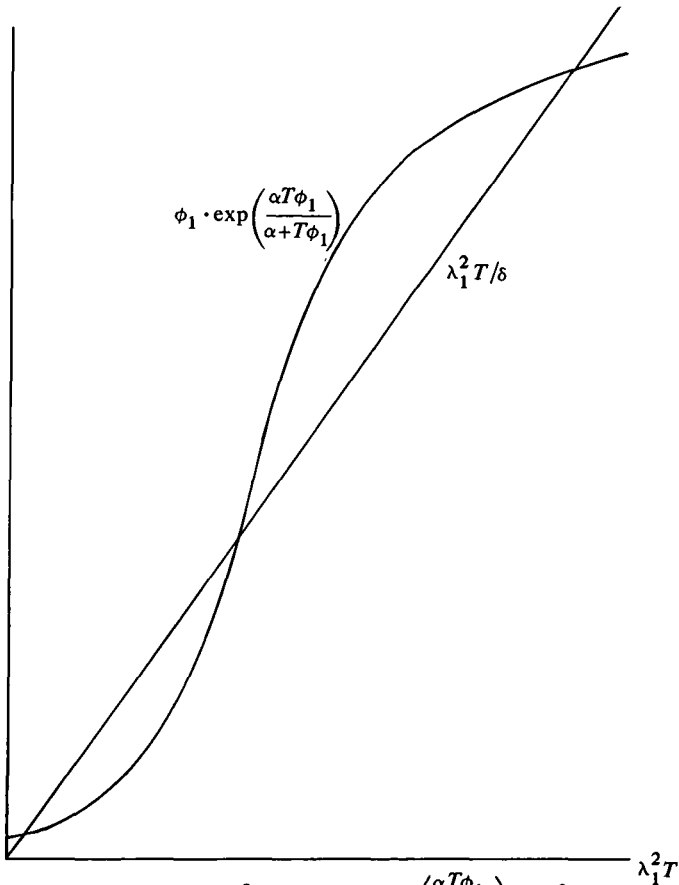

FIGURE II. $\lambda_1^2 T/\delta$ and $\phi_1 \cdot \exp\left(\dfrac{\alpha T\phi_1}{\alpha + T\phi_1}\right)$ vs. $\lambda_1^2 T$.

Plotting the two terms of (3-6) as solid lines in Figure II, we see that there are three solutions in the situation depicted. We label the minimum solution as sub-critical, and the maximum as super-critical. Clearly, there are two threshold values for δ, viz, δ_e, and δ_{cr}, such that for $\delta > \delta_{cr}$, there is only one (super-critical) solution; and for $\delta < \delta_e$, there is again only one (sub-critical) solution. We reiterate that at this point, the terms super-critical and sub-critical, as well as the symbols δ_e and δ_{cr}, refer to (3-5) only. Suppose $\delta_e < \delta < \delta_{cr}$. We first consider the particular case $h(\mathbf{x}) = c_1\varphi_1(\mathbf{x})$, where c_1 is a constant. We have $H(\mathbf{x}, t) = c_1 \exp(-\lambda_1^2 t)\varphi_1(\mathbf{x})$. Let $y = T(t) + c_1 \exp(-\lambda_1^2 t)$. In terms of y, (3-3) and (3-4) become

$$\frac{dy}{dt} = -\lambda_1^2 y + \delta \int_D \varphi_1(\xi)\exp\left\{\frac{\alpha y(t)\varphi_1(\xi)}{\alpha + y(t)\varphi_1(\xi)}\right\} dV, \qquad (3\text{-}7)$$

$$y(0) = c_1. \qquad (3\text{-}8)$$

Except for the changed initial data, $y(t)$ has the same equilibrium values as $T(t)$. Thus, it is clear from Figure II that if $y(0) = c_1 > v^*/\lambda_1^2$, where v^* is the middle intersection of the curve $\varphi_1 \exp[\alpha T\varphi_1/(\alpha + T\varphi_1)]$ with the straight line $\lambda_1^2 T/\delta$, then y reaches the super-critical state, and if $y(0) < v^*/\lambda_1^2$, y reaches the sub-critical state. We note that $y(0) \gtrless v^*/\lambda_1^2$ implies that

$$\delta \int_D \varphi_1(\xi)\exp\left\{\frac{\alpha h(\xi)}{\alpha + h(\xi)}\right\} dV \gtrless v^*. \qquad (3\text{-}9)$$

In terms of T, we have

$$\frac{dT(0)}{dt} = \delta \int_D \varphi_1(\xi)\exp\left\{\frac{\alpha h(\xi)}{\alpha + h(\xi)}\right\} dV.$$

Thus, we conclude that T reaches the super-critical state if $dT(0)/dt > v^*$, and it reaches the sub-critical state if $dT(0)/dt < v^*$.

We now consider the case of an arbitrary $h(\mathbf{x})$. We write

$$H(\mathbf{x}, t) = \sum_{n=1}^{\infty} c_n \exp(-\lambda_n^2 t)\varphi_n(\mathbf{x}) \equiv c_1 \exp(-\lambda_1^2 t)\varphi_1(\mathbf{x}) + J(\mathbf{x}, t)$$

where

$$c_n = \int_D \varphi_n(\mathbf{x})h(\mathbf{x}) dV.$$

We again let $y = T + c_1 e^{-\lambda_1^2 t}$. Equations (3-3) and (3-4) become

$$\frac{dy}{dt} = -\lambda_1^2 y + \delta\varphi_1 \cdot \exp\left\{\frac{\alpha[y\varphi_1 + J]}{\alpha + y\varphi_1 + J}\right\}, \qquad (3\text{-}10)$$

$$y(0) = c_1; \qquad \frac{dy(0)}{dt} = -\lambda_1^2 c_1 + \delta\varphi_1 \cdot e^{\alpha h/(\alpha + h)}. \qquad (3\text{-}11)$$

We seek conditions under which y tends to the super-critical state. Let $I(t)$ be defined as

$$I(t) = \varphi_1(\mathbf{x}) \cdot \exp\left\{\frac{\alpha\big(c_1\varphi_1(\mathbf{x})\big) + J(x,t)}{\alpha + c_1\varphi_1(x) + J(x,t)}\right\}.$$

Let $h(\mathbf{x})$ be such that the following two conditions are satisfied.

$$(1)\qquad I(0) = \varphi_1 \cdot \exp\left\{\frac{\alpha h}{\alpha + h}\right\} > \varphi_1 \cdot \exp\left\{\frac{\alpha c_1\varphi_1}{\alpha + c_1\varphi_1}\right\}\qquad (3\text{--}12a)$$

where $c_1 > v^*/\lambda_1^2$;

$$(2)\qquad \frac{dI(t)}{dt} < 0 \quad \text{for } 0 < t < \infty.\qquad (3\text{--}12b)$$

Since $\delta\varphi_1 \cdot \exp\{\alpha c_1\varphi_1/(\alpha + c_1\varphi_1)\} > -\lambda_1^2 c_1$, we have $dy(0)/dt > 0$.

Suppose y does not tend to the super-critical state. Then dy/dt must become negative and there exists $t_1 < \infty$ at which $y(t_1) = c_1$, with

$$\frac{dy(t_1)}{dt} = -\lambda_1^2 c_1 + \delta\varphi_1 \cdot \exp\left\{\frac{\alpha\big(c_1\varphi_1 + J(x,t_1)\big)}{\alpha + c_1\varphi_1 + J(x,t_1)}\right\} \leqslant 0.$$

However, by (3–12), $I(t)$ decreases monotonically to $\varphi_1 \exp[\alpha c_1\varphi_1/(\alpha + c_1\varphi_1)]$. Thus, we have

$$\delta\varphi_1 \cdot \exp\left\{\frac{\alpha\big(c_1\varphi_1 + J(x,t_1)\big)}{\alpha + c_1\varphi_1 + J(x,t_1)}\right\} - \lambda_1^2 c_1 > \delta\varphi_1 \cdot \exp\left\{\frac{\alpha c_1\varphi_1}{\alpha + c_1\varphi_1}\right\} - \lambda_1^2 c_1$$

$$> 0,$$

contradicting $dy(t_1)/dt \leqslant 0$. Thus, y cannot resume to value c_1, and so y tends to the super-critical state as $t \to \infty$. We summarize this result in

LEMMA 3. *Let v^* and $I(t)$ be as defined. If $h(\mathbf{x})$ satisfies the two conditions (3.12a) and (3.12b), then the solution of equations (3–10), (3–11), and hence T, tends to the super-critical value.*

In an analogous manner, we have

LEMMA 3'. *Let v^* and $I(t)$ be as defined in Lemma 3. If $h(\mathbf{x})$ satisfies the two conditions*
    (1) $I(0) = \varphi_1 \cdot \exp\{\alpha h/\alpha + h\} < \varphi_1 \cdot \exp\{\alpha c_1\varphi_1/(\alpha + c_1\varphi_1)\}$, *where* $c_1 < v^*/\lambda_1^2$;
    (2) $dT/dt < 0$ *for* $0 < t < \infty$,
then the solution of equations (3–10), (3–11), and hence T, tends to the subcritical value.

A different set of sufficient conditions giving the same conclusion as in the above Lemmas can be formulated. We observe that $J(\mathbf{x}, t)$ is the solution of the heat equation

$$\frac{\partial J}{\partial t} = \nabla^2 J; \qquad \mathbf{x} \in D; \ t > 0,$$

subject to

$$J(\mathbf{x}, 0) = \sum_{n=2}^{\infty} c_n \varphi_n(\mathbf{x}); \qquad J = 0 \quad \text{on } \partial D.$$

Suppose $-a\varphi_1(\mathbf{x}) < J(\mathbf{x}, 0) < A\varphi_1(\mathbf{x})$ for suitably determined non-negative a, A. It is readily seen that

$$-a \exp\left[-\lambda_1^2 t\right] \varphi_1(\mathbf{x}) < J(\mathbf{x}, t) < A \exp\left[-\lambda_1^2 t\right] \varphi_1(\mathbf{x}).$$

Now we consider the equation

$$\frac{d\tilde{y}}{dt} = -\lambda_1^2 \tilde{y} + \delta \int_D \varphi_1(\xi) \exp\left[\frac{\alpha\left(\tilde{y}(t)\varphi_1(\mathbf{x}) - a\exp(-\lambda_1^2 t)\varphi_1(\mathbf{x})\right)}{\alpha + \tilde{y}(t)\varphi_1(\mathbf{x}) - a\exp(-\lambda_1^2 t)\varphi_1(\mathbf{x})}\right] dV$$

with

$$\tilde{y}(0) = c_1, \qquad \frac{d\tilde{y}(0)}{dt} = -\lambda_1^2 c_1 + \delta\varphi_1 \cdot \exp\left\{\frac{\alpha(c_1 - a)\varphi_1}{\alpha + (c_1 - a)\varphi_1}\right\}.$$

Recalling that the function $\exp[\alpha u/(\alpha + u)]$ is an increasing function of u, it follows from a comparison theorem for ordinary differential equation that $\tilde{y} < y$. We note that the value v^* determined before is also the middle intersection of the curve $\varphi_1 \exp[\alpha \tilde{y}\varphi_1/(\alpha + \tilde{y}\varphi_1)]$ with $\lambda_1^2 \tilde{y}/\delta$. Thus, if we require the function $h(x)$ to satisfy the two conditions (1) $c_1 \geqslant v^*/\lambda_1^2$ and

(2) $\delta\varphi_1 \cdot \exp\{\alpha(c_1 - a)\varphi_1/(\alpha + (c_1 - a)\varphi_1)\} - \lambda_1^2 c_1 > 0,$

we have $d\tilde{y}(0)/dt > 0$. We claim $d\tilde{y}/dt$ can tend to zero only as $t \to \infty$. Suppose the contrary is true. Let $d\tilde{y}/dt = 0$ at t_1, with $d\tilde{y}/dt > 0$ for $0 \leqslant t < t_1$. Then at t_1, we have

$$\frac{d^2\tilde{y}}{dt^2} = \delta\alpha^2 \int_D \frac{a\lambda_1^2 \exp(-\lambda_1^2 t)\varphi_1^2(\mathbf{x})}{\left[\alpha + \tilde{y}(t)\varphi_1(\mathbf{x}) - a\exp(-\lambda_1^2 t)\varphi_1(\mathbf{x})\right]^2}$$

$$\cdot \exp\left[\frac{\alpha\left(\tilde{y}(t)\varphi_1(\mathbf{x}) - a\exp(-\lambda_1^2 t)\varphi_1(\mathbf{x})\right)}{\alpha + \tilde{y}(t)\varphi_1(\mathbf{x}) - a\exp(-\lambda_1^2 t)\varphi_1(\mathbf{x})}\right] dV$$

which is positive, leading to a contradiction. Thus $d\tilde{y}/dt > 0$ for all finite t, and $\tilde{y}$, and hence y and T all tend to the same super-critical value. The two conditions imposed on $h(\mathbf{x})$ together imply

$$\delta \int_D \varphi_1(\xi) \exp\left\{\frac{\alpha[h(\mathbf{x}) - J(\mathbf{x}, 0) - a\varphi_1(\mathbf{x})]}{\alpha + h(\mathbf{x}) - J(\mathbf{x}, 0) - a\varphi_1(\mathbf{x})}\right\} dV > \lambda_1^2 c_1 > v^*. \quad (3\text{-}13)$$

We can summarize the above result as:

LEMMA 4. *Let v^*, J and a be as defined above. If $h(\mathbf{x})$ satisfies (3–13), the solution of (3–10), (3–11), and hence T, tends to the super-critical value.*

In an analogous manner, we have:

LEMMA 4′. *Let v^*, J and A be as defined above. If $h(\mathbf{x})$ satisfies the condition*

$$\delta \int_D \varphi_1(\xi) \exp\left\{ \frac{\alpha[h(\mathbf{x}) - J(\mathbf{x}, 0) + A\varphi_1(\mathbf{x})]}{\alpha + h(\mathbf{x}) - J(\mathbf{x}, 0) + A\varphi_1(\mathbf{x})} \right\} dV < \lambda_1^2 c_1 < v^*, \quad (3\text{–}14)$$

the solution of (3–10), (3–11), and hence T, tends to the sub-critical value.

We observe that the quantity $\delta\varphi_1 \exp[\alpha h/(\alpha + h)]$ is bounded from below by the integral in (3–13), and from above by the integral in (3–14). These bounds are close when the derivations $(J(\mathbf{x}, 0) + a\varphi_1)$ and $(A\varphi_1 - j(\mathbf{x}, 0))$ are small.

We have considered in some detail the behaviour of a function $T(t)$. In the next section, we shall use this function in the construction of upper and lower solutions for $\theta(\mathbf{x}, t)$; and also obtain an approximate solution for $\theta(\mathbf{x}, t)$.

IIIb. Construction of upper and lower solutions

Let H and T be as defined in (3–1) and (3–5). Let $m(\mathbf{x}) > 0$ be a function to be determined. We consider the IBVP

$$\chi_t - \nabla^2\chi = \delta \exp\left[\frac{\alpha(mT\varphi_1 + H)}{\alpha + mT\varphi_1 + H} \right]$$

subject to $\chi(\mathbf{x}, 0) = h(\mathbf{x})$; $\chi(\mathbf{x}, t) = 0$, $\mathbf{x} \in \partial D$. Since the equation is linear, its solution can be obtained readily. We have (see [2], page 291)

$$\chi(\mathbf{x}, t) = H(\mathbf{x}, t) + \delta \int_0^t G(\mathbf{x}, \xi, t\tau) \cdot \exp\left[\frac{\alpha(mT(\tau)\varphi_1(\xi) + H(\xi, \tau))}{\alpha + mT(\tau)\varphi_1(\xi) + H(\xi, \tau)} \right] d\tau.$$

$$(3\text{–}15)$$

Our objective is to choose $m(\mathbf{x})$ so that $\chi(\mathbf{x}, t)$ is a lower solution of equation (1–1), (1–2) and (1–3), with $g(\mathbf{x}, t) = 0$, i.e. we wish to make

$$P\chi = \chi_t - \nabla^2\chi - \delta \exp\left(\frac{\alpha\chi}{\alpha + \chi} \right) \leqslant 0.$$

Now, we have

$$P\chi = \delta\left\{ \exp\left[\frac{\alpha(mT(t)\varphi_1(\mathbf{x}) + H(\mathbf{x}, t))}{\alpha + mT(t)\varphi_1(\mathbf{x}) + H(\mathbf{x}, t)} \right] - \exp\left[\frac{\alpha\chi(\mathbf{x}, t)}{\alpha + \chi(\mathbf{x}, t)} \right] \right\}.$$

Since the function $\exp[\alpha u/(\alpha + u)]$ is an increasing function of u, we will have $P\chi \leqslant 0$ if we can choose m to make

$$mT\varphi_1 + H \leqslant H + \delta \int_0^t G(\mathbf{x}, \xi, t - \tau) \cdot \exp\left[\frac{\alpha(mT(\tau)\varphi_1(\xi) + H(\xi, \tau))}{\alpha + mT(\tau)\varphi_1(\xi) + H(\xi, \tau)}\right] d\tau$$

or

$$mT(t)\varphi_1(\mathbf{x}) \leqslant \delta \int_0^t G(\mathbf{x}, \xi, t - \tau) \cdot \exp\left\{\frac{\alpha(mT(\tau)\varphi_1(\xi) + H(\xi, \tau))}{\alpha + mT(\tau)\varphi_1(\xi) + H(\xi, \tau)}\right\} d\tau.$$

$$(3-16)$$

From equation (3–5), we have

$$mT(t)\varphi_1(\mathbf{x}) = \delta \int_0^t G(\mathbf{x}, \xi, t - \tau) \cdot \exp\left[\frac{\alpha(mT(\tau)\varphi_1(\xi) + H(\xi, \tau))}{\alpha + mT(\tau)\varphi_1(\xi) + H(\xi, \tau)}\right] d\tau$$

$$+ \delta m \int_0^t e^{-\lambda_1^2(t-\tau)} \varphi_1(\mathbf{x})\varphi_1(\xi) \cdot \exp\left[\frac{\alpha(T(\tau)\varphi_1(\xi) + H(\xi, \tau))}{\alpha + T(\tau)\varphi_1(\xi) + H(\xi, \tau)}\right] d\tau$$

$$- \delta \int_0^t G(\mathbf{x}, \xi, t - \tau) \cdot \exp\left[\frac{\alpha(mT(\tau)\varphi_1(\xi) + H(\xi, \tau))}{\alpha + mT(\tau)\varphi_1(\xi) + H(\xi, \tau)}\right] d\tau. \quad (3-17)$$

Since the exponential function in the integrands is bounded between 1 and $\exp(\alpha)$, and both φ_1 and G are positive, it is clear that by choosing m sufficiently small, the sum of the second and third integrals on the right of the above equation can be made to be negative, thus ensuring that equation (3–16) is satisfied, and therefore implying that the function $\chi(\mathbf{x}, t)$ given by (3–15) is a lower solution. An entirely analogous procedure with $M > 0$ replacing m yields an upper solution. By examining equation (3–14) we see that the deviation of m and M from unity is governed by the magnitude of

$$\left\{\sum_{n=2}^{\infty} e^{-\lambda_n^2(t-\tau)} \varphi_n(\mathbf{x})\varphi_n(\xi)\right\} \cdot \exp\left\{\frac{\alpha(T(\tau)\varphi_1(\xi) + H(\xi, \tau))}{\alpha + T(\tau)\varphi_1(\xi) + H(\xi, \tau)}\right\}.$$

Since the shape of the exponential function is not too different from that of const. $\varphi_1(\xi)$, we expect the above quantity, and hence the deviation of m and M from unity, to be small. In view of this, we can consider χ as given by (3–15) with $m = 1$ as an approximation for θ, the solution of the original problem. An examination of this solution shows that if T becomes large, (labelled as super-critical), χ will also be large, and if T remains small (labelled as sub-critical), χ will also remain small. Thus, the dependence of T on δ and $h(\mathbf{x})$ implies a corresponding dependence of χ on δ and $h(\mathbf{x})$, and this is the essence of the two-step linearization that links T, χ and θ together.

As an illustration, we have computed $m(x)$ and $M(x)$ in the form of step functions, for the case of an infinite slab, $0 < x < 1$, taking $\alpha = 20$, and

$H(\xi, \tau) = 5 \exp(-\lambda_1^2 \tau)\varphi_1(\xi) = 5\sqrt{2} \exp(-\pi^2 \tau)\sin \pi\xi$. We carried out the computation for a time interval of $0 < t \leqslant 1.5$, with a step size of 0.1 in t as well as x. We find

$$m(x) = \begin{cases} 1.0, & 0 < x < 0.3, \quad 0.7 < x < 1, \\ 0.75, & 0.3 \leqslant x \leqslant 0.7, \end{cases}$$

and

$$M(x) = \begin{cases} 3.1, & 0 < x < 0.2, \quad 0.8 < x < 1, \\ 1.1, & 0.2 \leqslant x \leqslant 0.8. \end{cases}$$

Thus, for $0 < t < 1.5$, $\chi(m)$ and $\chi(M)$ are respectively lower and upper solutions of (1–1), (1–2) and (1–3) with $g = 0$ and $h = 5\sqrt{2} \sin \pi x$. In this case, to use $m = 1$ to provide an approximate solution gives quite reasonable results.

IIIc. Comparison with previous result

Previously, the author has considered the influence of the initial value by using an iteration scheme, together with certain large-time asymptotic analysis. We now demonstrate that the present approach yields similar results. Retaining the notation in [9], [10], [11] we have

$$\theta_{n+1} \sim K_n \delta U_1/\lambda_1^2$$

and

$$\theta_{n+2} \sim K_{n+1} \delta U_1/\lambda_1^2,$$

where

$$K_{n+1} = U_1 \cdot \exp\left\{ \frac{\alpha K_n \delta U_1/\lambda_1^2}{\alpha + K_n \delta U_1/\lambda_1^2} \right\}.$$

Here, U_1 is the first normalized eigenfunction with corresponding eigenvalue λ_1^2. If we rewrite $K_0\delta/\lambda_1^2$ as T, i.e. $K_n \equiv \lambda_1^2 T/\delta$, then the above equation becomes $K_{n+1} = U_1\exp[\alpha T U_1/(\alpha + T U_1)]$. Our previous result shows that if $K_{n+1} > K_n$, then the iterates $\{\theta_n\}$ increase to a limit, and if $K_{n+1} < K_n$, $\{\theta_n\}$ decrease to a limit. In particular, if δ is in a range permitting multiple steady-state solutions, then the solution will be super-critical if $K_n > v^*/\delta$, and sub-critical if $K_n < v^*/\delta$, where v^* is the middle intersection of the curve K_{n+1} with K_n plotted against $v = K_n\delta$. Referring to equation (3–6) and Figure II, it is clear that the comparison of K_n with K_{n+1} is exactly the comparison of the two terms of (3–6). Thus, the values for δ_e, δ_{cr} and v^* obtained from the previous asymptotic results are precisely those obtainable from the associated ordinary differential equation studied here. The initial data $\theta(x, 0) = h(x)$ were used previously to calculate $U_1\exp[\alpha h/(\alpha + h)]$, which was then compared with v^*/δ. The solution θ reaches the super-critical state if $\delta U_1\exp[\alpha h/(\alpha + h)] > v^*$, and reaches the sub-critical

state if $\delta U_1 \exp[\alpha h/(\alpha \in h)] < v^*$. In Lemmas 3.3′, 4.4′, we have obtained sufficient conditions on h for T to become sub- or super-critical. By inference, these are also sufficient conditions for χ to be sub- or super-critical. It does appear that the asymptotic result is consistent with the present results. We have thus obtained some relevant information on the initial data and parameter dependence for the original nonlinear parabolic equation by studying an associated ordinary differential equation.

IV. Influence of the boundary condition

In this section, we examine the influence of the boundary condition in determining the nature of the steady state. Thus, we set the initial value to zero. We first partition the problem into two parts so as to make the effect of the boundary condition more transparent.

Let $\theta = P + \psi$, where P is governed by the system

$$P_t - \nabla^2 P = 0, \tag{4-1}$$

$$P(\mathbf{x}, 0) = 0; \qquad P(\mathbf{x}, t) = g(\mathbf{x}, t) \geqslant 0, \mathbf{x} \in \partial D, \tag{4-2}$$

and ψ is governed by the system

$$\psi_t - \nabla^2 \psi = \delta f(P + \psi), \tag{4-3}$$

$$\psi(\mathbf{x}, 0) = 0; \qquad \psi(\mathbf{x}, t) = 0, \mathbf{x} \in \partial D. \tag{4-4}$$

The solution of (4–1) and (4–2) is given by (see [2], page 290)

$$P(\mathbf{x}, t) = -\int_0^t \int_S \frac{\partial G(\mathbf{x}, \xi, t - \tau)}{\partial n_\xi} g(\xi, \tau)\, dS_\xi\, d\tau. \tag{4-5}$$

We observe that $g \geqslant 0$ implies that $P(\mathbf{x}, t) \geqslant 0$, and we suppose that g is suitably restricted to ensure that $P(\mathbf{x}, \infty)$ exists. Here, we differentiate between the two cases $P(\mathbf{x}, \infty) = 0$ and $P(\mathbf{x}, \infty) = C(x) \not\equiv 0$.

To begin with, let $\delta < \delta_{cr}$, and let

$$B = \max_{0 < t < \infty} \int_D \varphi_1 \exp\left(\frac{\alpha P}{\alpha + P}\right) dV.$$

Suppose B occurs at $t = t_1 > 0$. Since $t_1 > 0$ implies $\psi(\mathbf{x}, t_1) > 0$, it is clear that at t_1, we have

$$\int_D \varphi_1 \exp\left(\frac{\alpha(P + \psi)}{\alpha + P + \psi}\right) dV > B. \tag{4-6}$$

With t_1 taken as the new initial point for (4–3) and (4–4), the same consideration as in section III shows that for ψ, and hence θ, to reach the super-critical state, it is sufficient to have $(P + \psi)t_1$ satisfying conditions imposed on h as detailed in

IIIa. However, since we do not know the value of ψ at t_1, we have to be content with the sufficiency condition being imposed on δB. For the case of $P(\mathbf{x}, \infty) = 0$, the above is the only information we can obtain. However, more can be said about the case of $P(\mathbf{x}, \infty) > 0$, $\mathbf{x} \in D$; and in what follows, our attention is confined to this case. We note that the problem of a slab with unsymmetric boundary temperatures considered by Tam and Chapman [13], among others, belongs to this case.

IVa. The associated ordinary differential equation

Let $T(t)$ be the solution of the ordinary differential equation

$$\frac{dT}{dt} = -\lambda_1^2 T + \delta \int_D \varphi_1(\xi) \exp\left[\frac{\alpha(T(t)\varphi_1(\xi) + P(\xi, t))}{\alpha + T(t)\varphi_1(\xi) + P(\xi, t)} \right] dV \qquad (4\text{–}7)$$

subject to $T(0) = 0$. Equivalently, T can be expressed as the solution of the integral equation

$$T = \delta \int_0^t e^{-\lambda_1^2(t-\tau)} \varphi_1(\xi) \cdot \exp\left[\frac{\alpha(T(\tau)\varphi_1(\xi) + P(\xi, \tau))}{\alpha + T(\tau)\varphi_1(\xi) + P(\xi, \tau)} \right] d\tau. \qquad (4\text{–}8)$$

Here, we have used T to denote a somewhat different quantity as in Section III, but the possibility of confusion is more than compensated for by avoiding a proliferation of symbols. In Figure III, we superpose on Figure II the new curve

$$\varphi_1 \cdot \exp\left[\frac{\alpha(T\varphi_1 + P(\xi, \infty))}{\alpha + T\varphi_1 + P(\xi, \infty)} \right].$$

It is clear that $P(\xi, \infty) > 0$ implies that the new curve always lies above the curve $\varphi_1 \cdot \exp[\alpha T\varphi_1/(\alpha + T\varphi_1)]$. Further, for a given $\delta < \delta_{cr}$, it is clear that if $P(\xi, \infty)$ is sufficiently large, the new curve will intersect the straight line $\lambda_1^2 T/\delta$ only at one point. Moreover, since

$$\varphi_1 \cdot \exp\left\{ \frac{\alpha(T\varphi_1 + P(\xi, \infty))}{\alpha + T\varphi_1 + P(\xi, \infty)} \right\}$$

is bounded from above by $\varphi_1 \cdot e^\alpha$ no matter how large P is, it is clear that the intersection point decreases monotonically to zero with δ.

We now pose the same question as in Section III: for a given δ, $\delta_e < \delta < \delta_{cr}$, such that $\lambda_1^2 T$ intersects

$$\varphi_1 \cdot \exp\left\{ \frac{\alpha T\varphi_1}{\alpha + T\varphi_1} \right\}$$

at three points, how large must $P(\mathbf{x}, \infty)$ be for the solution of (4–7) to be super-critical?

We construct the curve

$$\varphi_1 \cdot \exp\left(\frac{\alpha(T\varphi_1 + C)}{\alpha + T\varphi_1 + C}\right)$$

where C is a constant. We increase the value of C from zero to C^*, at which

$$\varphi_1 \cdot \exp\left(\frac{\alpha(T\varphi_1 + C^*)}{\alpha + T\varphi_1 + C^*}\right)$$

becomes tangent to $\lambda_1^2 T/\delta$ at the point $\lambda_1^2 T = b^*$, as depicted schematically in Figure III. Now, suppose for a given $g(x, t)$, we have

$$\varphi_1 \cdot \exp\left[\frac{\alpha(T\varphi_1 + P(\xi, \infty))}{\alpha + T\varphi_1 + P(\xi, \infty)}\right] > \varphi_1 \cdot \exp\left[\frac{\alpha(T\varphi_1 + C^*)}{\alpha + T\varphi_1 + C^*}\right] \quad \text{for } T > 0,$$
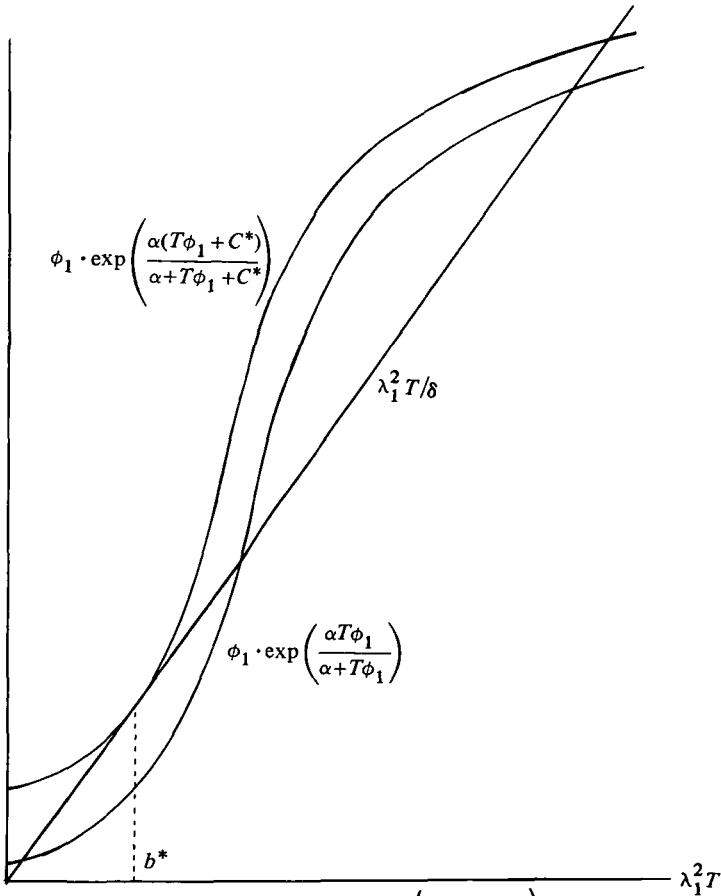

FIGURE III. $\lambda_1^2 T/\delta$ and $\phi_1 \cdot \exp\left(\dfrac{\alpha(T\phi_1 + C)}{\alpha + T\phi_1 + C}\right)$ vs. $\lambda_1^2 T$.

then it is clear from (4–7) and Figure III that there is only one equilibrium value for T, which is super-critical—in the sense that it is larger than the maximum of the intersections of $\lambda_1^2 T/\delta$ and

$$\varphi_1 \cdot \exp\left[\frac{\alpha T\varphi_1 + C^*}{\alpha + T\varphi_1 + C^*}\right].$$

Conversely, if

$$\varphi_1 \cdot \exp\left[\frac{\alpha(T\varphi_1 + P(\xi, \infty))}{\alpha + T\varphi_1 + P(\xi, \infty)}\right] < \varphi_1 \cdot \exp\left[\frac{\alpha(T\varphi_1 + C^*)}{\alpha + T\varphi_1 + C^*}\right],$$

then the equilibrium value of T will be sub-critical,—in the sense that it is less than b^*/λ_1^2. Thus, the number C^* provides a demarcation value in assessing the influence of the boundary data $g(x, t)$. Clearly, C^* depends on δ.

As an illustration, we consider the case of an infinite slab, $0 < x < 1$, with non-zero boundary conditions. The quantities λ_1 and φ_1 are known from previous studies to be $\lambda_1 = \pi$ and $\varphi_1 = \sqrt{2} \sin \pi x$. We take $\alpha = 20$, $\delta = 5.61 \times 10^{-3}$, and we compute C^* to be 11.91, and $b^* = 25.5$. Now, if we take the non-zero boundary conditions to $\theta(0, t) = A$ and $\theta(1, t) = 0$, we find that $P(x, \infty) = A(1 - x)$. By varying the value of A, we see that the solution is sub-critical if $A < 20$ and super-critical if $A > 20$, a result agreeing quite well with that given in Tam and Chapman [13].

IVb. Construction of upper and lower solutions

Let P and T be as defined in (4–5) and equation (4–8). Let $m(x) > 0$ be a function to be determined. We consider the IBVP

$$\chi_t - \nabla^2 \chi = \delta \exp\left[\frac{\alpha(mT\varphi_1 + P)}{\alpha + mT\varphi_1 + P}\right] \tag{4–9}$$

subject to $\chi(x, 0) = 0$; $\chi(x, t) = g(x, t) > 0$, $x \in \partial D$. The solution of (4–9) and (4–10) can be represented by (see [2], page 291)

$$\chi(x, t) = \delta \int_0^t G(x, \xi, t - \tau) \cdot \exp\left[\frac{\alpha(mT(\tau)\varphi_1(\xi) + P(\xi, \tau))}{\alpha + mT(\tau)\Phi_1(\xi) + P(\xi, \tau)}\right] d\tau$$
$$+ P(x, t). \tag{4–11}$$

As in section IIIb, we want to choose m so that $\chi(x, t)$ is a lower solution of equations (1–1), (1–2) and (1–3), with $h(x) = 0$. Thus, we are led to the requirement

$$mT(t)\varphi_1(x) \leqslant \delta \int_0^t G(x, \xi, t - \tau)$$
$$\cdot \exp\left[\frac{\alpha(mT(\tau)\varphi_1(\xi) + P(\xi, \tau))}{\alpha + mT(\tau)\varphi_1(\xi) + P(\xi, \tau)}\right] d\tau. \tag{4–12}$$

From equation (4–7) we have

$$mT(t)\varphi_1(x) = \delta m \int_0^t \exp\left[-\lambda_1^2(t - \tau)\right]\varphi_1(x)\varphi_1(\xi)$$

$$\cdot \exp\left[\frac{\alpha(T(\tau)\varphi_1(\xi) + P(\xi,\tau))}{\alpha + T(\tau)\varphi_1(\xi) + P(\xi,\tau)}\right]d\tau.$$

Proceeding in exactly the same manner as in section IIIb, we see that a function $m(x)$ can be chosen to satisfy (4–12), thus making $\chi(x, t)$ a lower solution. In an analogous manner, and as in section IIIb, we can construct an upper solution. In particular, we can consider the solution of (4–8) and (4–9) with $m = 1$ as a linearized approximate solution to the original problem.

IVc. The combined influence of the initial and boundary conditions

We now consider the full problem comprising equations (1–1), (1–2) and (1–3). It is clear that if either $h(x)$ or $g(x, t)$ is such that one of the criteria obtained in Sections III and IV for super-criticality is met, the solution will be super-critical. The question therefore is: suppose taken separately, neither $h(x)$ nor $g(x, t)$ causes super-criticality, will their combined effect be super-critical? To answer this, we let $S(x, t)$ be the solution of IBVP

$$S_t - \nabla^2 S = 0,$$

$$S(x,0) = h(x); \qquad S(x, t) = g(x, t), x \in \partial D.$$

If we now let $\theta = \psi + S$, equations (1–1), (1–2) and (1–3) become

$$\psi_t - \nabla^2\psi = \delta \exp\left[\frac{\alpha(\psi + S)}{\alpha + \psi + S}\right],$$

$$\psi(x,0) = 0; \qquad \psi(x, t) = 0, x \in \partial D,$$

and the solution can be expressed in terms of an integral equation.

$$\psi = \delta \int_0^t G(x, \xi, t - \tau) \cdot \exp\left[\frac{\alpha(\psi + S(\xi,\tau))}{\alpha + \psi + S(\xi,\tau)}\right]d\tau.$$

Now, if at any time t_1, $S(x, t)$ evolves to such an extent that $(\psi + S)_{t_1}$ satisfies the sufficiency conditions as imposed on h in III, then our previous results show that the solution of (1–1), (1–2) and (1–3) is super-critical. However, as remarked after (4–6), since we do not know ψ at t_1, we can only impose the sufficiency condition for super-criticality on S.

V. The influence of the Biot number in the case of general boundary conditions

Instead of the boundary condition (1–3), a more general condition, taking into account Newtonian cooling at the surface, is

$$\frac{\partial \theta}{\partial \nu} + \beta\theta = 0, \quad x \in \partial D, \tag{5–1}$$

where ν is the outward normal direction. The parameter $\beta > 0$ is referred to as the Biot number. The case $\beta \to \infty$ yields the homogeneous boundary condition $\theta = 0$. Now, with $0 < \beta < \infty$, the boundary temperature will not be fixed at zero. Consequently, the heat conduction away from the surface will be less pronounced, implying that more heat will be retained in the body. Thus, we expect the super-critical state will result with a smaller value of δ, and/or lower initial temperature profile. Because of the relative ease of the required computation, we shall treat the case with non-homogeneous initial data via the integral-equation approach.

Let $\{\chi_n\}$ and $\{\mu_n^2\}$ be the normalized eigenfunctions and eigenvalues, respectively, of the BVP

$$\nabla^2 \chi_n = -\mu_n^2 \chi_n, \qquad \mathbf{x} \in D,$$

$$\frac{\partial \chi_n}{\partial \nu} + \beta \chi_n = 0, \qquad \mathbf{x} \in \partial D,$$

and let $K(\mathbf{x}, \xi, t)$ be the Robin function for the system (see page 289, [2])

$$K_t - \nabla^2 K = 0,$$

$$K(\mathbf{x}, 0) = 0; \qquad \frac{\partial K}{\partial \nu} + \beta K = 0; \mathbf{x} \in \partial D.$$

We have

$$K(\mathbf{x}, \xi, t) = \sum_{n=1}^{\infty} \exp(-\mu_n^2 t) \chi_n(\mathbf{x}) \chi_n(\xi). \tag{5-2}$$

Clearly, the eigenvalues now depend on the parameter β. We rewrite equations (1–1), (1–2) and (5–1) as the integral equation

$$\theta(\mathbf{x}, t) = K(\mathbf{x}, \xi, t) \cdot h(\xi) + \delta \int_0^t K(\mathbf{x}, \xi, t - \tau) \cdot \exp\left\{ \frac{\alpha \theta(\xi, \tau)}{\alpha + \theta(\xi, \tau)} \right\} d\tau. \tag{5-3}$$

Following our previous work, we can define an iteration process, and then use an asymptotic argument to assess the influence of both $h(\mathbf{x})$ and β. For the three simple geometries (Class A), the respective eigenvalues and eigenfunctions are given as follows:

(a) Slab: $-1 < x < 1$

$$\chi_n = \left(\frac{2\mu_n}{2\mu_n + \sin^2 \mu_n} \right)^{1/2} \cos \mu_n x$$

where μ_n is determined from

$$\mu_n \tan \mu_n = \beta.$$

TABLE Ia. Values of δ_e, δ_{cr}, and v^*

for some values of α and β for the slab.

	$\beta = 1.0$		$\beta = 10$	
	δ	v^*	δ	v^*
	$\delta_e = 3.75 \times 10^{-2}$	48.2	$\delta_e = 0.10535$	119.8
$\alpha = 8$	0.25	3.05	0.5	14
	$\delta_{cr} = 0.314$	1.41	$\delta_{cr} = 0.855$	3.5
	$\delta_c = 1.5826 \times 10^{-6}$	372.2	$\delta_e = 4.47 \times 10^{-6}$	932
$\alpha = 20$	0.1	4.2	0.2	13.1
	$\delta_{cr} = 0.286$	1.2	$\delta_{cr} = 0.775$	2.7
	$\delta_e = 1.34 \times 10^{-14}$	1576	$\delta_e = 3.796 \times 10^{-14}$	3954.5
$\alpha = 40$	0.1	3.65	0.2	10
	$\delta_{cr} = 0.27027$	1.10	$\delta_{cr} = 0.7519$	2.6
	$\delta_e =$ overflow		$\delta_e =$ overflow	
$\alpha = 100$	0.1	3.3	0.2	9.2
	$\delta_{cr} = 0.2703$	1.0	$\delta_{cr} = 0.735$	2.55

TABLE Ib. Values of δ_e, δ_{cr}, and v^*

for some values of α and β for the slab.

	$\beta = 100$		$\beta = 10000$	
	δ	v^*	δ	v^*
	$\delta_e = 0.1247$	136	$\delta_e = 0.1274$	138.3
$\alpha = 8$	0.5	16	0.2	16.4
	$\delta_{cr} = 1.00$	4.0	$\delta_{cr} = 1.0204$	4.0
	$\delta_e = 5.293 \times 10^{-6}$	1058	$\delta_e = 5.4085 \times 10^{-6}$	1075
$\alpha = 20$	0.2	14.6	0.2	14.8
	$\delta_{cr} = 0.9174$	3.1	$\delta_{cr} = 0.926$	3.3
	$\delta_e = 4.494 \times 10^{-14}$	4488	$\delta_e = 4.592 \times 10^{-14}$	4561
$\alpha = 40$	0.2	12.1	0.2	13.7
	$\delta_{cr} = 0.885$	3.05	$\delta_{cr} = 0.910$	3.1
	δ_e overflow		δ_e overflow	
$\alpha = 100$	0.2	11.1	0.2	11.4
	$\delta_{cr} = 0.8696$	3.0	$\delta_{cr} = 0.885$	3.0

TABLE IIa. Values of δ_e, δ_{cr}, and v^*
for some values of α and β for the sphere.

	$\beta = 1.0$		$\beta = 10$	
	δ	v^*	δ	v^*
$\alpha = 8$	$\delta_e = 0.1252$	232.6	$\delta_e = 0.42986$	597.4
	$\delta = 0.8333$	15	$\delta = 1.67$	65.5
	$\delta_{cr} = 1.036$	7.3	$\delta_{cr} = 3.265$	16.8
$\alpha = 20$	$\delta_e = 5.28 \times 10^{-6}$	1796	$\delta_e = 1.8446 \times 10^{-5}$	4732.8
	$\delta = 0.5$	15.4	$\delta = 0.5$	64
	$\delta_{cr} = 0.952$	5.5	$\delta_{cr} = 2.91$	14.2
$\alpha = 40$	$\delta_e = 4.4746 \times 10^{-14}$	7609	$\delta_e = 1.571 \times 10^{-13}$	20170
	$\delta = 0.5$	13.7	$\delta = 0.67$	47
	$\delta_{cr} = 0.93$	5.1	$\delta_{cr} = 2.857$	13.8
$\alpha = 100$	δ_e overflow		δ_e overflow	
	$\delta = 0.5$	12.9	$\delta = 0.67$	43
	$\delta_{cr} = 0.92$	5.0	$\delta_{cr} = 2.759$	12.6

TABLE IIb. Values of δ_e, δ_{cr} and v^*
for some values of α and β for the sphere.

	$\beta = 100$		$\beta = 10000$	
	δ	v^*	δ	v^*
$\alpha = 8$	$\delta_e = 0.519$	629.2	$\delta_e = 0.5295$	632.4
	$\delta = 2$	70	$\delta = 2$	71.5
	$\delta_{cr} = 3.846$	18.75	$\delta_{cr} = 3.922$	19.5
$\alpha = 20$	$\delta_e = 2.23 \times 10^{-5}$	4980.4	$\delta_e = 2.275 \times 10^{-5}$	5005.2
	$\delta = 2$	317	$\delta = 2$	37.2
	$\delta_{cr} = 3.478$	14.4	$\delta_{cr} = 3.555$	15.5
$\alpha = 40$	$\delta_e = 1.8996 \times 10^{-13}$	21216	$\delta_e = 1.9379 \times 10^{-13}$	21321.8
	$\delta = 2$	32	$\delta = 2$	32.2
	$\delta_{cr} = 3.404$	14.0	$\delta_{cr} = 3.478$	13.8
$\alpha = 100$	δ_e overflow		δ_e overflow	
	$\delta = 2$	29.5	$\delta = 2$	30
	$\delta_{cr} = 3.265$	13.9	$\delta_{cr} = 3.333$	13.2

TABLE IIIa. Values of δ_e, δ_{cr} and v^*
for some values of α and β for the cylinder.

	$\beta = 1.0$		$\beta = 10$	
	δ	v^*	δ	v^*
$\alpha = 8$	$\delta_e = 0.1687$	241.6	$\delta_e = 0.285$	345
	0.333	15	$\delta = 1$	26
	$\delta_{cr} = 0.676$	4.0	$\delta_{cr} = 1.951$	10
$\alpha = 20$	$\delta_e = 7.1482 \times 10^{-6}$	1882	$\delta_e = 1.218 \times 10^{-5}$	2711
	0.2	8.1	$\delta = 0.5$	28.3
	$\delta_{cr} = 0.606$	3.1	$\delta_{cr} = 1.768$	7.7
$\alpha = 40$	$\delta_e = 6.066 \times 10^{-14}$	7991.7	$\delta_e = 1.0358 \times 10^{-13}$	11529
	0.2	9.8	$\delta = 0.25$	24.3
	$\delta_{cr} = 0.592$	3.0	$\delta_{cr} = 1.728$	7.3
$\alpha = 100$	δ_e overflow		δ_e overflow	
	$\delta = 0.2$	5.9	$\delta = 0.25$	22.4
	$\delta_{cr} = 0.588$	2.7	$\delta_{cr} = 1.684$	7.2

TABLE IIIb. Values of δ_e, δ_{cr} and v^*
for some values of α and β for the cylinder.

	$\beta = 100$		$\beta = 10000$	
	δ	v^*	δ	v^*
$\alpha = 8$	$\delta_e = 0.30274$	356	$\delta_e = 0.305$	357
	$\delta = 0.833$	57.5	0.986	50.7
	$\delta_{cr} = 2.286$	11.0	$\delta_{cr} = 2.350$	10.6
$\alpha = 20$	$\delta_e = 1.2934 \times 10^{-5}$	2795	$\delta_e = 1.30 \times 10^{-5}$	2804
	$\delta = 0.4$	38	.706	42.5
	$\delta_{cr} = 2.051$	8.5	$\delta_{cr} = 2.1086$	8.5
$\alpha = 40$	$\delta_e = 1.10 \times 10^{-13}$	11884	$\delta_e = 1.107 \times 10^{-13}$	11922
	$\delta = 0.667$	26.2	0.189	40
	$\delta_{cr} = 2.025$	8.0	$\delta_{cr} = 2.055$	7.96
$\alpha = 100$	δ_e overflow		δ_e overflow	
	$\delta = 0.667$	23.8	0.1130	40
	$\delta_{cr} = 1.975$	7.5	$\delta_{cr} = 2.01926$	7.697

TABLE IV. Comparison of α_{tr} and δ_{tr}
with results from Fenaughty et al.

Biot No.		Slab		Cylinder		Sphere	
		present result	Fenaughty et al.	present result	Fenaughty et al.	present result	Fenaughty et al.
$\beta = 0.01$	α_{tr}	4.00004	4	4.0015	4	4.000002	4
	δ_{tr}	0.005395	0.00542	0.03038	0.01085	0.016273	0.01629
$\beta = 0.1$	α_{tr}	4.00023	4.00016	4.0015	4.00016	4.00017	4.00016
	δ_{tr}	0.052359	0.05262	0.323925	0.10609	0.1591718	0.15992
$\beta = 1$	α_{tr}	4.01242	4.01220	4.0558	4.01494	4.01469	4.01445
	δ_{tr}	0.399377	0.40049	1.734455	0.85234	1.33080	1.33340
$\beta = 10$	α_{tr}	4.0642	4.06306	4.12352	4.11709	4.16843	4.16406
	δ_{tr}	1.083952	1.08454	2.826147	2 48202	4.14390	4.14461
$\beta = 100$	α_{tr}	4.069395	4.06851	4.12833	4.130013	4.19165	4.18708
	δ_{tr}	1.278937	1.28168	2.991818	2.94682	4 93478	4.94085
$\beta = 10000$	α_{tr}	4.0694	4.06868	4.12875	4.130354	4.19196	4.18761
	δ_{tr}	1.306765	1.30709	3.00997	3.00557	5.03346	5.03980

(b) infinite cylinder: $0 < r < 1$

$$\chi_n = \frac{\mu_n}{\sqrt{\pi\left(\mu_n^2 + \beta^2\right)} J_0(\mu_n)} J_0(\mu_n r)$$

where μ_n is determined from

$$\beta J_0(\mu_n) = \mu_n J_1(\mu_n).$$

(c) sphere: $0 < r < 1$

$$\chi_n = \left[\frac{\mu_n}{\pi\left(2\mu_n - \sin k^2\mu_n\right)}\right]^{1/2} \frac{\sin \mu_n r}{r}$$

where μ_n is determined from

$$\cos \mu_n = 0; \qquad \beta = 1,$$

$$\tan \mu_n = \frac{1}{1 - \beta}\mu_n, \qquad \beta \neq 1.$$

Representative computations for δ_{cr}, δ_e, and v^* for some values of β and α for these cases are presented in Tables I, II and III. The numbers in these three tables are obtained graphically. In Table IV, we list the transitional values of α, with the corresponding δ_{tr}, for the three class A geometries. These numbers are de-termined numerically on an IBM personal computer in an iterative manner. The

corresponding values obtained by Fenaughty *et al.* [3] are listed for comparison. The agreement is good except for the values of δ_{tr} for the cylinder for $\beta < 10$. We recall the transitional value α_{tr} is the value of a below which the time-independent version of equations (1–1), (1–3) has no multiple solutions, and δ_{tr} is the limit of δ_{cr} as α approaches α_{tr} from above.

VI. Concluding remarks

We have studied a nonlinear parabolic equation arising from the theory of autocatalytic reactions. For the case with homogeneous initial and Dirichlet boundary conditions, and when the parameter α is sufficiently large, we constructed upper and lower solutions from which definitive bounds on the critical value of δ are obtained. The role of the first eigenfunction is emphasized. We then consider the influence of the initial data. A two-step linearization method is used to demonstrate a procedure for the construction of upper and lower solutions, from which an approximate solution is obtained. The essence of the two-step method is that the dependence of the solution of the original problem on initial data and parameters can be inferred from the dependence of the solution of an associated ordinary differential equation on those quantities. Sufficient conditions for the solution of the ordinary differential equation to become super-critical, and hence the solution of the original nonlinear parabolic equation to become sub- or super-critical are obtained. Results obtained previously from an asymptotic method are shown to be consistent with the current results. Since the case with non-zero initial data includes the case with homogeneous initial data as a special case, the approximate evaluation of δ_{cr} can be checked against the definitive bounds obtained in II. The influence of the boundary condition is considered next and tied in with the influence of the initial data. Lastly, we introduce a new parameter, the Biot number, in the form of a Robin boundary condition, and consider its influence of the solution. Values of δ_e, δ_{cr}, v^* and α_{cr} as given by the asymptotic analysis are computed for a wide range of the Biot number, for the class A geometries.

Of some interest is the two-step linearization method introduced in III. It is a capitalization of the recognition that the first eigenfunction associated with the linear problem plays a significant role in the nonlinear problem. Indeed, the derivation of the ordinary differential equation is based on what we have learned about the behaviour of the given parabolic equation, and is in the sense a culmination of our past efforts. The idea of tackling a difficult problem in two steps is not a new one, and conceivably this method may be used in other situations.

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