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HOPF'S ERGODIC THEOREM FOR PARTICLES WITH DIFFERENT VELOCITIES AND THE "STRONG SWEEPING OUT PROPERTY"

A. BELLOW, A. P. CALDERÓN AND U. KRENGEL

ABSTRACT. In an earlier paper we provided a counterexample to an old conjecture of Hopf. In this note we show that the "strong sweeping out property" obtains for the Hopf operators (T_t) both when $t \longrightarrow +\infty$ and when $t \longrightarrow 0+$, that is a.e. convergence fails in the worst possible way.

1. **Introduction.** Let $(\Omega, \mathfrak{F}, \mu)$ be a probability space and $\{\tau_t \mid t \in \mathbb{R}\}$ a measurable measure-preserving flow on it (see [5]). Let $\tilde{\Omega} = \Omega \times [0, +\infty)$ and let $\tilde{\mu} = \mu \otimes \lambda$ be the product of μ and Lebesgue measure λ. For $f \in L^1(\tilde{\mu})$ and $h \in L^\infty(\tilde{\mu})$ Hopf [3] defined the operators

$$(T_tf)(\omega) = \int_{[0,+\infty)} f(\tau_{tv}\omega, v)h(\omega, v)\,d\lambda(v)$$

and showed that T_tf converges in L^1-norm as $t \longrightarrow \infty$. (As noted in [1], there is also a "local version" of Hopf's Ergodic Theorem, namely: T_tf converges in L^1-norm as $t \longrightarrow 0+$.) Hopf conjectured that, as $t \longrightarrow \infty$, T_tf also converges a.e. for $f \in L^1(\tilde{\Omega}) = L^1(\tilde{\mu})$. In [1] we provided a counterexample to this conjecture. The example we constructed was the indicator function $f = 1_E$, where the set E was of finite $\tilde{\mu}$ measure but unbounded (in the v-coordinate); the construction also showed that for $f \geq 0$, $h \geq 0$, the lim inf coincides with the L^1-limit. Thus the possibility of demiconvergence is not ruled out. Here we strengthen the above result about 1_E and show that, restricting ourselves to functions with support in the product $\tilde{\Omega}_0 = \Omega \times [0, 1]$ we can even obtain the "strong sweeping out property" for (T_t) in both cases, $t \longrightarrow +\infty$ and $t \longrightarrow 0+$.

We recall that for a sequence (T_n) of operators with $T_n 1 = 1$, we say that the "strong sweeping out property" holds if, given any $\varepsilon > 0$ there is a set E of measure less than ε such that

$$\limsup_n T_n 1_E = 1 \text{ a.e.}$$
$$\liminf_n T_n 1_E = 0 \text{ a.e.}$$

In other words, convergence fails in the worst possible way.

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We are indebted to Benjamin Weiss for bringing to our attention the Perron tree construction.

2. **An application of the Perron tree construction.** Let $A > 0$ and let $R^{(A)} = [0,1] \times [0,A]$ be the rectangle of height A over the base $[0,1]$. By a "closed strip in $R^{(A)}$" we mean a set of the form

$$S^{(A)} = S^{(A)}_{[a,b];t} = \{(\omega + tv, v) \mid \omega \in [a,b], v \in [0,A]\}$$

where $0 \le a < b \le 1$ and $t \in \mathbb{R}$. Note that S may have points outside $R^{(A)}$.

Consider now the circle group. For notational convenience we shall write it in the form $\Omega = [0,1] \pmod 1$ (keeping in mind that 0 and 1 are identified and shall denote by $x \dot{+} y$ addition $\pmod 1$).

Let $I = [0,1]$ and let $\tilde{\Omega}_0 = \Omega \times I$ be the corresponding cylinder of height 1. By a "cylindrical closed strip" we mean a set of the form

$$\mathbb{S} = \mathbb{S}_{[a,b];t} = \{(\omega \dot{+} tv, v) \mid \omega \in [a,b], v \in [0,1]\}$$

where $0 \le a < b \le 1$ and $t \in \mathbb{R}$.

For $M \in \mathbb{R}$, $M \neq 0$, consider the map $\sigma = \sigma_M \colon \tilde{\Omega}_0 \longrightarrow \tilde{\Omega}_0$ given by

$$\sigma(\omega, v) = (\omega \dot{+} Mv, v).$$

This is an automorphism (measurable, measure-preserving, invertible) of $\tilde{\Omega}_0$. Its inverse is

$$\sigma^{-1}(\omega, v) = (\omega \dot{-} Mv, v).$$

For fixed $\omega_0 \in \Omega$, $t_0 \in \mathbb{R}$, the "cylindrical line segment"

$$\ell_{\omega_0,t_0} = \{(\omega_0 \dot{+} t_0 v, v) \mid v \in [0,1]\}.$$

under σ becomes the "cylindrical line segment"

$$\ell_{\omega_0,t_0+M} = \{(\omega_0 \dot{+} t_0 v \dot{+} Mv, v) \mid v \in [0,1]\}.$$

REMARK. The "cylindrical closed strip" $\mathbb{S}_{[a,b];t}$ is mapped under $\sigma = \sigma_M$ onto the "cylindrical closed strip" $\mathbb{S}_{[a,b];t+M}$.

We now recall a classical lemma in differentiation theory, whose proof is based on the Perron tree construction (see, for instance, M. de Guzmán [2] p. 215, Lemma 8.5.1).

LEMMA. *Let $0 < \varepsilon < 1$, $1 < A$ and consider the rectangles*

$$R_1 = [0,1] \times [0, \varepsilon/2] \quad (= R^{(\varepsilon/2)})$$
$$R_2 = [0,1] \times [0, A] \quad (= R^{(A)}).$$

There is then a finite collection of "closed strips in $R^{(A)}$," $S_1^{(A)}, S_2^{(A)}, \ldots, S_k^{(A)}$, $S_i^{(A)} = S^{(A)}_{[a_i,b_i];t_i}$, such that

$$S_i^{(A)} \subset R_2 \quad \text{for } i = 1, 2, \ldots, k \tag{1}$$

$$R_1 \subset \bigcup_{i=1}^{k} S_i^{(A)} \tag{2}$$

$$\textit{Lebesgue measure of}\left(\left(\bigcup_{i=1}^{k} S_i^{(A)}\right) \cap (R_2 \setminus R_1)\right) \leq \varepsilon/2 \tag{3}$$

$$\textit{Lebesgue measure of}\left(\bigcup_{i=1}^{k} S_i^{(A)}\right) \leq \varepsilon. \tag{4}$$

REMARKS. 1) Note that from (1) above it follows that $|t_i| \leq 1/A$ for $i = 1, 2, \ldots, k$. In fact we have

$$t_i A \leq b_i + t_i A \leq 1$$
$$a_i + t_i A \geq 0 \Rightarrow t_i \geq -a_i \geq -1.$$

2) Note that from (2) above it follows that

$$\bigcup_{i=1}^{k} [a_i, b_i] = [0, 1].$$

This has the following picturesque interpretation:

Think of the vertical strip R_2 as a (two-dimensional) piece of cheese. Then one can cut out finitely many strips through R_2, such that from every point of the base one can "see the sky" and the total area of the hollow strips is less than ε.

With the above notation we have:

COROLLARY. *Let $0 < \varepsilon < 1$ and $\alpha < \beta$, $\alpha, \beta \in \mathbb{R}$ be given. Then there is a finite collection of "cylindrical closed strips", $V_1, V_2, \ldots, V_k$, $V_i = \mathbb{S}_{[a_i,b_i];t_i'}$ such that*

$$t_i' \in [\alpha, \beta] \quad \text{for } i = 1, 2, \ldots, k \tag{1'}$$

$$[0, 1] = \bigcup_{i=1}^{k} [a_i, b_i] \tag{2'}$$

$$\textit{Lebesgue measure of}\left(\bigcup_{i=1}^{k} V_i\right) \leq \varepsilon \tag{3'}$$

PROOF. Choose $A > 1$, $A \geq 2/(\beta - \alpha)$. Observe that for the closed strips of the lemma determined by $[a_i, b_i]$ and t_i, the corresponding "closed strip in $R^{(1)}$" and the "cylindrical closed strip" coincide

$$S^{(1)}_{[a_i,b_i];t_i} = \mathbb{S}_{[a_i,b_i];t_i}.$$

Consider now the automorphism $\sigma = \sigma_M$ with $M = \alpha + 1/A$ and define

$$V_i = \sigma(\mathbb{S}_{[a_i,b_i];t_i})$$

then

$$V_i = \mathbb{S}_{[a_i,b_i];t'_i}, \quad \text{where } t'_i = t_i + M.$$

Since $[-1/A, 1/A]+M \subset [\alpha, \beta]$ and $|t_i| \leq A$, $(1')$ follows; $(2')$ follows from (2) and $(3')$ follows from (4) and the fact that $\sigma = \sigma_M$ is measure preserving. ∎

3. **Hopf's ergodic theorem and the "strong sweeping out property".** In the remainder of this note we assume that $\Omega = [0, 1] \pmod 1$, that μ is the Lebesgue measure on Ω, and that $\tau_t(\omega) = \omega + t$. We also take $h \equiv 1$ in our example.

For the operators T_t defined in the introduction, note that we have

(a) $$T_t: L^1(\tilde{\Omega}_0) \longrightarrow L^1(\Omega)$$

(b) $$f \geq 0 \Rightarrow T_t f \geq 0$$

(c) $$T_t(1_{\tilde{\Omega}_0}) = 1 \quad (= 1_\Omega).$$

THEOREM 1. *For each $\varepsilon > 0$ and $\delta > 0$ there is a set $E \subset \tilde{\Omega}_0$ and a finite collection of numbers $t'_1, t'_2, \ldots, t'_k$ such that*

$$\tilde{\mu}(E) \leq \varepsilon, \quad 0 < t'_i \leq \delta$$

and

(∗) $$\mu\Big(\{\omega \mid \sup_{1\leq i\leq k} T_{t'_i}(1_E)(\omega) = 1\}\Big) = 1.$$

In particular, the operators $T_t: L^1(\tilde{\Omega}_0) \longrightarrow L^1(\Omega)$ satisfy the "strong sweeping out property" as $t \longrightarrow 0+$.

PROOF. By a well-known criterion of del Junco and Rosenblatt (see [4], Theorem 1.3), it suffices to check (∗). We apply the Corollary in Section 2 with $\alpha = \delta/2$, $\beta = \delta$. Let $E = \bigcup_{i=1}^k V_i$. By $(3')$, $\tilde{\mu}(E) \leq \varepsilon$ and by $(1')$, $\delta/2 \leq t'_i \leq \delta$. By $(2')$

$$\begin{aligned} \omega_0 \in [0,1] \pmod 1 &\Rightarrow \omega_0 \in [a_j, b_j] \quad \text{for some } j,\ 1 \leq j \leq k \\ &\Rightarrow 1_E(\omega_0 + t'_j v, v) = 1 \quad \text{for all } v \in [0,1] \\ &\Rightarrow T_{t'_j} 1_E(\omega_0) = 1 = \sup_{1\leq i\leq k} T_{t'_i}(1_E)(\omega_0) = 1. \end{aligned}$$

This proves (∗) and, consequently, also our theorem. ∎

THEOREM 2. *For each $\varepsilon > 0$ and $M > 0$, there is a set E, $E \subset \tilde{\Omega}_0$, and a finite collection of numbers $t'_1, t'_2, \ldots, t'_k$ such that $\tilde{\mu}(E) \leq \varepsilon$, $t'_i \geq M$ and*

(∗∗) $$\mu\Big(\{\omega \mid \sup_{1\leq i\leq k} T_{t'_i}(1_E)(\omega) = 1\}\Big) = 1.$$

In particular, the operators $T_t: L^1(\tilde{\Omega}_0) \longrightarrow L^1(\Omega)$ satisfy the "strong sweeping out property" as $t \longrightarrow \infty$.

PROOF. Entirely analogous to that of Theorem 1, except that here we take $[\alpha, \beta] = [M, M+1]$. ∎

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Department of Mathematics
Northwestern University
Evanston, Illinois 60208
U.S.A.

Department of Mathematics
University of Chicago
Chicago, Illinois 60637
U.S.A.

Department of Mathematical Stochastics
University of Göttingen
Lotzestr. 13
3400 Göttingen
Germany