Can. J. Math., Vol. XXXIV, No. 2, 1982, pp. 349–355

SIMPLE LINKS IN LOCALLY COMPACT CONNECTED HAUSDORFF SPACES ARE NONDEGENERATE

DAVID JOHN

1. Introduction. The fact that simple links in locally compact connected metric spaces are nondegenerate was probably first established by C. Kuratowski and G. T. Whyburn in [**2**], where it is proved for Peano continua. J. L. Kelley in [**3**] established it for arbitrary metric continua, and A. D. Wallace extended the theorem to Hausdorff continua in [**4**]. In [**1**], B. Lehman proved this theorem for locally compact, locally connected Hausdorff spaces. We will show that the locally connected property is not necessary.

2. Definitions. A *continuum* is a compact connected Hausdorff space. For any two points a and b of a connected space M, $E(a, b)$ denotes the set of all points of M which separate a from b in M. The *interval ab* of M is the set $E(a, b) \cup \{a, b\}$.

The following theorem appears in [**7**], where it is proved only for the metric case. The proof of the non-metric case was established in [**5**].

THEOREM 1. *If M is a connected space, a and b are points of M, p is a point of M not in the interval ab of M, and M is semi-locally connected at p, then there exists a closed connected subset N of M such that a and b are points in N and N is a subset of M-$\{p\}$.*

The following theorem appears in [**6**]; however, it is stated only for metric spaces and only a suggestion for the proof is given which does not generalize to non-metric spaces.

THEOREM 2. *If the connected space M is semi-locally connected at p and M-$\{p\}$ has exactly k components, then for each open set U containing p, there exists an open set V containing p such that $V \subset U$ and $M - V$ has exactly k components.*

Proof. Let

$$M - \{p\} = \bigcup_{i=1}^{k} K_i,$$

where K_i is a component of M-$\{p\}$ for $i = 1, \ldots, k$. Let U be an open set containing p and let p_i be a point of K_i for $i = 1, \ldots, k$. There exists

Received May 10, 1980 and in revised form October 7, 1981.

an open set W such that $p \in W$, $W \subset U$, $p_i \notin W$ for $i = 1, \ldots, k$, and

$$M - W = \bigcup_{j=1}^{n} C_j,$$

where C_j is a component of $M - W$ for $j = 1, \ldots, n$. Now $n \geqq k$, and each C_j is a subset of some K_i. Let x_j be a point in C_j for $j = 1, \ldots, n$, and let $\{C_j | 1 \leqq j \leqq m\}$ be the collection of all components of $M - W$ that are subsets of K_1. Since p is not in the interval $x_1 x_j$ for $j = 2, \ldots, m$, it follows from Theorem 1 that there exist closed connected sets A_j for $j = 2, \ldots, m$ such that x_1 and x_j are points in A_j and $p \notin A_j$. Let

$$B_1 = \bigcup_{j=2}^{m} A_j.$$

Thus B_1 is a closed connected set.

Now for each $i = 1, \ldots, k$, there exists a closed connected set B_i such that if $x_j \in K_i$, then $x_j \in B_i$ and $p \notin B_i$. There exists an open set G such that

$$p \in G, G \subset W, G \cap \bigcup_{i=1}^{k} B_i = \emptyset,$$

and $M - G$ has only finitely many components, $D_1, \ldots, D_h$. Clearly $h \geqq k$. Since B_i is a connected subset of $M - G$ for $i = 1, \ldots, k$, let B_i be a subset of D_i for $i = 1, \ldots, k$.

Let $\{C_r | r = 1, \ldots, l\}$ be the components of $M - W$ such that $C_r \cap B_i \neq \emptyset$ for a fixed i, where $1 \leqq i \leqq k$. Since $C_r \subset M = W$ and $\bigcup_{r=1}^{l} C_r \cup B_i$ is connected,

$$\bigcup_{r=1}^{l} C_r \cup B_i \subset D_i.$$

Let

$$V = W - \bigcup_{i=1}^{k} D_i.$$

V is open and

$$M - V = M - W \cup \bigcup_{i=1}^{k} D_i.$$

Now

$$M - W = \bigcup_{j}^{n} C_j \quad \text{and}$$

$$\bigcup_{j}^{n} C_j \subset \bigcup_{i=1}^{k} D_i.$$

https://doi.org/10.4153/CJM-1982-022-2 Published online by Cambridge University Press

Hence

$$M - V = \bigcup_{i=1}^{k} D_i,$$

and it follows that $M - V$ has exactly k components.

THEOREM 3. *If p is a non-cut point of the connected space M, and M is semi-locally connected at p, then for each open set U containing p, there exists an open subset V of U containing p such that $M - V$ is connected.*

Proof. Since M-$\{p\}$ has only one component, let $k = 1$ in Theorem 2.

Definitions. If $\{M_{\mathfrak{a}}, \mathfrak{a} \in \mathscr{A}\}$ is a net of sets, then lim inf $M_{\mathfrak{a}}$ is the set of all points p such that for each open set U containing p, there exists $\mathfrak{b} \in \mathscr{A}$ such that for each $\mathfrak{a} > \mathfrak{b}$, $\mathfrak{a} \in \mathscr{A}$, $M_{\mathfrak{a}}$ intersects U. The lim sup $M_{\mathfrak{a}}$ is the set of all points p such that for each open set U containing p and for each $\mathfrak{b} \in \mathscr{A}$, there exists $\mathfrak{a} \in \mathscr{A}$ such that $\mathfrak{a} > \mathfrak{b}$ and $M_{\mathfrak{a}}$ intersects U. If

$$\liminf M_{\mathfrak{a}} = \limsup M_{\mathfrak{a}} = L,$$

then L is denoted by $\lim M_{\mathfrak{a}}$ and $\{M_{\mathfrak{a}}, \mathfrak{a} \in \mathscr{A}\}$ is said to *converge* to L. A nondegenerate continuum K in a space M is a *continuum of convergence* if and only if there is a net $\{K_{\mathfrak{a}}, \mathfrak{a} \in \mathscr{A}\}$ of continua such that for each $\mathfrak{a}$ in $\mathscr{A}$,

$$K \cap K_{\mathfrak{a}} = \emptyset \quad \text{and} \quad K = \lim K_{\mathfrak{a}}.$$

THEOREM 4. *If M is a locally compact connected space, and M is not semi-locally connected at the point p in M, then there exists an open set U containing p such that if V is a proper open subset of U containing p, then $M - V$ has infinitely many components that intersect both ∂U and ∂V.*

Proof. Since M is locally compact and not semi-locally connected at p, there exists a proper open subset U of M such that $p \in U$, $\bar{U}$ is compact, and for each open set G with $p \in G$ and $G \subset U$, $M - G$ has infinitely many components. Let G be an open subset of U containing p. Assume there exist only finitely many components that intersect both ∂U and ∂G. Let $\mathscr{C}$ be the collection of all components of $M - G$. Let

$$\mathscr{H} = \{C \in \mathscr{C} / C \cap \partial G = \emptyset\},$$
$$\mathscr{K} = \{C \in \mathscr{C} / C \cap \partial U = \emptyset \quad \text{and} \quad C \cap \partial G \neq \emptyset\}, \quad \text{and}$$
$$\mathscr{L} = \{C \in \mathscr{C} / C \cap \partial U \neq \emptyset \quad \text{and} \quad C \cap \partial G \neq \emptyset\}.$$

By assumption, $\mathscr{L}$ is finite.

If $H \in \mathscr{H}$, then H does not contain a limit point of $\cup \mathscr{K}$. If $K \in \mathscr{K}$, then K does not contain a limit point of $\cup \mathscr{H}$. Suppose there exists a point x in $\overline{\cup \mathscr{H}} \cap \cup \mathscr{K}$. Let $\mathscr{W}$ be the collection of all open sets containing x. For each W in $\mathscr{W}$, let $x_w \in \cup \mathscr{H} \cap U \cap W$ and let C_w be the component

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of $\bar{U} - G$ containing x_w. Since C_w is a connected subset of $M - G$, C_w lies in some element of $\mathscr{H}$. Now, $C_w \cap \partial G = \emptyset$, and hence

$$C_w \cap \partial U \neq \emptyset.$$

Consider the net $\{C_w, w \in \mathscr{W}\}$. Since $x \in \lim\sup C_w$, some subnet of $\{C_w, w \in \mathscr{W}\}$ converges to a continuum C containing x. Now $C \cap \partial U \neq \emptyset$, but C lies in some element of $\mathscr{K}$, which is a contradiction. Hence

$$\overline{\cup\mathscr{H}} \cap \cup\mathscr{K} = \emptyset.$$

Suppose there exists a point y in $\cup\mathscr{H} \cap \overline{\cup\mathscr{K}}$. Let $\mathscr{E}$ be the collection of all open sets containing y. For each E in $\mathscr{E}$, let $y_E \in \cup\mathscr{K} \cap U \cap E$, and let C_E be the component of $\bar{U} - G$ containing y_E. Since C_E is a connected subset of $M - G$, C_E lies in some element of $\mathscr{K}$. Now $C_E \cap \partial U = \emptyset$, and hence $C_E \cap \partial G \neq \emptyset$. Consider the net $\{C_E, E \in \mathscr{E}\}$. Since $y \in \lim\sup C_E$, some subnet of $\{C_E, E \in \mathscr{E}\}$ converges to a continuum D containing y, $D \cap \partial G \neq \emptyset$, but D lies in some element of $\mathscr{H}$, which is a contradiction. Hence $\cup\mathscr{H} \cap \overline{\cup\mathscr{K}} = \emptyset$. Therefore $\cup\mathscr{H}$ is separated from $\cup\mathscr{K}$.

Suppose $\mathscr{H}$ is infinite. Since $\mathscr{L}$ is finite, there exist separated sets A and B such that

$$\cup\mathscr{H} \cup \cup\mathscr{L} = A \cup B \quad \text{and} \quad \cup\mathscr{L} \subset B.$$

Then $M = A \cup \cup\mathscr{K} \cup B \cup G$ and A is separated from $\cup\mathscr{K} \cup B \cup G$, which is a contradiction. Hence $\mathscr{H}$ is finite. Suppose $\mathscr{H} \neq \emptyset$. Then $\cup\mathscr{H}$ is separated from $\cup\mathscr{L}$, and hence $\cup\mathscr{H}$ is separated from $\cup\mathscr{K} \cup \cup\mathscr{L} \cup G$. However,

$$M = \cup\mathscr{H} \cup \cup\mathscr{K} \cup \cup\mathscr{L} \cup G,$$

which is a contradiction, and thus $\mathscr{H} = \emptyset$. Therefore

$$M = \cup\mathscr{K} \cup \cup\mathscr{L} \cup G.$$

Let

$$V = M - \cup\mathscr{L} = \cup\mathscr{K} \cup G.$$

Now V is the union of G and all components of $M - G$ lying entirely in U. Since each element L in $\mathscr{L}$ is closed, V is open, and $p \in V$ and $V \subset U$. Now $M - V = \cup\mathscr{L}$ has only a finite number of components, which implies M is semi-locally connected at p, but this is a contradiction. Hence the theorem is proved.

THEOREM 5. *Let M be a connected space, and let U be a proper open subset of M such that $\bar{U}$ is compact. If V is an open subset of U and C is a component of $M - V$ such that $C \cap \partial V \neq \emptyset$ and $C \cap \partial U \neq \emptyset$, then there exists a component K of $\bar{U} - V$ such that $K \subset C$, $K \cap \partial V \neq \emptyset$, and $K \cap \partial U \neq \emptyset$.*

Proof. Let C, U, and V be as described in the hypothesis. Assume no component of $C \cap \bar{U}$ intersects both ∂U and ∂V. Suppose L is a component of $C \cap \bar{U}$ such that $L \cap \partial V = \emptyset$ and $L \cap \partial U = \emptyset$. Then L is a component of $U - \bar{V}$. $U - \bar{V}$ is a proper open subset of M, and $\overline{U - \bar{V}}$ is compact, which implies L has a limit point in $\partial \overline{U - \bar{V}}$. Hence L has a limit point in ∂U or in ∂V. Since L is a closed subset of $C \cap \bar{U}$, $C \cap \partial U \neq \emptyset$ or $C \cap \partial V \neq \emptyset$, which is a contradiction. Thus each component of $C \cap \bar{U}$ intersects ∂U or ∂V.

Let $\mathscr{K}$ be the collection of all components of $C \cap \bar{U}$ that intersect ∂U, and let $\mathscr{H}$ be the collection of all components of $C \cap \bar{U}$ that intersect ∂V. Suppose $\mathscr{K} = \emptyset$. If $\mathscr{H}$ is finite, then $\cup \mathscr{H}$ is separated from $C - \bar{U}$ and $\cup \mathscr{H} \cup (C - \bar{U}) = C$, which is a contradiction. Hence $\mathscr{H}$ is infinite. Since $\cup \mathscr{H} \cup (C - \bar{U}) = C$ and $\cup \mathscr{H}$ is not separated from $C - \bar{U}$, $\cup \mathscr{H}$ has a limit point p in ∂U. Since each element of $\mathscr{H}$ is a continuum in $\bar{U}$, some net of elements of $\mathscr{H}$ converges to a continuum A containing p. Each member of this net contains a point of ∂V. Hence A intersects both ∂U and ∂V. $C \cap \bar{U}$ is closed, and therefore $A \subset C \cap \bar{U}$, which contradicts our assumption that $\mathscr{K} = \emptyset$. If $\mathscr{H} = \emptyset$, then $C = \cup \mathscr{K} \cup (C - \bar{U})$ does not intersect ∂U. Hence $\mathscr{H} \neq \emptyset$.

Suppose $\cup \mathscr{H}$ contains a limit point q of $\cup \mathscr{K}$. Then some net of elements of K converges to a continuum B containing q. Since each element of this net contains a point in ∂U, $B \cap \partial U \neq \emptyset$. However, B must be a subset of the element of $\mathscr{H}$ that contains q, which implies some element of $\mathscr{H}$ intersects ∂U, and this is a contradiction. Hence $\overline{\cup \mathscr{K}} \cap \cup \mathscr{H} = \emptyset$. Similarly, $\cup \mathscr{K} \cap \overline{\cup \mathscr{H}} = \emptyset$. Now $\cup \mathscr{H}$ is separated from $C - \bar{U}$, and thus $\cup \mathscr{H}$ is separated from $\cup \mathscr{K} \cup (C - \bar{U})$. However,

$$\cup \mathscr{H} \cup \cup \mathscr{K} \cup (C - \bar{U}) = C,$$

which is a contradiction. Hence some component of $C \cap \bar{U}$ intersects both ∂U and ∂V.

Whyburn [**6**] established the following:

THEOREM 6. *If the locally compact connected metric space M is not semi-locally connected at a point $p \in M$, then p lies on a continuum of convergence of M.*

Theorem 6 need not be true for non-metric spaces. However, we do get the somewhat weaker result.

THEOREM 7. *If M is a locally compact space and M is not semi-locally connected at the point p, then there exists a net that converges to a nondegenerate continuum K containing p such that each member of the net is a continuum of convergence.*

Proof. By Theorem 4, there exists an open set U containing p such that $\bar{U}$ is compact and if V is an open set with $p \in V$ and $\bar{V} \subset U$, then

$M - V$ has infinitely many components that intersect both ∂U and ∂V. Let $\mathscr{V}$ be the collection of all open subsets of U such that for each element V in $\mathscr{V}$, $p \in V$ and $\bar{V} \subset U$. Let V be a fixed element of $\mathscr{V}$, and let $\mathscr{C}$ be the collection of all components of $\bar{U} - V$ that intersect both ∂U and ∂V. By Theorem 5, each component of $M - V$ contains at least one member of $\mathscr{C}$. Hence $\mathscr{C}$ is infinite. $\bar{U} - V$ is closed, and so each member of $\mathscr{C}$ is a continuum in the compact space $\bar{U}$. Hence there exists a net of elements of $\mathscr{C}$ that converges to a continuum K_v. Since each member of the net intersects both ∂U and ∂V, K_v intersects both ∂U and ∂V. There exists at most one element of C of this net that intersects K_v, and therefore if we delete C from this net and denote the resulting net by N, then K_v is a continuum of convergence of N in $\bar{U} - V$.

Consider the net $\{K_v, v \in \mathscr{V}\}$ of continua in the compact space $\bar{U}$. Some subset of $\{K_v, V \in \mathscr{V}\}$ converges to a continuum K in $\bar{U}$. Since each K_v intersects ∂U, K intersects ∂U. Now let W be an open set containing p. There exists an element V in $\mathscr{V}$ such that $p \in V$ and $\bar{V} \subset W$. Hence $K_v \cap W \neq \emptyset$. Therefore $p \in \lim K_v = K$, $p \notin \partial U$, and thus K is nondegenerate.

Definitions. Two points a and b of a connected space M are said to be *conjugate* if no point of M separates a from b in M. If p is neither a cut point nor an end point of a connected space M and $p \in M$, then the set consisting of p and all points of M conjugate to p is called the *simple link of M generated by p*.

Theorem 8 was established in [**6**] for metric spaces. In proving this theorem, Whyburn utilizes Theorem 6, which is proved using sequences. His proof does not generalize to non-metric spaces.

THEOREM 8. *If a point p of the locally compact connected space M is neither a cut point nor an end point of M, then there exists a point q of M such that $p \neq q$ and p is conjugate to q.*

Proof. Let p be a non-cut point of M. Suppose p is not conjugate to any other point of M. Assume M is not semi-locally connected at p. Then by Theorem 7, there exists a net of continua $\{K_\mathfrak{a}, \mathfrak{a} \in \mathscr{A}\}$ such that each $K_\mathfrak{a}$ is a continuum of convergence and $\{K_\mathfrak{a}, \mathfrak{a} \in \mathscr{A}\}$ converges to a nondegenerate continuum K containing p. Let q be a point in K different from p. Since p is not conjugate to q, there exist a point x in M and two separated sets A and B such that $M - \{x\} = A \cup B$, $q \in A$, and $p \in B$. Now x must be a point in K. A is an open set containing q and $q \in \lim K_\mathfrak{a}$, and so there exists an $\mathfrak{a}$ in $\mathscr{A}$ such that for all $\mathfrak{a} \geqq \mathfrak{a}_1$, $K_\mathfrak{a} \cap A \neq \emptyset$. Similarly, there exists an $\mathfrak{a}_2$ in $\mathscr{A}$ such that for all $\mathfrak{a} \geqq \mathfrak{a}_2$, $K_\mathfrak{a} \cap B \neq \emptyset$. Let $\mathfrak{b} \in \mathscr{A}$ such that $\mathfrak{b} \geqq \mathfrak{a}_1$ and $\mathfrak{b} \geqq \mathfrak{a}_2$. Then

$$K_\mathfrak{b} \cap A \neq \emptyset \quad \text{and} \quad K_\mathfrak{b} \cap B \neq \emptyset.$$

Suppose $x \notin K_{\mathfrak{b}}$. Then $K_{\mathfrak{b}}$ is a connected subset of M-$\{x\}$ which implies $K_{\mathfrak{b}} \subset A$ or $K_{\mathfrak{b}} \subset B$, but this is a contradiction. Hence $x \in K_{\mathfrak{b}}$. Now x separates two points in $K_{\mathfrak{b}}$, and $K_{\mathfrak{b}}$ is a continuum of convergence, but this is impossible. Hence M is semi-locally connected at p.

Let U be an open set containing p such that $\bar{U}$ is compact. By Theorem 3, there exists an open set V such that $p \in V$, $V \subset U$, and $M - V$ is connected. Also, $\bar{V}$ is compact. Let C be the component of V containing p. Since $\partial V \cap \bar{C} \neq \emptyset$, there is a point z in $\partial V \cap \bar{C}$. Now since z is not conjugate to p, there exist a point y and two separated sets E and F such that M-$\{y\} = E \cup F$, $p \in E$, and $z \in F$. If $y \notin C$, then C is a connected subset of M-$\{y\}$, and $C \cup \{z\}$ is a connected subset of M-$\{y\}$, which is a contradiction. Hence $y \in C$, and therefore $M - V \subset F$, which implies $E \subset V$. E is open and $\partial E = \{y\}$. Thus p is an end point of M, and the theorem is proved.

THEOREM 9. *Every simple link of a locally compact connected space M is nondegenerate, and every point of M is a cut point, an end point, or a point of a simple link of M.*

Proof. Let L_p be the simple link of M generated by p. By Theorem 8, p is conjugate to some point q in M different from p. This implies $q \in L_p$, and hence L_p is nondegenerate. If $p \in M$, and p is not a cut point and not an end point of M, then there exists a point q in M such that p is conjugate to q. Hence p is a point in the simple link of M generated by q.

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Missouri Western State College,
Saint Joseph, Missouri