AN ENUMERATION OF THE FIVE PARALLELOHEDRA

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A parallelohedron is a convex polyhedron, in real affine three-dimensional space, which can be repeated by translation to fill the whole space without interstices. It has centrally symmetrical faces [4, p. 120] and hence is centrally symmetrical.[1]

Let F_i denote the number of faces each having exactly i edges, V_i denote the number of vertices each incident with exactly i edges, E denote the number of edges, n denote the number of sets of parallel edges, F denote the total number of faces, V denote the total number of vertices. Then $F_i = 0$ for odd i, F_i is even for even i,

$$F = \Sigma F_i, \quad V = \Sigma V_i,$$

$$\Sigma i V_i = \Sigma i F_i = 2E$$

and

$$V - E + F = 2.$$

Hence

$$E = \frac{1}{3} \Sigma i V_i + \frac{1}{6} \Sigma i F_i,$$

so that

$$\Sigma V_i - \frac{1}{3} \Sigma i V_i - \frac{1}{6} \Sigma i F_i + \Sigma F_i = 2,$$

or

$$2\Sigma (3 - i)V_i + \Sigma (6 - i)F_i = 12,$$

or

$$F_4 = 6 + F_8 + 2F_{10} + 3F_{12} + \ldots + V_4 + 2V_5 + 3V_6 + \ldots,$$

which implies

$$F_4 \geq 6.$$

[1] This theorem is due to Alexandroff. See Burckhardt [1, pp. 149-154].

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Of course, these statements apply to the larger class of polyhedra known as zonohedra [2, pp. 27-30], for which we also have

$$F_4 + 3F_6 + 6F_8 + 10F_{10} + \dots + \frac{k(k-1)}{2} F_{2k} + \dots = n(n-1).$$

Voronoï [5, p. 278] and Minkowski [4, p. 120] showed that for a parallelohedron $F \leq 14$ and that the faces must in fact be parallelograms or parallel-sided hexagons, i. e., that $F_{2k} = 0$ $(k = 4, 5, 6, \dots)$. Thus, for a parallelohedron

$$F = F_4 + F_6 \leq 14, \quad F_4 \geq 6, \quad F_4 + 3F_6 = n(n-1).$$

It follows that

$$6 \leq n(n-1) = F_4 + 3F_6 = F_4 + 3(F - F_4)$$

$$= 3F - 2F_4 \leq 3.14 - 2.6 = 30$$

and hence $3 \leq n \leq 6$. Furthermore, these inequalities imply

$$\frac{n(n-1) - 14}{2} \leq F_6 \leq \frac{n(n-1) - 6}{3}$$

Thus,

when $n = 3$, $F_6 = 0$;

when $n = 4$, $F_6 \leq 2$;

when $n = 5$, $F_6 = 4$;

when $n = 6$, $F_6 = 8$.

We have the following possible parallelohedra [3, pp. 688-689].

n	n(n - 1)	F_4	F_6	parallelohedron
3	6	6	0	parallelepiped
4	12	12	0	rhombic dodecahedron
		6	2	hexagonal prism
5	20	8	4	elongated dodecahedron
6	30	6	8	truncated octahedron

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