Forum of Mathematics, Pi (2025), **Vol. 13**:e14 1–21
doi:10.1017/fmp.2025.6

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RESEARCH ARTICLE

The local geometry of idempotent Schur multipliers

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Received: 10 April 2024; **Revised:** 4 February 2025; **Accepted:** 26 February 2025

2020 Mathematics Subject Classification: *Primary* – 22D15, 42B15, 46L07

Abstract
A Schur multiplier is a linear map on matrices which acts on its entries by multiplication with some function, called the symbol. We consider idempotent Schur multipliers, whose symbols are indicator functions of smooth Euclidean domains. Given $1 < p \neq 2 < \infty$, we provide a local characterization (under some mild transversality condition) for the boundedness on Schatten p-classes of Schur idempotents in terms of a lax notion of boundary flatness. We prove in particular that all Schur idempotents are modeled on a single fundamental example: the triangular projection. As an application, we fully characterize the local L_p-boundedness of smooth Fourier idempotents on connected Lie groups. They are all modeled on one of three fundamental examples: the classical Hilbert transform and two new examples of Hilbert transforms that we call affine and projective. Our results in this paper are vast noncommutative generalizations of Fefferman's celebrated ball multiplier theorem. They confirm the intuition that Schur multipliers share profound similarities with Euclidean Fourier multipliers – even in the lack of a Fourier transform connection – and complete, for Lie groups, a longstanding search of Fourier L_p-idempotents.

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1. Introduction

Schur multipliers are linear maps on matrix algebras with a great impact on geometric group theory, operator algebras and functional analysis. Their definition is rather simple on discrete spaces $S_M(A) = (M(j,k)A_{jk})_{jk}$. It easily extends to nonatomic σ-finite measure spaces (Ω, μ) by restricting to operators A in $L_2(\Omega, \mu)$ admitting a kernel representation over $\Omega \times \Omega$. Their role in geometric group theory and operator algebras was first analyzed by Haagerup. His pioneering work on free groups [20] and the research thereafter on semisimple lattices [4, 13] encoded deep geometric properties via approximation properties with Schur multipliers. Other interesting links can be found in [1, 22, 37, 39, 41, 42, 44].

In 2011, stronger rigidity properties of high rank lattices were discovered by studying L_p-approximations [27, 28]. First, there are no L_p-approximations by means of Fourier or Schur multipliers over $\mathrm{SL}_n(\mathbf{R})$ for $p > 2 + \alpha_n$, with $\alpha_n \to 0$ as $n \to \infty$. Second, it turns out that this unprecedented pathology leads to a strong form of nonamenability which is potentially useful to distinguish the group von Neumann algebras of $\mathrm{PSL}_n(\mathbf{Z})$ for different values of $n \geq 3$, the most iconic form of Connes' rigidity conjecture. This has strongly motivated our recent work [10, 35] with several forms of the Hörmander-Mikhlin theorem. Nevertheless, there is still much to learn about less regular multipliers. A key point in [28] was a careful analysis of Schur multipliers over the n-sphere for symbols of the form $M_\varphi(x,y) = \varphi(\langle x, y\rangle)$. More precisely, the boundedness of S_{M_φ} on the Schatten class S_p for $p > 2 + \frac{2}{n-1}$ imposes Hölder regularity conditions on φ. This article grew from the analysis of the *spherical Hilbert transform*

$$H_{\mathbf{S}} : A \mapsto \Big(-i \operatorname{sgn} \langle x, y\rangle A_{xy} \Big)_{xy}.$$

Is it S_p-bounded for some $\frac{2n}{n+1} < p \neq 2 < \frac{2n}{n-1}$? Its S_p-boundedness is equivalent to that of $\frac{1}{2}(1+iH_{\mathbf{S}})$ – whose symbol is χ_Σ with $\Sigma = \{(x,y) : \langle x, y\rangle > 0\}$ – and it is worth noting the analogy with the ball multiplier problem, which was only known to be unbounded for p outside this range before Fefferman's celebrated contribution [15]. Our main result completely solves this problem: $H_{\mathbf{S}}$ is S_p-unbounded unless $n = 1$ or $p = 2$. We characterize S_p-boundedness for a lot more idempotents.

Let M, N be two differentiable manifolds with the Lebesgue measure coming from any Riemannian structure on them. Consider a $\mathcal{C}^1$-domain $\Sigma \subset M \times N$ so that its boundary $\partial\Sigma$ is a smooth hypersurface, which is locally represented by level sets of some real-valued $\mathcal{C}^1$-functions with nonvanishing gradients. We say that $\partial\Sigma$ is transverse at a point (x,y) when the tangent space of $\partial\Sigma$ at (x,y) maps surjectively on each factor T_xM and T_yN. In that case, both sections

$$\partial\Sigma_x = \big\{ y' \in N \mid (x, y') \in \partial\Sigma \big\} \quad \text{and} \quad \partial\Sigma^y = \big\{ x' \in M \mid (x', y) \in \partial\Sigma \big\}$$

become codimension 1 manifolds on some neighbourhood of y and x, respectively.

Theorem A. *Let $p \in (1,\infty) \setminus \{2\}$ and consider a $\mathcal{C}^1$-domain $\Sigma \subset M \times N$. Then the following statements are equivalent for any transverse point $(x_0, y_0) \in \partial\Sigma$:*

(1) *S_p-boundedness. The idempotent Schur multiplier S_Σ whose symbol equals 1 on Σ and 0 elsewhere is bounded on $S_p(L_2(U), L_2(V))$ for some pair of neighbourhoods U, V of x_0, y_0 in M, N.*
(2) *Zero-curvature condition. There are neighbourhoods U, V of x_0, y_0 in M, N such that the tangent spaces $T_y(\partial\Sigma_{x_1})$ and $T_y(\partial\Sigma_{x_2})$ coincide for any pair of points $(x_1, y), (x_2, y) \in \partial\Sigma \cap (U \times V)$.*
(3) *Triangular truncation representation. There are neighbourhoods U, V of the points x_0, y_0 in M, N and $\mathcal{C}^1$-functions $f_1 : U \to \mathbf{R}$ and $f_2 : V \to \mathbf{R}$, such that $\Sigma \cap (U \times V) = \big\{(x,y) \in U \times V : f_1(x) > f_2(y)\big\}$.*

Theorem A characterizes the local geometry of S_p-bounded idempotent Schur multipliers and vastly amplifies the ball multiplier theorem [15]. A first interesting consequence is that this property does not depend on the value of p. It is important to insist here that the characterization is local. If the global aspects are taken into account, the S_p-boundedness of idempotent Schur multipliers does depend on

https://doi.org/10.1017/fmp.2025.6 Published online by Cambridge University Press

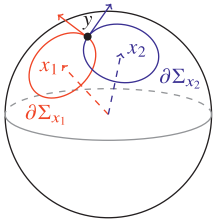

Figure 1. *Failure of (2) for spherical Hilbert transforms $H_{\mathbf{S},\delta}$.*
Here $H_{\mathbf{S},\delta} = -i(2S_{\Sigma_\delta} - \mathrm{id})$ with $\Sigma_\delta = \big\{(x,y) \in \mathbf{S}^n \times \mathbf{S}^n : \langle x,y\rangle > \delta\big\}$ for $n = 2$.

p: in the discrete setting, whenever $p < q$ with p is an even integer, there is an idempotent Schur multiplier that is S_p-bounded but not S_q-bounded [22]. Remark 2.7 provides other examples where the local theorem fails to be global.

Theorem A is only interesting when M and N have dimension at least 2. When M or N is a discrete space – in other words, $\dim M = 0$ or $\dim N = 0$ – it is obvious just by taking $U = \{x_0\}$ or $V = \{y_0\}$. Similarly, when $\dim M = 1$ or $\dim N = 1$, condition (3) always holds by the implicit function theorem, and the other conditions are also easily seen to always hold. However, in dimension at least 2, the conditions become restrictive. A simple example of a transverse domain that fails to satisfy (2) or (3) at any boundary point is shown in Figure 1 above.

The implication (3)⇒(1) is the easy one. By known techniques, it follows from the classical S_p-boundedness of the triangular projection $(A_{jk}) \mapsto (\chi_{j\geq k}A_{jk})$ with $j,k \in \mathbf{N}$, closely related to the L_p-boundedness of the Hilbert transform. On the contrary, the converse implication (1)⇒(3) is certainly unexpected. It says that the triangular projection is the only local model for S_p-bounded idempotents. Our proof decomposes in two independent parts. The implication (1)⇒(2) is very much analytical. By well-known Fourier-Schur transference results, Fefferman's theorem corresponds to the case $M = N = \mathbf{R}^n$ and domains $\Sigma = \big\{(x,y) : x - y \in \Omega\big\}$ for a Euclidean $\mathcal{C}^1$-domain Ω. Transversality trivially holds at every boundary point in this case. In the general case, the main idea (Lemma 2.3) is a new connection between Schur and Fourier multipliers: it gives an L_p square-function inequality for half-space multipliers out of the S_p-boundedness of S_Σ. This is a noncommutative form of Meyer's lemma, which derived such a square inequality from L_p-bounded Fourier multipliers, and which was a key part in the proof of the ball multiplier theorem. However, the implication (2)⇒(3) is a purely geometric statement about hypersurfaces in product manifolds (Theorem 2.5) to which we failed to find a straightforward proof.

It is rather surprising to us that Theorem A holds for Schur multipliers on general manifolds which – contrary to Euclidean spaces, where Fefferman's result held so far – lack to admit a Fourier transform connection. Also observe that when we take $M = N = \mathbf{R}^n$ and write

$$\mathbf{n}(x,y) = \big(\mathbf{n}_1(x,y), \mathbf{n}_2(x,y)\big)$$

for a normal vector to $\partial\Sigma$ at (x,y), transversality means that both n-dimensional components $\mathbf{n}_1, \mathbf{n}_2$ are nonzero. The zero-curvature condition means that $\mathbf{n}_2(x_1,y)$ and $\mathbf{n}_2(x_2,y)$ are parallel—equivalent forms in terms of $\mathbf{n}_1(x,y_1)$ and $\mathbf{n}_1(x,y_2)$ instead, or simpler formulations for $\mathcal{C}^2$-domains will be also discussed. In a different direction, a global (nonlocal) characterization of S_p-bounded idempotent Schur multipliers also follows for relatively compact fully transverse domains Σ.

Theorem A has profound consequences for Fourier multipliers on Lie group von Neumann algebras. Smooth Fourier multipliers on group algebras were intensively investigated over the last decade [5, 11, 18, 24, 25, 32, 35]. The nonsmooth theory concerns a longstanding search to classify idempotent Fourier L_p-multipliers, but their geometric behavior is very sensitive to the underlying group. Harcharras investigated noncommutative $\Lambda(p)$-sets in [22]. Bożejko and Fendler [3] studied an analog of Fefferman's ball multiplier theorem in the free group for $|1/p - 1/2| > 1/6$. More recently, Mei and Ricard found a

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large class of free Hilbert transforms in their remarkable work [31]. The search for Hilbert transforms on general groups also includes crossed products and groups acting on tree-like structures [19, 36].

In this paper, we shall give a complete characterization of the local boundary behavior for completely bounded idempotent Fourier multipliers on arbitrary Lie groups. We say that a function $m : \mathrm{G} \to \mathbf{C}$ defines locally at $g_0 \in \mathrm{G}$ a completely bounded Fourier L_p-multiplier if there is a function $\varphi : \mathrm{G} \to \mathbf{C}$ equal to 1 on a neighbourhood of g_0 such that φm defines (globally) a completely bounded Fourier L_p-multiplier. Our result is more easily stated for simply connected groups. We refer to Section 3 for various characterizations of local Fourier multipliers and for the statement of our result below on general Lie groups (Theorem 3.3).

Theorem B. *Let $p \in (1,\infty) \setminus \{2\}$. Let* G *be a simply connected Lie group,* $\Omega \subset \mathrm{G}$ *a* $\mathcal{C}^1$*-domain and* $g_0 \in \partial\Omega$ *a point in the boundary of* Ω*. The following are equivalent:*

(1) χ_Ω *defines locally at* g_0 *a completely bounded Fourier* L_p*-multiplier.*
(2) *There is a smooth action* $\mathrm{G} \to \mathrm{Diff}(\mathbf{R})$ *by diffeomorphisms on the real line, such that* Ω *coincides on a neighbourhood of* g_0 *with* $\{g \in \mathrm{G} \mid g \cdot 0 > g_0 \cdot 0\}$.

Alternatively, this means that $\partial\Omega$ is locally a coset of a codimension 1 subgroup. There are two ingredients in the proof of Theorem B. The first is Theorem A. The second is a general result relating local complete L_p-boundedness of Fourier and Schur multipliers for arbitrary locally compact groups. Such a result is known to be true globally at the endpoints $p = 1, \infty$ [3] or when the group G is amenable [7, 34]. The eventuality that it could be true locally is a recent observation [35] for even integers $p \in 2\mathbf{Z}_+$ and unimodular groups G. In Theorem 3.1 below, we manage to prove it in full generality as a crucial step towards Theorem B.

Lie himself classified Lie groups admitting (local) actions by diffeomorphisms on the real line [30]. This classification into three types (translation, affine and projective) gives rise to the following three fundamental examples of a group G with a smooth domain Ω:

i) The real line $\mathrm{G}_1 = \mathbf{R}$ with $\Omega_1 = (0,\infty)$.
ii) The affine group $\mathrm{G}_2 = \mathrm{Aff}_+(\mathbf{R})$[1] and $\Omega_2 = \{ax + b : b > 0\}$.
iii) The universal covering group $\mathrm{G}_3 = \widetilde{\mathrm{PSL}}_2(\mathbf{R})$[2] with $\Omega_3 = \{g : \alpha_g(0) > 0\}$.

The domains Ω_j define global (not just local) completely bounded idempotent Fourier L_p-multipliers for $1 < p < \infty$. Ω_1 gives the classical Hilbert transform $H = -i(2T_{\chi_{\Omega_1}} - \mathrm{id})$. L_p-boundedness for the domain Ω_3 follows from recent Cotlar identities for unimodular groups [19]. G_2 is nonunimodular, and L_p-boundedness for Ω_2 is properly justified in Example 3.8. The basic models $H_j = -i(2T_{\chi_{\Omega_j}} - \mathrm{id})$ will be referred to as classical, affine and projective Hilbert transforms. We find very surprising that every other idempotent Fourier multiplier locally comes from a surjective homomorphism on one of these three groups.

Corollary B1. *Conditions (1) and (2) in Theorem B are equivalent to the following*:

(3) *There is* $j \in \{1,2,3\}$ *and a smooth surjective homomorphism* $f : \mathrm{G} \to \mathrm{G}_j$ *such that the domain* Ω *coincides on a neighbourhood of* g_0 *with* $g_0 f^{-1}(\Omega_j)$.

When $\mathrm{G} = \mathbf{R}^n$, a homomorphism as in Corollary B1 above is of the form $\mathbf{R}^n \ni \xi \mapsto c\langle \xi, u\rangle \in \mathbf{R}$ for $c \in \mathbf{R}^*$ and $u \in \mathbf{S}^{n-1}$. Thus, Corollary B1 recovers that L_p-bounded idempotent Fourier multipliers in the Euclidean setting locally correspond (up to translations) to half-space multipliers with symbol $m_u(\xi) = \chi_{\langle \xi, u\rangle > 0}$ for some $u \in \mathbf{S}^{n-1}$. This is well understood since Fefferman's solution to the Ball multiplier problem. Note in passing that these half-space multipliers are directional extensions of the Riesz projection $R = \frac{1}{2}(iH + \mathrm{id})$. By analogy, we could rephrase Corollary B1 by saying that every

[1] Affine increasing bijections $x \mapsto ax + b$ for $a \in \mathbf{R}_+^*$ and $b \in \mathbf{R}$, isomorphic to $\mathbf{R} \rtimes \mathbf{R}_+^*$.

[2] The action $\alpha : \widetilde{\mathrm{PSL}}_2(\mathbf{R}) \curvearrowright \mathbf{R}$ is obtained by lifting the standard action of $\mathrm{PSL}_2(\mathbf{R})$ on the projective line to the universal covers. If $p : \mathbf{R} \to P^1(\mathbf{R})$ denotes the universal cover, then the universal cover of $\mathrm{SL}_2(\mathbf{R})$ is identified with the group of homeomorphisms $g : \mathbf{R} \to \mathbf{R}$ for which there is $A \in \mathrm{PSL}_2(\mathbf{R})$ such that $p \circ g = A \cdot p$.

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Fourier L_p-idempotent on an arbitrary Lie group arises as a *directional amplification* of one of the three fundamental models of Riesz projections above.

Theorem B and Corollary B1 give very satisfactory descriptions of completely bounded Fourier idempotents in arbitrary Lie groups. It is certainly surprising that these multipliers are modeled out of exactly three fundamental examples, the classical Hilbert transform and its affine and projective variants. It also shows that every $\mathcal{C}^1$-idempotent is automatically $\mathcal{C}^\infty$. This rigidity property collides head-on with the much more flexible scenario of Theorem A.

Corollary B2. *Let* $p \in (1,\infty) \setminus \{2\}$ *and let* G *be a Lie group*:

i) *If* G *is simply connected and nilpotent, every cb-*L_p*-bounded smooth Fourier idempotent is locally of the form* $R \circ \varphi$*, for the classical Riesz projection* R *and some continuous homomorphism* $\varphi : \mathrm{G} \to \mathbf{R}$.
ii) *If* G *is a simple Lie group which is not locally isomorphic to* $\mathrm{SL}_2(\mathbf{R})$*, then* G *does not carry any smooth Fourier idempotent which is locally completely* L_p*-bounded on its group von Neumann algebra.*
iii) *If* G *is locally isomorphic to* $\mathrm{SL}_2(\mathbf{R})$*, then* G *carries a unique local Fourier idempotent which is completely* L_p*-bounded on its group algebra (up to left/right translations) given by* $g \mapsto \frac{1}{2}\big(1 + \operatorname{sgn}\operatorname{Tr}(ge_{12})\big)$.

As an illustration for stratified Lie groups, φ corresponds on the Lie algebra level with the projection onto any 1-dimensional subspace of the first stratum. The second statement spotlights the singular nature of harmonic analysis over simple Lie groups. It also yields an alternative way to justify that the spherical Hilbert transform $H_{\mathbf{S}}$ is not L_p-bounded for any $p \neq 2$. Finally, as we shall justify, the third statement gives a straightforward solution (in the negative) to Problem A in [19]. We refer to [38, 40] for the operator space background necessary for this paper.

The plan of the paper is as follows. Section 2 is devoted to idempotent Schur multipliers. It contains the proof of Theorem A and several discussions, including our analysis of the spherical Hilbert transform. Section 3 is devoted to Fourier multipliers. It contains the proof of Theorem B and its corollaries. The proof relies on a result of independent interest on the local transference between Fourier and Schur multipliers for arbitrary locally compact groups, Theorem 3.1.

2. Idempotent Schur multipliers

In this section, we give a complete proof of Theorem A. We begin by recalling some particularly flexible changes of variables for Schur symbols, which preserve the S_p-norm of the corresponding Schur multipliers on nonatomic spaces. Then, we prove the implications (1) ⇒ (2) ⇒ (3) ⇒ (1) in Theorem A separately. We shall finish with some comments and applications to spherical Hilbert transforms.

2.1. Schur multipliers

Let (X,μ) and (Y,ν) be σ-finite measure spaces. Given $1 \le p < \infty$, let $S_p(L_2(Y), L_2(X))$ be the space all of bounded linear operators $T : L_2(Y) \to L_2(X)$ with $\operatorname{Tr}|T|^p < \infty$, which is a Banach space for the norm below

$$\|T\|_{S_p} = \big(\operatorname{Tr}|T|^p\big)^{\frac{1}{p}}.$$

When $p = 2$, the Schatten class $S_2(L_2(Y), L_2(X))$ is the space of Hilbert-Schmidt operators $L_2(Y) \to L_2(X)$. It coincides with $L_2(X \times Y)$, regarding any L_2-function $(x,y) \mapsto K(x,y)$ as the kernel of the corresponding Hilbert-Schmidt operator

$$T_K f(x) = \int_Y K(x,y) f(y) d\nu(y).$$

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Given a bounded measurable function $m\colon X\times Y\to\mathbf{C}$, the Schur S_p-multiplier with symbol m is defined (when it exists) as the unique bounded linear map S_m on $S_p(L_2(Y),L_2(X))$ which assigns $T_K=(K(x,y))_{x\in X,y\in Y}\in S_2\cap S_p$ to $(m(x,y)K(x,y))_{x\in X,y\in XY}=S_m(T_K)$. We shall write $\|m\|_{MS_p}$ for its norm, with the convention $\|m\|_{MS_p}=\infty$ if S_m does not exist. By definition, if it exists, S_m is unchanged when m is modified on a measure 0 subset, so we can and will often consider S_m for $m\in L_\infty(X\times Y)$.

The following general fact will be crucial in our proof of Theorem A. It evidences a much greater flexibility of Schur multipliers compared to Fourier multipliers. The proof follows from [28], we include the argument below.

Lemma 2.1. *Let $(X,\mu),(X',\mu'),(Y,\nu),(Y',\nu')$ be atomless σ-finite measure spaces and $f\colon X\to X'$ and $g\colon Y\to Y'$ be measurable maps. Assume the pushforward measures $f_*\mu$ and $g_*\nu$ are absolutely continuous with respect to the measures μ' and ν', respectively. Then, for every $m\in L_\infty(X'\times Y')$,*

$$\|m\circ(f\times g)\|_{MS_p(L_2(Y,\nu),L_2(X,\mu))}\leq\|m\|_{MS_p(L_2(Y',\nu'),L_2(X',\mu'))}\,.$$

The absolute continuity of $f_*\mu$ and $g_*\nu$ is necessary. Indeed, otherwise there would exist a bounded measurable function $m\colon X'\times Y'\to\mathbf{C}$ with $m=0\mu'\otimes\nu'$-almost everywhere but $\widetilde{m}:(x,y)\mapsto m(f(x),g(y))$ does not vanish almost $\mu\otimes\nu$-almost everywhere. And in particular,

$$0<\|m\circ(f\times g)\|_{MS_p(L_2(Y,\nu),L_2(X,\mu))}\not\leq\|m\|_{MS_p(L_2(Y',\nu'),L_2(X',\mu'))}=0.$$

Proof. The inequality

$$\|m\|_{MS_p(L_2(Y',g_*\nu),L_2(X',f_*\mu))}\leq\|m\|_{MS_p(L_2(Y',\nu'),L_2(X',\mu'))}$$

follows directly from [28, Lemma 1.9] and the absolute continuity assumption. So our goal will be to prove the following equality:

$$\|m\circ(f\times g)\|_{MS_p(L_2(Y,\nu),L_2(X,\mu))}=\|m\|_{MS_p(L_2(Y',g_*\nu),L_2(X',f_*\mu))}\,.$$

To lighten the notation, let us assume that $(X,\mu)=(Y,\nu)$, $(X',\mu')=(Y',\nu')$ and $f=g$. Let $\mathcal{B},\mathcal{B}'$ be the underlying σ-algebras, and consider $\mathcal{A}:=f^{-1}(\mathcal{B}')$. Then f allows to identify $L_2(X',\mathcal{B}',f_*\mu)$ with $L_2(X,\mathcal{A},\mu)$. In particular,

$$\|m\circ(f\times f)\|_{MS_p(L_2(X,\mathcal{A},\mu))}=\|m\|_{MS_p(L_2(X',\mathcal{B}',f_*\mu))}\,,$$

and similarly for the cb-norm. However, [28, Lemma 1.13] implies that the cb norms of $m\circ(f\times f)$ on $S_p(L_2(\mathcal{A},\mu))$ and $S_p(L_2(\mathcal{B},\mu))$ coincide, so we deduce

$$\|m\circ(f\times f)\|_{\mathrm{cb}MS_p(L_2(X,\mathcal{B},\mu))}=\|m\|_{\mathrm{cb}MS_p(L_2(X',\mathcal{B}',f_*\mu))}\,.$$

Then [28, Theorem 1.18] allows us to conclude. Indeed, our assumptions that μ and μ' have no atoms imply that both cb-norms are equal to their norms. □

2.2. Proof of Theorem A: Boundedness implies zero-curvature

In this paragraph, we prove (1) ⇒ (2) from the statement of Theorem A. The proof of this implication follows the same path as in Fefferman's solution of the ball multiplier theorem [15]. The first step in his argument is a reduction due to Yves Meyer [15, Lemma 1] from the ball multiplier to square function estimates for half-space multipliers. Given a nonzero vector $u\in\mathbf{R}^n$, let H_u denote the corresponding u-directional half-space multiplier

$$\widehat{H_uf}(\xi)=\chi_{\langle\xi,u\rangle>0}\widehat{f}(\xi).$$

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Lemma 2.2 (Meyer). *Assume that the ball multiplier in $\mathbf{R}^2$ is L_p-bounded with norm $\le C$. For every integer N, every sequence of unit vectors $u_1, u_e, \ldots, u_N \in \mathbf{R}^2$ and functions $f_1, f_2, \ldots, f_N \in L_p(\mathbf{R}^2)$, the following inequality holds*:

$$\Big\|\Big(\sum_{j=1}^{N}\big|H_{u_j}(f_j)\big|^2\Big)^{\frac{1}{2}}\Big\|_{L_p(\mathbf{R}^2)} \le C\Big\|\Big(\sum_{j=1}^{N}|f_j|^2\Big)^{\frac{1}{2}}\Big\|_{L_p(\mathbf{R}^2)}. \tag{2.1}$$

The second step in the argument in [15] is the proof that (2.1) does not hold if $p \geq 2$. The argument relies on Besicovitch's construction for the Kakeya needle problem. The key new idea we introduce is a form of Meyer's Lemma 2.2 that is valid for Schur multipliers. It shows that, under the assumption that the indicator function of a domain Σ defines an S_p-bounded Schur multiplier with norm $\le C$, the square function estimate (2.1) will hold whenever $u_1, \ldots, u_N$ are normal vectors to $\partial\Sigma_{x_j}$ at a given point y. To make this precise, we introduce some notation: if $\Sigma \subset \mathbf{R}^n$ is a $\mathcal{C}^1$-domain and $z \in \partial\Sigma$, let $\mathbf{n}(z) = (\mathbf{n}_1(z), \mathbf{n}_2(z)) \in \mathbf{R}^n \oplus \mathbf{R}^n$ be a normal to $\partial\Sigma$ at z pointing away from Σ.

Lemma 2.3. *Let $U, V \subset \mathbf{R}^n$ be open subsets and $\Sigma \subset U \times V$ a $\mathcal{C}^1$-domain. Assume that the Schur multiplier S_Σ whose symbol is the characteristic function of Σ is bounded on $S_p(L_2(U), L_2(V))$ with norm C. Let $x_1, x_2, \ldots, x_N \in U$ and $y \in V$ such that $z_j = (x_j, y)$ is a transverse point in the boundary $\partial\Sigma$ for every $j = 1, 2, \ldots, N$. Define $u_j = \mathbf{n}_2(z_j)$ and consider functions $f_1, f_2, \ldots, f_N \in L_p(\mathbf{R}^n)$. Then, we have*

$$\Big\|\Big(\sum_{j=1}^{N}\big|H_{u_j}(f_j)\big|^2\Big)^{\frac{1}{2}}\Big\|_{L_p(\mathbf{R}^n)} \le C\Big\|\Big(\sum_{j=1}^{N}|f_j|^2\Big)^{\frac{1}{2}}\Big\|_{L_p(\mathbf{R}^n)}.$$

Proof. Given $u \in \mathbf{R}^n \setminus \{0\}$, define

$$m_u(\xi,\eta) = \chi_{\langle \xi-\eta, u\rangle > 0} = \begin{cases} 1 & \text{if } \langle \xi-\eta, u\rangle > 0 \\ 0 & \text{otherwise.} \end{cases}$$

Then, the proof relies on the following two claims:

(A) Let (x, y) be a transverse point in the boundary of $\partial\Sigma$ and let $T \in \mathrm{GL}_n(\mathbf{R})$ be such that $T^*\mathbf{n}_1(x,y) = -\mathbf{n}_2(x,y)$. Then, the following identity holds for almost every $\xi, \eta \in \mathbf{R}^n$:

$$\lim_{\varepsilon\to 0^+} \chi_\Sigma\big(x+\varepsilon T\xi, y+\varepsilon\eta\big) = m_{\mathbf{n}_2(x,y)}(\xi,\eta).$$

(B) Let u_j be as in the statement. Then the Schur multiplier

$$M : \big((\xi, j), \eta\big) \in \big(\mathbf{R}^n \times \{1,\ldots,N\}\big) \times \mathbf{R}^n \mapsto m_{u_j}(\xi,\eta)$$

is bounded on $S_p(L_2(\mathbf{R}^n), L_2(\mathbf{R}^n \times \{1, 2, \ldots, N\}))$ with norm $\le C$.

Assuming the validity of the above claims, we may now conclude the proof using standard transference ideas that go back at least to the work of Bożejko and Fendler [2]. Consider $f_1, f_2, \ldots, f_N \in L_p(\mathbf{R}^n)$. If $e_{j,1}$ denotes the standard elementary matrices, if $C = \sum_{j=1}^N c_j e_{j,1}$, we have $|C|^p = (C^*C)^{\frac{p}{2}} = (\sum_j |c_j|^2)^{\frac{p}{2}} e_{1,1}$. Taking $c_j = f_j(x)$ and integrating with respect to x, we get

$$\Big\|\Big(\sum_{j=1}^{N}|f_j|^2\Big)^{\frac{1}{2}}\Big\|_{L_p(\mathbf{R}^n)} = \Big\|\sum_{j=1}^{N} f_j \otimes e_{j,1}\Big\|_{L_p(\mathbf{R}^n; S_p)}.$$

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We know from [7, Theorem 5.2] that there is an ultrafilter $\mathcal{U}$ on $\mathbf{N}$ and a completely isometric map

$$j_p \colon L_p(\mathbf{R}^n) \to \prod_{\mathcal{U}} S_p(L_2(\mathbf{R}^n))$$

that intertwines Fourier and Schur multipliers. The notation $\prod_{\mathcal{U}} S_p(L_2(\mathbf{R}^n))$ stands for the Banach space ultraproduct, that is the quotient of

$$\prod_{\alpha\in\mathbf{N}} S_p(L_2(\mathbf{R}^n)) := \Big\{(A_\alpha)_{\alpha\in\mathbf{N}} \mid A_\alpha \in S_p(L_2(\mathbf{R}^n)), \sup_\alpha \|A_\alpha\|_p < \infty\Big\}$$

by its closed subspace $\{(A_\alpha)_\alpha \mid \lim_{\alpha\to\mathcal{U}} \|A_\alpha\|_p = 0\}$. Pick a representative $(A_{j,\alpha})_{\alpha\in\mathbf{N}}$ of $j_p(f_j)$. This gives that $(S_{m_u}(A_{j,\alpha}))_{\alpha\in\mathbf{N}}$ is a representative of $j_p(H_u(f_j))$ for every $u \in \mathbf{R}^n$. That j_p is a complete isometry gives

$$\Big\|\Big(\sum_{j=1}^N |f_j|^2\Big)^{\frac{1}{2}}\Big\|_{L_p(\mathbf{R}^n)} = \lim_{\alpha\to\mathcal{U}} \Big\|\sum_{j=1}^N A_{j,\alpha}\otimes e_{j,1}\Big\|_{S_p}, \tag{2.2}$$

and applying it to $H_{u_j}(f_j)$ instead of f_j, we obtain

$$\Big\|\Big(\sum_{j=1}^N |H_{u_j}(f_j)|^2\Big)^{\frac{1}{2}}\Big\|_{L_p(\mathbf{R}^n)} = \lim_{\alpha\to\mathcal{U}} \Big\|\sum_{j=1}^N S_{m_{u_j}}(A_{j,\alpha})\otimes e_{j,1}\Big\|_{S_p}.$$

In the preceding equality and in what follows, if $A_1,\ldots,A_N \in S_p(L_2(\mathbf{R}^n))$, we see the sum $\sum_{j=1}^N A_j \otimes e_{j,1}$ as the element of $S_p(L_2(\mathbf{R}^n), L_2(\mathbf{R}^n \times \{1,2,\ldots,N\}))$ mapping $g \in L_2(\mathbf{R}^n)$ to the function $(\xi, j) \in \mathbf{R}^n \times \{1,2,\ldots,N\} \mapsto (A_j g)(\xi)$. In the particular case when $A_j \in S_p \cap S_2$ has kernel $K_j \in L_2(\mathbf{R}^n \times \mathbf{R}^n)$, $\sum_{j=1}^N A_j \otimes e_{j,1}$ maps g to the function $(\xi, j) \mapsto \int K_j(\xi,\eta) g(\eta) d\eta$, so it is the Hilbert-Schmidt operator with kernel $((\xi, j),\eta) \mapsto K_j(\xi,\eta)$. Similarly, $\sum_{j=1}^N S_{m_j}(A_j)\otimes e_{j,1}$ is the Hilbert-Schmidt operators with kernel $((\xi, j),\eta) \mapsto m_j(\xi,\eta)K_j(\xi,\eta)$. Therefore, we have

$$S_M\Big(\sum_{j=1}^N A_j \otimes e_{j,1}\Big) = \sum_{j=1}^N S_{m_j}(A_j)\otimes e_{j,1}, \tag{2.3}$$

where S_M is the Schur multiplier appearing in claim (B). Claim (B) therefore implies that (2.3) holds for every $A_1,\ldots,A_N \in S_p(L_2(\mathbf{R}^n))$ and that the S_p-norm of (2.3) is $\leq C\|\sum_{j=1}^N A_j \otimes e_{j,1}\|_p$.

Let us apply this with $A_j = A_{j,\alpha}$. According to claim (2.2), we get

$$\begin{aligned}\Big\|\Big(\sum_{j=1}^N |H_{u_j}(f_j)|^2\Big)^{\frac{1}{2}}\Big\|_{L_p(\mathbf{R}^n)} &\leq C \lim_{\alpha\to\mathcal{U}} \Big\|\sum_{j=1}^N A_{j,\alpha}\otimes e_{j,1}\Big\|_{S_p}\\ &= C\Big\|\Big(\sum_{j=1}^N |f_j|^2\Big)^{\frac{1}{2}}\Big\|_{L_p(\mathbf{R}^n)}.\end{aligned}$$

Thus, the assertion is a consequence of claim (B), for which we need to justify claim (A) first. To do so, we can assume that $\Sigma = f^{-1}(-\infty, 0)$ for a $\mathcal{C}^1$-submersion $f \colon U \times V \to \mathbf{R}$. Then $\nabla f(x,y) = (\nabla_x f(x,y), \nabla_y f(x,y))$ is a normal vector to the boundary $\partial\Sigma = f^{-1}(0)$ at every $(x,y) \in \partial\Sigma$, pointing away from Σ. Thus, replacing f by a positive multiple, we can assume that its gradient is $(\mathbf{n}_1(x,y), \mathbf{n}_2(x,y))$. Then, the Taylor expansion of f gives

$$\begin{aligned} f\big(x+\varepsilon T\xi, y+\varepsilon\eta\big) &= \varepsilon\big\langle \mathbf{n}_1(x,y), T\xi\big\rangle + \varepsilon\big\langle \mathbf{n}_2(x,y), \eta\big\rangle + o(\varepsilon)\\ &= \varepsilon\big\langle \mathbf{n}_2(x,y), \eta-\xi\big\rangle + o(\varepsilon).\end{aligned}$$

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Therefore, if $\eta - \xi$ is not orthogonal to $\mathbf{n}_2(x, y)$ (a condition that holds for almost every ξ and η), we have $\chi_\Sigma(x + \varepsilon T\xi, y + \varepsilon\eta) = m_{\mathbf{n}_2(x,y)}(\xi, \eta)$ for every $\varepsilon > 0$ small enough. This proves claim (A).

To prove claim (B) we apply (A). More precisely, let $T_j \in \mathrm{GL}_n(\mathbf{R})$ be such that $T_j^* \mathbf{n}_1(x_j, y) = -\mathbf{n}_2(x_j, y) = -u_j$ for every $j = 1, 2, \ldots, N$. The existence of these maps is clear, because by the transversality assumption, both $\mathbf{n}_1(x_j, y)$ and $\mathbf{n}_2(x_j, y)$ are nonzero vectors in $\mathbf{R}^n$ and $\mathrm{GL}_n(\mathbf{R})$ acts transitively on them. By Lemma 2.1, the Schur multiplier with symbol

$$m_\varepsilon\big((\xi, j), \eta\big) = \chi_\Sigma\big(x_j + \varepsilon T_j \xi, y + \varepsilon\eta\big)$$

is bounded with norm $\leq C$ for every $\varepsilon > 0$. Taking $\varepsilon \to 0^+$, we obtain that the almost everywhere limit of m_ε is S_p-bounded with norm $\leq C$. However, this limit is $m_{u_j}(\xi, \eta)$ from claim (A). This proves claim (B). □

Remark 2.4. Taking

$$\Sigma = \big\{(x, y) : x - y \in \Omega\big\}$$

for a smooth domain Ω, Lemma 2.3 reduces to the classical Meyer's lemma.

Proof of (1)⇒(2) in Theorem A. Let us assume that (1) in Theorem A holds. By taking charts, we can and will assume that M and N are open subsets of $\mathbf{R}^m$ and $\mathbf{R}^n$, respectively. We shall further assume that $m = n$. For instance, if $m < n$, we can replace M by $M \times \mathbf{R}^{n-m}$ and Σ by

$$\Big\{\big((x, x'), y\big) \mid (x, y) \in \Sigma, x' \in \mathbf{R}^{n-m}\Big\}.$$

By the transversality assumption, the map $z \mapsto \mathbf{n}_2(z)/\|\mathbf{n}_2(z)\|$ is continuous on a neighbourhood of the transverse point (x_0, y_0) in Theorem A. Moreover, for y close to y_0, we have that $\partial\Sigma^y$ is locally a manifold, so is connected. Thus, if (2) was not true, there would exist y close to y_0 such that the subset of the sphere $X = \{\mathbf{n}_2(x', y)/\|\mathbf{n}_2(x', y)\| : x' \in \partial\Sigma^y \cap U\}$ contains a connected subset not reduced to a point. According to (1) and Lemma 2.3, this would imply that the square function inequality holds uniformly in $L_p(\mathbf{R}^n)$ for any finite set in X. However, Fefferman's main result in his proof of the ball multiplier theorem [15] claims that such a uniform inequality cannot hold. In fact, Fefferman stated it for $n = 2$, but the result in arbitrary dimension follows from the 2-dimensional case by K. de Leeuw's restriction theorem [29]. Hence, the zero-curvature condition (2) must hold. □

2.3. Proof of Theorem A: Zero-curvature implies triangular truncations

The implication (2)⇒(3) is a general geometric statement concerning transverse hypersurfaces in manifolds of product type. Let M, N be manifolds of dimension m, n. Following the terminology in the Introduction, we say that a $\mathcal{C}^1$-submanifold $\Pi \subset M \times N$ of codimension 1 is said to be transverse at $z = (x, y) \in \Pi$ if the tangent space of Π at z maps surjectively on each factor $T_x M$ and $T_y N$. In that case, $\Pi_x = \{y' \in N \mid (x, y') \in \Pi\}$ and $\Pi^y = \{x' \in M \mid (x', y) \in \Pi\}$ are manifolds on a neighbourhood y and x, respectively.

Theorem 2.5. *Let $\Pi \subset M \times N$ be a $\mathcal{C}^1$-submanifold of codimension 1 that is transverse at $z_0 = (x_0, y_0) \in \Pi$. Then, the following are equivalent:*

(a) *There are neighbourhoods U, V of x_0 and y_0 in M, N such that for every $x, x' \in U$ and $y \in V$ with $(x, y), (x', y) \in \Pi$, $T_y \Pi_x = T_y \Pi_{x'}$.*
(b) *There are neighbourhoods U, V of x_0 and y_0 in M, N such that for every $x \in U$ and $y, y' \in V$ with $(x, y), (x, y') \in \Pi$, $T_x \Pi^y = T_x \Pi^{y'}$.*
(c) *There are neighbourhoods U, V of x_0 and y_0 in M, N and C^1 submersions $f : U \to \mathbf{R}$ and $g : V \to \mathbf{R}$ with $\Pi \cap (U \times V) = \{(x, y) \in U \times V \mid f(x) = g(y)\}$.*

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The difficult direction in Theorem 2.5 is (a)⇒(c). Both conditions are invariant by diffeomorphisms of product type, that is of the form $(x,y) \mapsto (\phi(x), \psi(y))$. It will be useful to have a description of a local normal form (that is of an element in every orbit) of transverse manifolds.

Lemma 2.6. *Consider a $\mathcal{C}^1$-submanifold $\Pi \subset M \times N$ of codimension 1 that is transverse at $z_0 = (x_0, y_0) \in \Pi$. Then, there are diffeomorphisms ϕ and ψ from neighbourhoods U and V of x_0 and y_0, respectively, into $\mathbf{R}^m$ and $\mathbf{R}^n$ satisfying that $\phi(x_0) = 0 = \psi(y_0)$ and such that*

$$\Pi \cap (U \times V) = (\phi \times \psi)^{-1}\big\{(x,y) \mid x_1 = g(x_2, \ldots, x_m, y)\big\}$$

for some $\mathcal{C}^1$ function $g\colon \mathbf{R}^{m-1} \times \mathbf{R}^n \to \mathbf{R}$ satisfying $g(0,y) = y_1$ for every y.

Proof. By the transversality assumption that $T_{z_0}\Pi$ surjects onto $T_{y_0}M$, we see that $T_{z_0}\Pi \cap (T_{x_0}M \oplus 0) \neq T_{x_0}M \oplus 0$. Thus, by applying a local diffeomorphism $\phi : M \to \mathbf{R}^m$, we can assume that $M = \mathbf{R}^m$, $x_0 = 0$ and $(1, 0, \ldots, 0) \notin T_{z_0}\Pi$. Then by the implicit function theorem, there is a $\mathcal{C}^1$ function $h\colon \mathbf{R}^{m-1} \times N \to \mathbf{R}$ such that, on a neighbourhood of $(0, y_0)$,

$$\Pi = \big\{(x,y) \mid x_1 = h(x_2, \ldots, x_m, y)\big\}.$$

The function $h(0,\cdot)$ vanishes at y_0 and, by the second half of the transversality assumption, has nonzero differential at y_0. By the implicit function theorem (or the surjection theorem) again, there is a diffeomorphism ψ from a neighbourhood of y_0 into $\mathbf{R}^n$ vanishing at y_0 and such that $h(0,y) = \psi(y)_1$ for every y close enough to y_0. This proves the lemma with $g(0,y) = h(0, \psi^{-1}(y))$. □

Proof of Theorem 2.5. By symmetry of the two variables, it is enough to prove (a)⇔(c). The implication (c)⇒(a) is clear with the same U and V because in that case, $T_y\Pi_x$ is the kernel of $d_y g$, which is independent of x. It remains to prove the implication (a)⇒(c). Observe that both conditions are unchanged if we replace (x_0, y_0, Π) by $(\phi(x_0), \psi(y_0), \phi \times \psi(\Pi))$ for local diffeomorphisms. Therefore, by the normal form Lemma 2.6, we may assume that $M \times N = \mathbf{R}^m \times \mathbf{R}^n$, $(x_0, y_0) = (0,0)$ and

$$\Pi \cap (U \times V) = \Big\{(x,y) \in U \times V \mid x_1 = g(x_2, \ldots, x_m, y)\Big\}$$

for some $\mathcal{C}^1$ function $g\colon \mathbf{R}^{m-1} \times \mathbf{R}^n \to \mathbf{R}$ satisfying $g(0,y) = y_1$. Then, for every $(x,y) \in \Pi$ with $x = (x_1, \tilde{x})$, we have $T_y\Pi_x = \ker(d_y g(\tilde{x}, y))$. Let $\tilde{U} \subset \mathbf{R}^{m-1}, \tilde{V} \subset V$ be square neighbourhoods of 0 such that $(g(\tilde{x},y), \tilde{x}) \in U$ for $(\tilde{x}, y) \in \tilde{U} \times \tilde{V}$. Then for every such $\tilde{x}, y$, condition (a) applied to $x = (g(\tilde{x},y), \tilde{x})$ and $x' = (g(0,y), 0)$ yields $\ker d_y g(\tilde{x}, y) = \ker d_y g(0,y) = \operatorname{span}(e_2, \ldots, e_n)$. In particular, $\partial_{y_j} g(\tilde{x}, y) = 0$ for every $j \geq 2$, so (since $\tilde{V}$ is a square) $g(\tilde{x}, y) = w(\tilde{x}, y_1)$ for certain $\mathcal{C}^1$ function $w : \mathbf{R}^{m-1} \times \mathbf{R} \to \mathbf{R}$ satisfying $w(0,s) = s$ for all s. By the implicit function theorem, we get

$$\big\{(x_1, \tilde{x}, s) \mid x_1 = w(\tilde{x}, s)\big\} = \big\{(x_1, \tilde{x}, s) \mid s = u(x_1, \tilde{x})\big\}$$

locally for a $\mathcal{C}^1$ function $u\colon \mathbf{R}^m \to \mathbf{R}$. This completes the proof of Theorem 2.5. □

Proof of (2)⇒(3) in Theorem A. This is (a)⇒(c) in Theorem 2.5 for $\Pi = \partial\Sigma$. □

2.4. *Proof of Theorem A: Transference on triangular truncations*

We record finally the easy last implication, which completes the proof of Theorem A.

Proof of (3)⇒(1) in Theorem A. This is immediate from the classical boundedness of the triangular projection on Schatten p-classes for $1 < p < \infty$ (due to Macaev [48]; see also [17, Chap III, §6]) and the transference Lemma 2.1 above. □

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2.5. Relatively compact domains

Using a partition of unity argument, it is not difficult to prove that Theorem A holds globally for relatively compact fully transverse domains Σ. More precisely, let $p \in (1,\infty) \setminus \{2\}$ and consider a relatively compact domain Σ in $M \times N$ which is transverse at every point of $\partial\Sigma$. Then S_Σ is an S_p-bounded multiplier if and only if any of the equivalent conditions (2) and (3) in the statement of Theorem A holds at every point of the boundary.

Remark 2.7. The fact that Σ is relatively compact is crucial in the preceding argument. For instance, (2) holds trivially at every boundary point for every fully transverse $\mathcal{C}^1$-domain of $\mathbf{R} \times \mathbf{R}$. But there are examples of such domains – which are Toeplitz, arising from Fourier symbols – that do not define an S_p multiplier for any $p \neq 2$. An explicit construction is given in [6, Appendix A].

At this point, it is interesting to observe the difference here between Fourier and Schur idempotents. We know from Fefferman's theorem [15] that there are no Fourier L_p-idempotents associated to smooth compact domains. However, there are plenty such Schur idempotents: necessarily nonToeplitz, since Toeplitz symbols give rise to Fourier idempotents. A funny instance is precisely given by other forms of ball multipliers $\Sigma_R = \{(x,y) \in \mathbf{R}^n \times \mathbf{R}^n \colon |x|^2 + |y|^2 < R^2\}$, which are clearly S_p-bounded and have been recently used by Chuah-Liu-Mei in their recent paper [9, Example 4.4]. Theorem A proves in addition that the spheres $\partial\Sigma_R$ satisfy the zero-curvature condition (2). More intriguing examples are the spherical Hilbert transforms defined in the Introduction as

$$H_{\mathbf{S}} : A \mapsto \Big(-i \,\mathrm{sgn}\langle x,y\rangle A_{xy} \Big)_{x,y\in\mathbf{S}^n}.$$

More generally, we also define $H_{\mathbf{S},\delta} = -i(2S_{\Sigma_\delta} - \mathrm{id})$ with $S_{\Sigma_\delta}(A) = (\chi_{\langle x,y\rangle>\delta} A_{xy})$ for $\delta \in (-1,1)$. The case $\delta = 0$ corresponds to the spherical transform $H_{\mathbf{S}}$ above.

Corollary 2.8. *Let us fix $1 < p \neq 2 < \infty$. Then, the n-dimensional spherical Hilbert transforms $H_{\mathbf{S},\delta}$ are all S_p-bounded for $n = 1$ and S_p-unbounded for $n \geq 2$.*

Proof. Spherical Hilbert transforms $H_{\mathbf{S},\delta}$ arise from relatively compact domains Σ_δ whose boundary is fully transverse. In particular, we may apply Theorem A. In dimension 1, the assertion follows immediately since the zero-curvature condition (2) is trivially satisfied. Alternatively, the symbol can be expressed as a triangular truncation in terms of the polar coordinates of x and y. When $n \geq 2$, it is easily checked that the tangent spaces at $\partial\Sigma_{x_1}$ and $\partial\Sigma_{x_2}$ differ at their intersection points. This was illustrated for $n = 2$ in Figure 1. Theorem A implies the assertion. □

Remark 2.9. Alternatively, Corollary 2.8 also follows as a special case of Corollary B2. Indeed, Lemma 2.1 implies that $H_{\mathbf{S},\delta}$ has the same norm as the Schur multiplier on $\mathrm{SO}(n+1) \times \mathrm{SO}(n+1)$ with symbol $(g,h) \mapsto \mathrm{sgn}((g^{-1}h)_{1,1})$, which by [7] coincides with the cb-norm of the Fourier multipler with symbol $g \mapsto \mathrm{sgn}(g_{1,1})$. But for $n \geq 2$, $\mathrm{SO}(n+1)$ is a simple Lie group not locally isomorphic to $\mathrm{SL}_2(\mathbf{R})$, so it does not carry *any* idempotent multiplier.

Remark 2.10. We may also consider symbols $\Sigma_\delta = \{(x,y) \in \mathbf{R}^n : \langle x,y\rangle > \delta\}$ in the full Euclidean space for $n \geq 2$. In this case, Theorem A gives S_p-unboundedness for $(n,\delta) \neq (2,0)$. By [28, Theorem 1.18] and since $S_{\Sigma_0} = H_{\mathbf{S},0} \otimes \mathrm{id}_{\mathbf{R}_+}$, it turns out that S_p-boundedness for $(n,\delta) = (2,0)$ follows from Corollary 2.8.

2.6. Curvature on smoother domains

Our curvature condition (2) admits an alternative formulation under additional regularity. Let Σ be a $\mathcal{C}^2$-domain. Then $\Sigma \cap (U \times V) = \big\{(x,y) : F(x,y) > 0\big\}$ for some $\mathcal{C}^2$-function $F : M \times N \to \mathbf{R}$ and small enough neighbourhoods U, V. Our curvature condition holds if and only if we have

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$$\Big\langle d_x d_y F(x,y), u \otimes v \Big\rangle := u^{\mathrm{t}} \cdot \Big(\partial_{x_j} \partial_{y_k} F(x,y) \Big)_{j,k} \cdot v = 0$$

for $(u,v) \in \ker d_x F(x,y) \times \ker d_y F(x,y)$ at every $(x,y) \in \partial\Sigma \cap (U \times V)$. The argument is quite simple. By fixing boundary points (x,y) and vectors (u,v) as specified above, let $\gamma : [0,1] \to \partial\Sigma^y \cap U$ be a curve with $\gamma(0) = x$ and $\gamma'(0) = u$, and set $h(s) = d_y F(\gamma(s), y)$. The curvature condition (2) means that $h(s) = \alpha(s) h(0)$ for some nonvanishing function $\alpha : [0,1] \to \mathbf{R}$. In particular, we get

$$\Big\langle d_x d_y F(x,y), u \otimes v \Big\rangle = \big\langle h'(0), v \big\rangle = \alpha'(0) \big\langle h(0), v \big\rangle = 0.$$

Reciprocally, assume that the $\mathcal{C}^2$-curvature condition above holds. Consider a curve $\gamma\colon [0,1] \to \partial\Sigma^y \cap U$ and define h as above. Since we have $\gamma'(s) \in \ker d_x F(\gamma(s), y)$ and $h'(s) = \gamma'(s)^{\mathrm{t}} \cdot d_x d_y F(\gamma(s), y)$ by construction, it turns out that $\langle h'(s), v \rangle$ equals $\langle d_x d_y F(\gamma(s), y), \gamma'(s) \otimes v \rangle$ for any $v \in \ker d_y F(\gamma(s), y)$. Applying the $\mathcal{C}^2$-curvature condition, this implies that $h'(s)$ is parallel to $h(s)$ for every s, which leads to the ODE

$$\left. \begin{array}{rl} h'(s) & = \lambda(s) h(s) \\ h(0) & = d_y F(x,y) \end{array} \right\} \Rightarrow h(s) = \exp\Big(\int_0^s \lambda(t) dt \Big) h(0) = \alpha(s) h(0)$$

for a nonvanishing $\alpha : [0,1] \to \mathbf{R}$. This implies condition (2) in Theorem A.

Remark 2.11. In this form, *(2)* is invariant under exchanging x and y, which is clear a posteriori without the $\mathcal{C}^2$ assumption, since both *(1) and (3)* are. However, condition *(2)* in Theorem A seems new, while its $\mathcal{C}^2$-form above is quite similar to the *rotational curvature* $\det[d_x d_y F(x,y)]$ defined by Stein in [45, XI.3.1].

2.7. *On the transversality condition*

The transversality assumption has been essential in our proofs of (1)⇒(2)⇒(3) in Theorem A, but it is not clear that it is really needed for the statement. Indeed, conditions (1) and (3) make sense without it, and (2) is already meaningful if one only assumes that $\mathbf{n}_2(x_0, y_0) \neq 0$, and we do not have an example where the equivalence fails. It is likely that such examples can be found, but probably not for domains with analytic boundary. We leave these questions as open problems. In the degenerate case where $\mathbf{n}_1$ is identically 0, or equivalently when Σ is locally of the form $\Sigma = \{(x,y) : y \in \Omega\}$, all conditions in Theorem A hold. The S_p-boundedness is in that case even true for $1 \leq p \leq \infty$ because the Schur multiplier whose symbol is the indicator function of Σ is just the right-multiplication by the orthogonal projection on $L_2(\Omega)$.

3. Idempotent Fourier multipliers on Lie groups

Let G be a Lie group, that we equip with a left Haar measure. As to every locally compact group, we can associate to it the following:

- Its von Neumann algebra $\mathcal{L}\mathrm{G}$.
- The noncommutative L_p spaces $L_p(\mathcal{L}\mathrm{G})$ for $1 \leq p < \infty$.
- The Fourier L_p-multipliers T_m with symbol $m\colon \mathrm{G} \to \mathbf{C}$.

The group von Neumann algebra $\mathcal{L}\mathrm{G}$ is the weak-$*$ closure in $B(L_2(\mathrm{G}))$ of the algebra of convolution operators $\lambda(f) : \xi \in L_2(\mathrm{G}) \mapsto f * \xi$ for $f \in \mathcal{C}_c(\mathrm{G})$. When G is unimodular, its L_p-theory is quite elementary: $\mathcal{L}\mathrm{G}$ carries a natural semifinite trace τ given by $\tau(\lambda(f)^*\lambda(f)) = \int |f(g)|^2 dg$ for every $f \in L_2(\mathrm{G})$ with $\lambda(f) \in \mathcal{L}\mathrm{G}$; $L_p(\mathcal{L}\mathrm{G})$ is then defined as the completion of $\{x \in \mathcal{L}\mathrm{G} : \|x\|_p < \infty\}$ for the norm $\|x\|_p = \tau(|x|^p)^{1/p}$. It turns out that $L_p(\mathcal{L}\mathrm{G})$ contains $\{\lambda(f) : f \in \mathcal{C}_c(\mathrm{G}) * \mathcal{C}_c(\mathrm{G})\}$ as a dense subspace. A bounded measurable $m\colon \mathrm{G} \to \mathbf{C}$ defines a Fourier L_p-multiplier if $\lambda(f) \mapsto \lambda(mf)$ extends

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to a bounded map T_m on $L_p(\mathcal{L}\mathrm{G})$. These definitions are more involved for nonunimodular groups and will be recalled in Section 3.3 below.

When $p = 1, \infty$, a bounded measurable function $m\colon \mathrm{G} \to \mathbf{C}$ defines a completely bounded Fourier L_p-multiplier if and only if the Schur multiplier associated to the symbol $(g,h) \mapsto m(gh^{-1})$ – called the Herz-Schur multiplier with symbol m and denoted S_m – is completely S_p-bounded, with same norms [3]. For amenable groups, the same holds for $1 < p < \infty$ [7, 34], and it is an intriguing open problem whether this holds beyond amenable groups. We shall use that this always holds locally. This phenomenon was discovered recently [35, Theorem 1.4] when p is an even integer and G unimodular, and the following generalizes this to the general case; see [8] for other local results of similar nature. In what follows, the Fourier support of an element $x \in L_p(\mathcal{L}\mathrm{G})$ will refer to the smallest closed subset Λ such that $T_m(x) = 0$ for every Fourier L_p-multiplier with symbol m whose support is a compact subset of $\mathrm{G} \setminus \Lambda$. When G is unimodular and $x = \lambda(f)$ for $f \in \mathcal{C}_c(\mathrm{G}) * \mathcal{C}_c(\mathrm{G})$, it is easy to see that this coincides with the support of the function f.

Theorem 3.1. *Let* G *be a locally compact group and consider a bounded measurable function* $m\colon \mathrm{G} \to \mathbf{C}$. *Then, the following are equivalent for* $p \in (1,\infty)$ *and* $g_0 \in \mathrm{G}$:

(a) *There is a neighbourhood* U *of* g_0 *such that the restriction* $T_{m,U}$ *of* T_m *to the space of elements of* $L_p(\mathcal{L}\mathrm{G})$ *Fourier supported in* U *is completely bounded.*
(b) *There exists a function* $\varphi\colon \mathrm{G} \to \mathbf{C}$ *which equals* 1 *on a neighbourhood of* g_0 *such that* φm *defines a completely bounded Fourier multiplier on* $L_p(\mathcal{L}\mathrm{G})$.
(c) *There are open sets* $V, W \subset \mathrm{G}$ *with* $g_0 \in VW^{-1}$ *such that the function* $(g,h) \in V \times W \mapsto m(gh^{-1})$ *defines a completely bounded Schur multiplier on the Schatten class* $S_p(L_2(V), L_2(W))$.

When these conditions hold, we say that m defines locally at g_0 a completely bounded Fourier L_p-multiplier. The proof is given in Section 3.2. We can record the following consequence, which is immediate by looking at condition (c).

Corollary 3.2. *Let* G *be a connected Lie group and denote by* $\widetilde{\mathrm{G}}$ *its universal cover. Let* $\tilde{g}_0 \in \widetilde{\mathrm{G}}$ *be any lift of* $g_0 \in \mathrm{G}$. *Then* $m\colon \mathrm{G} \to \mathbf{C}$ *defines locally at* g_0 *a completely bounded Fourier* L_p*-multiplier over* $\mathcal{L}\mathrm{G}$ *if and only its lift* $\widetilde{m}$ *defines locally at* $\tilde{g}_0$ *a completely bounded Fourier* L_p*-multiplier as well.*

3.1. Idempotent multipliers

Now we are ready to prove Theorem B and also Corollaries B1 and B2 from the Introduction. In fact, we shall prove a slightly expanded version of Theorem B which includes non-simply connected groups and Corollary B1 at once. The groups $\mathrm{G}_1, \mathrm{G}_2, \mathrm{G}_3$ in the statement below refer to the real line $\mathbf{R}$, $\mathrm{Aff}_+(\mathbf{R})$ and $\widetilde{\mathrm{PSL}}_2(\mathbf{R})$ as in the Introduction.

Theorem 3.3. *Let* $p \in (1,\infty) \setminus \{2\}$. *Let* G *be a connected Lie group,* $\Omega \subset \mathrm{G}$ *a* $\mathcal{C}^1$*-domain and* $g_0 \in \partial\Omega$ *a point in the boundary of* Ω. *Consider the following conditions*:

(1) χ_Ω *defines locally at* g_0 *a completely bounded Fourier* L_p*-multiplier.*
(2) *There is a smooth action* $\mathrm{G} \to \mathrm{Diff}(\mathbf{R})$ *by diffeomorphisms on the real line, such that* Ω *coincides on a neighbourhood of* g_0 *with* $\{g \in \mathrm{G} \mid g \cdot 0 > g_0 \cdot 0\}$.
(3) *There is* $j \in \{1,2,3\}$ *and a smooth surjective homomorphism* $f\colon \mathrm{G} \to \mathrm{G}_j$ *such that the domain* Ω *coincides on a neighbourhood of* g_0 *with* $g_0 f^{-1}(\Omega_j)$.
(4) $\partial\Omega = g_0 \exp(\mathfrak{h})$ *locally near* g_0 *for some codimension* 1 *Lie subalgebra* $\mathfrak{h} \subset \mathfrak{g}$.

Then $(1) \Leftrightarrow (4) \Leftarrow (2) \Leftrightarrow (3)$. *If* G *is simply connected, then we also have* $(4) \Rightarrow (2)$.

Proof. The main difficulty is to prove the equivalence $(1) \Leftrightarrow (4)$, which we leave to the end of the proof. The implication $(2) \Rightarrow (4)$ is clear, with $\mathrm{H} = \exp(\mathfrak{h})$ the stabilizer of 0. Under the assumption that G is simply connected, the converse $(4) \Rightarrow (2)$ holds by a Theorem of Mostow [33], which implies that H is a closed subgroup. Therefore, G/H is a 1-dimensional manifold that is simply connected,

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and so is diffeomorphic to $\mathbf{R}$. The implication (3) $\Rightarrow$ (2) is also clear because G_j is given as a group of diffeomorphisms of $\mathbf{R}$ with $\Omega_j = \{g \in \mathrm{G}_j \mid g \cdot 0 > 0\}$. The converse (2) $\Rightarrow$ (3) follows from Lie's classification of (local) actions by diffeomorphism on the real line [30]; see also [47] for modern presentations and [16] for the global aspect. More precisely, the fact that g_0 belongs to the boundary of Ω implies that 0 is not fixed by G and – here we use that G is connected – the G-orbit of 0 is an open interval, so by identifying it with $\mathbf{R}$, we can assume that the G-action is transitive. In that case, the image of G in $\mathrm{Diff}(\mathbf{R})$ is one of the three groups in condition (3) of Corollary B1, see [16, Section 4.1] for the details.

Next, let us focus on the equivalence (1) $\Leftrightarrow$ (4) for general Lie groups. If we translate Ω by g_0^{-1}, we may assume that $g_0 = e$ and the tangent space of G at g_0 identifies with its Lie algebra $\mathfrak{g}$. Also, the tangent space of $\partial\Omega$ at g_0 identifies with a codimension 1 subspace $\mathfrak{h}$ of $\mathfrak{g}$. Define the $\mathcal{C}^1$-manifold

$$\widetilde{\Omega} = \Big\{(g,h) \in \mathrm{G} \times \mathrm{G} \mid gh \in \Omega\Big\}.$$

Its sections $\widetilde{\Omega}_g$ and $\widetilde{\Omega}^h$ are left and right translates of the $\mathcal{C}^1$-domain Ω

$$\widetilde{\Omega}_g := \big\{h : (g,h) \in \widetilde{\Omega}\big\} = g^{-1}\Omega \quad \text{and} \quad \widetilde{\Omega}^h := \big\{g : (g,h) \in \widetilde{\Omega}\big\} = \Omega h^{-1}. \tag{3.1}$$

In particular, $\widetilde{\Omega}$ is transverse at every point of its boundary. By Lemma 2.1 and Theorem 3.1, we know that (1) is equivalent to the existence of a neighbourhood of the identity $U \subset \mathrm{G}$ such that $\chi_{\widetilde{\Omega}}$ defines a Schur multiplier on $S_p(L_2(U))$. By Theorem A, this is equivalent to the existence of a neighbourhood of the identity $V \subset \mathrm{G}$ such that both conditions below hold:

$$T_h \partial\widetilde{\Omega}_{g_1} = T_h \partial\widetilde{\Omega}_{g_2} \text{ for every } g_1, g_2, h \in V \text{ such that } g_1 h, g_2 h \in \partial\Omega. \tag{3.2}$$

$$T_g \partial\widetilde{\Omega}^{h_1} = T_g \partial\widetilde{\Omega}^{h_2} \text{ for every } g, h_1, h_2 \in V \text{ such that } g h_1, g h_2 \in \partial\Omega. \tag{3.3}$$

By the above idenfications (3.1), if we denote by $L_x, R_x : \mathrm{G} \to \mathrm{G}$ the left and right multiplication by x, these conditions are equivalent to the existence of a neighbourhood of the identity W such that

$$d_{x_1} L_{x_2 x_1^{-1}}(T_{x_1}\partial\Omega) = T_{x_2}\partial\Omega \text{ for every } x_1, x_2 \in \partial\Omega \cap W. \tag{3.4}$$

$$d_{x_1} R_{x_1^{-1} x_2}(T_{x_1}\partial\Omega) = T_{x_2}\partial\Omega \text{ for every } x_1, x_2 \in \partial\Omega \cap W. \tag{3.5}$$

Indeed, taking $x_j = g_j h$ for $j = 1, 2$ we have

$$T_h(g_j^{-1}\partial\Omega) = d_{x_j} L_{g_j^{-1}}(T_{x_j}\partial\Omega).$$

Composing by $(d_{x_2} L_{g_2^{-1}})^{-1} = d_h L_{g_2}$, and using $(d_h L_{g_2}) \circ (d_{x_1} L_{g_1^{-1}}) = d_{x_1} L_{x_2 x_1^{-1}}$ by the chain rule, we see that (3.2) is equivalent to (3.4). The equivalence for right multiplication maps is entirely similar. Next, recalling that $T_e\partial\Omega = \mathfrak{h}$, the above conditions can be written in the equivalent forms:

$$d_e L_x(\mathfrak{h}) = T_x\partial\Omega \text{ for every } x \in \partial\Omega \cap W. \tag{3.6}$$

$$d_e R_x(\mathfrak{h}) = T_x\partial\Omega \text{ for every } x \in \partial\Omega \cap W. \tag{3.7}$$

If we remember that $\mathrm{Ad}_x = d_e(R_{x^{-1}} L_x)$, we obtain that this system is equivalent to $d_e L_x(\mathfrak{h}) = T_x\partial\Omega$ and $\mathrm{Ad}_x \mathfrak{h} = \mathfrak{h}$ for every $x \in \partial\Omega \cap W$. Therefore, we have proved that (1) at $g_0 = e$ is equivalent to the existence of a neighbourhood of the identity W such that $T_x\partial\Omega = d_e L_x(\mathfrak{h})$ and $\mathrm{Ad}_x \mathfrak{h} = \mathfrak{h}$ for every $x \in \partial\Omega \cap W$. These conditions clearly hold if $\mathfrak{h}$ is a Lie algebra and $\partial\Omega$ locally coincides with the exponential of a neighbourhood of 0 in $\mathfrak{h}$. Conversely, assume $T_x\partial\Omega = d_e L_x(\mathfrak{h})$ and $\mathrm{Ad}_x(\mathfrak{h}) = \mathfrak{h}$ for every $x \in \partial\Omega \cap W$. Making x go to the identity element e in the second condition, we deduce that $\mathrm{ad}_{\mathrm{X}}(\mathfrak{h}) \subset \mathfrak{h}$ for every $\mathrm{X} \in \mathfrak{h}$. That is, $\mathfrak{h}$ is a Lie subalgebra. By the local uniqueness of a manifold in

G containing e and whose tangent space at x is $d_eL_x(\mathfrak{h})$ (Frobenius' theorem), we deduce that $\partial\Omega$ is locally the exponential of a neighbourhood of 0 in $\mathfrak{h}$. This completes the proof. □

Remark 3.4. By a partition of the unity argument, the following global form of Theorem 3.3 holds: if $p \in (1,\infty)\setminus\{2\}$, G is a connected Lie group and $\Omega \subset$ G a relatively compact $\mathcal{C}^1$-domain, then χ_Ω defines a Fourier cb-L_p-multiplier if and only if the condition (4) holds for every point g_0 in the boundary of Ω.

Proof of Corollary B2. Assertion i) follows since the quotient of a nilpotent Lie algebra remains nilpotent, so the nonnilpotent examples in Theorem 3.3 (3) cannot happen when G is nilpotent. Assertions ii) and iii) follow immediately from Lie's classification [30]: up to isomorphism, there is a unique pair $(\mathfrak{h},\mathfrak{g})$ where $\mathfrak{g}$ is a simple Lie algebra and $\mathfrak{h}$ is a codimension 1 subalgebra. It is given by $\mathfrak{g} = \mathfrak{sl}_2$ and $\mathfrak{h}$ the subalgebra of upper-triangular matrices. This completes the proof. □

3.2. Local Fourier-Schur transference

The rest of this paper will be devoted to justify Theorem 3.1. We will sometimes consider the Fourier algebra $A(\mathrm{G})$ of G [14] – that is,

$$A(\mathrm{G}) = \Big\{g \mapsto \int \phi(gh)\psi(h)\,dh : \phi,\psi \in L_2(\mathrm{G})\Big\}.$$

A form of the following lemma was proved in [35, Lemma 1.3] for p an even integer and G unimodular, which was enough for the applications there. Here, we need a form valid for every p and every locally compact group.

Lemma 3.5. *Let $V, W \subset$ G be open sets and $g_0 \in VW^{-1}$. Then, there are a neighbourhood U of g_0, a constant C, and maps $J_p\colon L_p(\mathcal{L}\mathrm{G}) \to S_p(L_2(V), L_2(W))$ for $1 \le p \le \infty$ intertwining Fourier and Herz-Schur multipliers and such that*

$$C^{-1}\|x\|_p \le \|J_p(x)\|_p \le C\|x\|_p$$

for every $n \ge 1$ and every $x \in M_n \otimes L_p(\mathcal{L}\mathrm{G})$ which is Fourier supported in U.

Remark 3.6. The proof of Lemma 3.5 that we present was kindly communicated to us by Éric Ricard. Our original proof was more complicated but worked whenever U is a relatively compact subset of VW^{-1}. The simpler version above is, however, enough to prove the main implication (c)⇒(a) in Theorem 3.1, and by using a partition of the unity argument in the Fourier algebra of G [14], it is not hard to deduce that this implication holds actually whenever U is a relatively compact subset of VW^{-1}.

For the reader's convenience, we first prove Lemma 3.5 and Theorem 3.1 for unimodular groups and explain in the next paragraph how to modify the definition of Fourier multiplier and the argument for nonunimodular groups.

Proof of Lemma 3.5 for G *unimodular.* Translating V and W, we can assume that the identity belongs to V and W and $g_0 = e$. Let U be a neighbourhood of e such that $U \subset V$ and $U^{-1}U \subset W$. Let $\phi = \frac{1}{|U|}\chi_U$ and $\psi = \chi_{U^{-1}U}$, so that

$$\int \phi(gh)\psi(h)dh = 1 \quad \text{for every} g \in U.$$

Consider the map

$$J_p\colon L_p(\mathcal{L}\mathrm{G}) \ni x \mapsto \phi^{\frac{1}{p}} x \psi^{\frac{1}{p}} \in S_p(L_2(V), L_2(W)),$$

where we identify ϕ, ψ with the operators of multiplication by ϕ, ψ. The convention is that $0^{\frac{1}{\infty}} = 0$. We claim that the maps J_p are completely bounded with cb-norm

$$\|J_p\|_{\mathrm{cb}(L_p, S_p)} \le \|\phi\|_{L_2(\mathrm{G})}^{\frac{1}{p}} \|\psi\|_{L_2(\mathrm{G})}^{\frac{1}{p}}$$

whenever $1 \le p \le \infty$. By interpolation, it is enough to justify the extreme cases $p = 1$ and $p = \infty$. The case $p = \infty$ is clear. For the case $p = 1$, we factorize $x = x_1 x_2$ so that $J_1(x) = \phi x_1 \cdot x_2 \psi$. Take them so that $\|x\|_1 = \|x_1\|_2 \|x_2\|_2$, and it suffices to show that both factors are bounded in $S_2(L_2(\mathrm{G})) \simeq L_2(\mathrm{G}\times\mathrm{G})$ by $\|\phi\|_{L_2(\mathrm{G})} \|x_1\|_{L_2(\mathcal{L}\mathrm{G})}$ and $\|\psi\|_{L_2(\mathrm{G})} \|x_2\|_{L_2(\mathcal{L}\mathrm{G})}$, respectively. Using that

$$J_\infty : x \mapsto \Big(\widehat{x}(gh^{-1})\Big),$$

the expected bounds follow from Plancherel theorem $L_2(\mathcal{L}\mathrm{G}) \simeq L_2(\mathrm{G})$. Moreover when $x = \lambda(f)$, the operator $J_p(x)$ has kernel $(\phi(g)^{1/p} f(gh^{-1}) \psi(h)^{1/p})$. Thus, it is clear that the map J_p intertwines the Fourier multiplier with symbol $g \mapsto m(g)$ and the Schur multiplier with symbol $(g,h) \mapsto m(gh^{-1})$.

The inequality $C^{-1}\|x\|_p \le \|J_p(x)\|_p$ is a bit more involved. Let $f \in M_n \otimes \mathcal{C}_c(U)$ with $\lambda(f) \in L_p(M_n \otimes \mathcal{L}\mathrm{G})$ and assume that $x = \lambda(f)$ by density. Let q be the conjugate exponent of p and $\gamma \in M_n \otimes \mathcal{C}_c(\mathrm{G})$ with $\lambda(\gamma) \in L_q(M_n \otimes \mathcal{L}\mathrm{G})$. Then we have

$$\begin{aligned}
\mathrm{Tr} \otimes \mathrm{Tr}_n \big(J_p(\lambda(f)) J_q(\lambda(\gamma))^*\big) &= \mathrm{Tr}_n \int_{\mathrm{G}\times\mathrm{G}} \phi(g) f \gamma^*(gh^{-1}) \psi(h) ds dt \\
&= \int_{\mathrm{G}} \mathrm{Tr}_n \big(f(g)\gamma(g)^*\big) \Big[\int_{\mathrm{G}} \phi(gh)\psi(h)\, dh\Big] dg \\
&= \int_{\mathrm{G}} \mathrm{Tr}_n \big(f(g)\gamma(g)^*\big)\, dg = \tau \otimes \mathrm{Tr}_n \big(\lambda(f)\lambda(\gamma)^*\big).
\end{aligned}$$

In the last line, we used that $\int_{\mathrm{G}} \phi(gh)\psi(h)\, dh = 1$ on $\mathrm{supp} f \subset U$. By Hölder's inequality, we get

$$\begin{aligned}
\big|\tau \otimes \mathrm{Tr}_n(\lambda(f)\lambda(\gamma)^*)\big| &\le \|J_p(\lambda(f))\|_p \|J_q(\lambda(\gamma))\|_q \\
&\le \|J_q\|_{\mathrm{cb}} \|J_p(\lambda(f))\|_p \|\lambda(\gamma)\|_q \\
&\le \|J_q\|_{\mathrm{cb}} \|\lambda(\gamma)\|_q \|J_p(\lambda(f))\|_p.
\end{aligned}$$

Taking the sup over γ, we get $C^{-1}\|x\|_p \le \|J_p(x)\|_p$ for $C = \|J_q\|_{\mathrm{cb}} < \infty$. □

With the same argument as in [35], we deduce the following:

Proof of Theorem 3.1. The implication (a)⇒(b) is easy. Indeed, if (a) holds and $\varphi \in A(\mathrm{G})$ is supported in U and equal to 1 on a neighbourhood of g_0 (for the construction of φ, see the proof of Lemma 3.5), then T_φ is completely bounded on $L_p(\mathcal{L}\mathrm{G})$ for every $1 \le p \le \infty$, and takes values in the space of elements Fourier supported in U. In particular, $T_{m\varphi} = T_{m,U} \circ T_\varphi$ is also completely bounded. The implication (b)⇒(c) follows from [7, Theorem 4.2], which implies that

$$(g,h) \in \mathrm{G}\times\mathrm{G} \mapsto m(gh^{-1})\varphi(gh^{-1})$$

defines a completely bounded Fourier L_p-multiplier. Thus, we get (c) if V, W are chosen so that $\varphi = 1$ on VW^{-1}. Finally, (c)⇒(a) follows from Lemma 3.5. □

3.3. Nonunimodular groups

Let G be an arbitrary locally compact group with modular function $\Delta\colon \mathrm{G} \to \mathbf{R}_+$. Our choice of Δ is characterized by the following identity for all $f \in \mathcal{C}_c(\mathrm{G})$

$$\int_{\mathrm{G}} f(hg)\,dh = \Delta(g)^{-1}\int_{\mathrm{G}} f(h)\,dh.$$

When G is not unimodular – that is, Δ is not the constant 1 function – the natural weight $\lambda(f)^*\lambda(f) \mapsto \int |f|^2$ on $\mathcal{L}\mathrm{G}$ is not tracial. Even when $\mathcal{L}\mathrm{G}$ is semifinite, it is better to work with the general definition of L_p spaces associated to a von Neumann algebra. Several concrete descriptions are possible: Haagerup's original one [21], Kosaki's complex interpolation [26], Connes-Hilsum's [12, 23]...; see [43]. Here, we will use the Connes-Hilsum spatial description because we want to rely on some results from [7, 46], to which we refer for precise definitions. In that case, $L_p(\mathcal{L}\mathrm{G})$ is realized as a space of unbounded operators on $L_2(\mathrm{G})$.

In [7], Caspers and the second-named author defined Fourier L_p-multipliers for symbols that ensure that the Fourier multiplier is completely bounded for every $1 \leq p \leq \infty$. Here, we extend the definition, allowing to talk about Fourier multipliers for a single p, and possibly only bounded. The shortest way to do so properly in this context is by using Terp's Hausdorff-Young inequality [46].

Informally, a typical element of $L_p(\mathcal{L}\mathrm{G})$ is of the form $\lambda(f)\Delta^{\frac{1}{p}}$ – where we are identifying the function Δ with the densely defined operator of multiplication by Δ on $L_2(\mathrm{G})$ – for some *suitable* function *f*. Keeping at the informal level, the Fourier multiplier with symbol $m\colon \mathrm{G} \to \mathbf{C}$ should be, whenever it exists, the operator acting as follows:

$$\lambda(f)\Delta^{\frac{1}{p}} \mapsto \lambda(mf)\Delta^{\frac{1}{p}}.$$

Making this definition precise requires some lengthy and unpleasant discussions about domains/cores of unbounded operators, but fortunately, we can rely on the results from [46], where these discussions have been performed. We shall need to distinguish the cases $p \geq 2$ and $p \leq 2$. Let $q = \frac{p}{p-1}$ be the conjugate exponent of p. When $p \geq 2$, the Fourier transform

$$\mathcal{F}_q\colon L_q(\mathrm{G}) \to L_p(\mathcal{L}\mathrm{G})$$

is an injective norm 1 linear map with dense image, where $\mathcal{F}_q(f)$ is defined as a suitable extension of $\lambda(f)\Delta^{1/p}$; see [46, Theorem 4.5]. When $p \leq 2$, the adjoint of $\mathcal{F}_p$ gives a norm 1 injective map with dense image $\overline{\mathcal{F}}_p\colon L_p(\mathcal{L}\mathrm{G}) \to L_q(\mathrm{G})$. If I_q denotes the isometry of $L_q(\mathrm{G})$ defined by $I_q(f)(g) = f(g^{-1})\Delta(g)^{-1/q}$, we know from [46, Proposition 1.15] that every element x of $L_p(\mathcal{L}\mathrm{G})$ is a suitable extension of $\lambda(f)\Delta^{1/p}$ for $f = I_q \circ \overline{\mathcal{F}}_p(x) \in L_q(\mathrm{G})$. In the particular case $p = 2$, these two statements together yield Plancherel's formula: $\mathcal{F}_2$ is a unitary. If $p = 1$, the image of $\overline{\mathcal{F}}_1$ is the Fourier algebra $A(\mathrm{G})$, and following standard notation, we write

$$\mathrm{tr}(x) = \varphi(e) \quad \text{if} \quad \overline{\mathcal{F}}_1(x) = \varphi. \tag{3.8}$$

Definition 3.7. Let $1 < p < \infty$ and $m \in L_\infty(\mathrm{G})$. We say that m defines a bounded Fourier L_p-multiplier when the condition below holds according to the value of p:

- Case $p \geq 2$. The map

$$\mathcal{F}_q(f) \mapsto \mathcal{F}_q(mf)$$

  (densely defined on $\mathcal{F}_q(L_q(\mathrm{G}))$) extends to a bounded map T_m on $L_p(\mathcal{L}\mathrm{G})$.
- Case $p \leq 2$. The multiplication by m preserves the image of $I_q \circ \overline{\mathcal{F}}_p$ when the map $T_m : x \mapsto (I_q \circ \overline{\mathcal{F}}_p)^{-1}(m(I_q \circ \overline{\mathcal{F}}_p(x)))$ is a bounded map on $L_p(\mathcal{L}\mathrm{G})$.

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We say that m defines a completely bounded Fourier L_p-multiplier when m defines a bounded Fourier L_p-multiplier and the Fourier multiplier T_m is completely bounded.

It follows from the above definition that m defines a (completely) bounded L_p multiplier if and only it defines a (completely) bounded L_q multiplier, and in that case,

$$\operatorname{tr}(T_m(x)y^*) = \operatorname{tr}(x(T_{\overline{m}}(y))^*) \quad \text{for all} \quad x \in L_p(\mathcal{L}\mathrm{G}), y \in L_q(\mathcal{L}\mathrm{G}).$$

Once we have polished the definition of Fourier L_p-multipliers in nonunimodular group von Neumann algebras, we can extend a Cotlar identity from [19] to arbitrary locally compact groups.

Example 3.8. Let $\mathrm{G} \to \mathrm{Homeo}_+(\mathbf{R})$ be a continuous action of a connected Lie group. Then, the indicator function m of $\{g \in \mathrm{G} \mid g \cdot 0 > 0\}$ defines a completely bounded L_p Fourier multiplier on G with completely bounded norm $\leq 2\max\{p, \frac{p}{p-1}\}$.

Proof. Let m be the indicator function of $\{g \in \mathrm{G} \mid g \cdot 0 > 0\}$. It suffices to prove the following implication for every $2 \leq p < \infty$: if m defines a completely bounded Fourier L_p-multiplier with norm $\leq C_p$, then it defines a completely bounded Fourier L_{2p}-multiplier with norm $\leq 2C_p$. Indeed, using that $C_2 = \|m\|_\infty = 1$, we deduce $C_{2^N} \leq 2^N$ for every integer N, so by interpolation, $C_p \leq 2p$ for all $p \geq 2$. By duality, the conclusion also holds for $p \leq 2$.

Let r be the dual exponent of $2p$. Let $f \in \mathcal{C}_c(\mathrm{G})$ and consider $X = \mathcal{F}_r(f)$ and $Y = \mathcal{F}_r(mf)$; these are well-defined elements of $L_{2p}(\mathcal{L}G)$ by [46]. Then, we claim that the equality below holds:

$$Y^*Y = T_m(Y^*X) + T_m(Y^*X)^*. \tag{3.9}$$

Indeed, this inequality is equivalent to the almost everywhere equality

$$(mf)^* * (mf) = m\big((mf)^* * f\big) + \big(m\big((mf)^* * f\big)\big)^*,$$

which follows for the fact that $m(g^{-1})m(g^{-1}h) = m(h)m(g^{-1}) + m(h^{-1})m(g^{-1}h)$ for almost every $g, h \in \mathrm{G}$. If the whole group G fixes 0, this is obvious because m is identically 0. Otherwise, the stabilizer of 0 is a closed subgroup, so it has measure 0 and it is enough to justify the equality for $h \cdot 0 \neq 0$. Set $(\alpha, \beta) = (g \cdot 0, h \cdot 0)$ and observe that $m(g^{-1})m(g^{-1}h) = 1$ if and only if $\alpha < \min\{0, \beta\}$. Similarly, we have $m(h)m(g^{-1}) = 1$ iff $\alpha < 0 < \beta$ and $m(h^{-1})m(g^{-1}h) = 1$ iff $\alpha < \beta < 0$. Therefore, the expected identity reduces to the trivial one $\chi_{\alpha < 0 \wedge \beta} = \chi_{\alpha < 0 < \beta} + \chi_{\alpha < \beta < 0}$. This justifies (3.9), both sides of which are in $L_p(\mathcal{L}\mathrm{G})$. Thus, taking the norm and applying the triangle inequality, the hypothesis and Hölder's inequality leads to

$$\|Y\|_{2p}^2 \leq 2C_p \|X\|_{2p} \|Y\|_{2p}.$$

We deduce $\|Y\|_{2p} \leq 2C_p \|X\|_{2p}$. Since $\mathcal{C}_c(\mathrm{G})$ is dense in $L_r(\mathrm{G})$, we obtain that m defines a Fourier L_{2p}-multiplier with norm $\leq 2C_p$. A similar argument gives the same bound for the completely bounded norm, which concludes the proof. □

Remark 3.9. The Cotlar-type identity from [19] is refined in some cases by (3.9).

The following summarizes the properties that we need.

Lemma 3.10. *Let $1 \leq p \leq \infty$ and consider functions $\phi, \psi \in L_{2p}(\mathrm{G})$, which we identify with (possibly unbounded) multiplication operators on $L_2(\mathrm{G})$. Then*

- *Given $x \in L_{2p}(\mathcal{L}\mathrm{G})$, $x\phi$ is densely defined and closable. In fact, its closure $[x\phi]$ belongs to $S_{2p}(L_2(\mathrm{G}))$ and has S_{2p}-norm $\leq \|\phi\|_{L_{2p}(\mathrm{G})} \|x\|_{L_{2p}(\mathcal{L}\mathrm{G})}$.*
- *There exists a bounded linear map $L_p(\mathcal{L}\mathrm{G}) \to S_p(L_2(\mathrm{G}))$[3] sending y^*x to $[y\psi^*]^*[x\phi]$ for every $x, y \in L_{2p}(\mathcal{L}\mathrm{G})$. It has norm $\leq \|\phi\|_{L_{2p}(\mathrm{G})} \|\psi\|_{L_{2p}(\mathrm{G})}$.*

[3] That, with a slight abuse of notation, we denote $z \mapsto \psi z \phi$.

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- *If q denotes the conjugate exponent of p, consider $\phi', \psi' \in L_{2q}(\mathrm{G})$ and $y \in L_q(\mathcal{L}\mathrm{G})$. Then, we have*

$$\mathrm{Tr}\left(\phi x \psi (\phi' y \psi')^*\right) = \mathrm{tr}(T_m(x) y^*), \tag{3.10}$$

where $m \in A(\mathrm{G})$ is the function $m(g) = \int_{\mathrm{G}} (\phi\overline{\phi'})(h)(\psi\overline{\psi'})(g^{-1}h)\, dh.$

Proof. When $\phi = \psi$ are indicator functions, the first two points were proved in [7, Proposition 3.3, Theorem 5.2]. The same argument applies in our case. Let us justify identity (3.10). First, observe that $\phi\overline{\phi'}$ and $\psi\overline{\phi'}$ belong to $L_2(\mathrm{G})$ by Hölder's inequality, so that m indeed belongs to $A(\mathrm{G})$. In particular, m defines a completely bounded L_1 and L_∞ Fourier multiplier. Thus, it also defines a Fourier L_p-multiplier [7, Definition-Proposition 3.5]. Therefore, by interpolation [7, Section 6], it suffices to prove (3.10) for $p = 1$ and $p = \infty$. These two cases are formally equivalent, and we just consider $p = \infty$. In that case, $y \in L_1(\mathcal{L}\mathrm{G})$ corresponds to an element $f \in A(\mathrm{G})$ and $\phi' y \psi'$ is the trace class operator with kernel

$$\Big(\phi'(g) f(hg^{-1}) \psi'(h)\Big)_{g,h\in\mathrm{G}};$$

see [7, Lemma 3.4]. By a weak-$*$ density argument, it is enough to prove (3.10) for $x = \lambda(g_0)$ for some $g_0 \in \mathrm{G}$. In that case, $T_m(x) = m(g_0)\lambda(g_0)$ and we obtain $\mathrm{tr}(T_m(x)y^*) = m(g_0)\overline{f(g_0^{-1})}$. We can compute

$$\begin{aligned}\mathrm{Tr}\left(\phi x \psi (\phi' y \psi')^*\right) &= \mathrm{Tr}\left(\lambda(g_0)\psi\overline{\psi'} y^* \phi\overline{\phi'}\right) \\ [5pt] &= \mathrm{Tr}\left[\Big((\psi\overline{\psi'})(g_0^{-1}g)\overline{f(g_0^{-1}gh^{-1})}(\phi\overline{\phi'})(h)\Big)_{g,h\in\mathrm{G}}\right] \\ &= \int_{\mathrm{G}} (\psi\overline{\psi'})(g_0^{-1}g)\overline{f(g_0^{-1})}(\phi\overline{\phi'})(g)\, dg = m(g_0)\overline{f(g_0^{-1})}.\end{aligned}$$

This justifies the identity (3.10) and completes the proof of the lemma. □

Lemma 3.10 allows us to adapt the proof of Lemma 3.5 from the unimodular case.

Proof of Lemma 3.5, general case. We take $\phi, \psi \in \mathcal{C}_c(\mathrm{G})$ as in the proof in the unimodular case, and set $m(g) = \int \phi(gh)\psi(h) dh$. By Lemma 3.10, we can define completely bounded maps

$$J_p \colon L_p(\mathcal{L}\mathrm{G}) \ni x \mapsto \phi^{\frac{1}{p}} x \psi^{\frac{1}{p}} \in S_p(L_2(V), L_2(W)),$$

which intertwine Fourier and Schur multipliers. Now, if $x \in L_p(M_n \otimes \mathcal{L}\mathrm{G})$ is Fourier supported in U and $y \in L_q(M_n \otimes \mathcal{L}\mathrm{G})$ – for q being the dual exponent of p – we get

$$\begin{aligned}\mathrm{tr}(xy^*) &= \mathrm{tr}\left(T_m(x)y^*\right) = \mathrm{Tr}\left(J_p(x)J_q(y)^*\right) \\ &\le \|J_p(x)\|_{S_p}\|J_q(y)\|_{S_q} \le \|J_q\|_{\mathrm{cb}}\|y\|_{L_q(\mathcal{L}\mathrm{G})}\|J_p(x)\|_{S_p}.\end{aligned}$$

The first line is because $m = 1$ on U and x is Fourier supported in U, and by (3.10). The last line is Hölder's inequality. Taking suprema over y in the unit ball of $L_q(\mathcal{L}\mathrm{G})$ gives $\|x\|_{L_p(\mathcal{L}\mathrm{G})} \le \|J_q\|_{\mathrm{cb}}\|J_p(x)\|_{S_p}$. □

3.4. The group $\mathrm{SL}_2(\mathbf{R})$

Consider the symbol

$$m_0\left[\begin{pmatrix} a & b \\ c & d\end{pmatrix}\right] = \frac{1}{2}\Big(1 + \mathrm{sgn}(ac + bd)\Big).$$

This was identified in [19] as the canonical Hilbert transform (Riesz projection would be more accurate though) in $\mathrm{SL}_2(\mathbf{Z})$. Its complete L_p-boundedness follows for $1 < p < \infty$ from a Cotlar-type identity.

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The same problem in $\mathrm{SL}_2(\mathbf{R})$ was left open in [19, Problem A]. Now this is solved by condition (4) in Theorem 3.3, which disproves cb-L_p-boundedness for any $p \neq 2$. However, according to Corollary B2, the map

$$m(g) = m\left[\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right] := \frac{1}{2}\big(1 + \mathrm{sgn}(c)\big) = m_0(gg^{\mathrm{t}})$$

does define, locally at every point of its boundary, a completely bounded Fourier L_p-multiplier for every $1 < p < \infty$. But, is it globally L_p-bounded? Is it completely L_p-bounded as well? We leave these problems open for future attempts.

3.5. Stratified Lie groups

A Lie algebra $\mathfrak{g}$ is called graded when there exists a finite family of subspaces $\mathrm{W}_1, \mathrm{W}_2, \ldots, \mathrm{W}_N$ of the Lie algebra satisfying conditions below:

$$\mathfrak{g} = \bigoplus_{j=1}^{N} \mathrm{W}_j \qquad \text{and} \qquad [\mathrm{W}_j, \mathrm{W}_k] \subset \mathrm{W}_{j+k}.$$

A simply connected Lie group G is called stratified when its Lie algebra $\mathfrak{g}$ is graded and the first stratum W_1 generates $\mathfrak{g}$ as an algebra. Stratified Lie groups are nilpotent and include, among many other examples, Heisenberg groups. According to Corollary B2, idempotent multipliers are of the form $R \circ \varphi$, for the classical Riesz projection $R = \frac{1}{2}(iH + \mathrm{id})$ and some continuous homomorphism $\varphi : \mathrm{G} \to \mathbf{R}$. A quick look at Theorem 3.3 shows that φ corresponds on the Lie algebra with the projection onto any 1-dimensional subspace of the first stratum, since codimension 1 Lie subalgebras are exactly those codimension 1 subspaces leaving out a vector in the first stratum.

Competing interest. The authors have no competing interests to declare.

Financial support. The research of JP and ET was partially supported by the Spanish Research Grant PID2022-141354NB-I00 ‘Fronteras del Análisis Armónico’ (MCIN) as well as Severo Ochoa Grant CEX2019-000904-S (ICMAT), funded by MCIN/AEI 10.13039/501100011033. The research of MdlS was partially supported by the Charles Simonyi Endowment at the Institute for Advanced Study, and the ANR projects ANCG ANR-19-CE40-0002 and PLAGE ANR-24-CE40-3137. ET was supported as well by Spanish Ministry of Universities with a FPU Grant with reference FPU19/00837.

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