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NOTE ON AN ASYMPTOTIC FORMULA FOR A CLASS OF DIGRAPHS

BY

M. R. SRIDHARAN

Self-complementary digraphs, and oriented type of these were counted by Read [4] and Sridharan [5] respectively. In [3] Palmer obtained an asymptotic formula for the number of self-complementary digraphs following a method of Oberschelp [2]. An asymptotic formula for the number of self-complementary oriented graphs is given here. We refer to [1] for definitions and details not mentioned here.

§1. **Basic definitions.** A directed graph or a digraph consists of a finite nonempty set of distinct elements called vertices together with a prescribed collection of ordered pairs of these distinct vertices. Each ordered pair is called an edge. The complement $\bar{D}$ of a digraph D has the same set of vertices and an edge belongs to $\bar{D}$ if and only if it is not in D. If $e=(a, b)$ is an edge of D, then a is adjacent to b and b is adjacent from a. Two digraphs D_1 and D_2 are said to be isomorphic if there is a one-to-one correspondence between their vertex sets that preserves adjacency. A digraph D is said to be self-complementary if it is isomorphic to its complement. An oriented graph is a graph in which the edges are of the form either (u, v) or (v, u) but not both. Let S_n denote the symmetric group on n elements. Any permutation $\alpha \in S_n$ which has j_1 cycles of length 1, j_2 cycles of length 2, or in general j_i cycles of length i is written as $\alpha=(1^{j_1}2^{j_2}\cdots n^{j_n})$ where $j_1\cdot 1+j_2\cdot 2+\cdots+j_n\cdot n=n$. Let $[x]$ denote the greatest integer less than or equal to x. Let (r, s) and $\langle r, s\rangle$ denote the greatest common divisor and the least common multiple of r and s respectively.

§2. **Self-Complementary oriented graphs.** It is known [5] that the contributions to self-complementary oriented graphs come only from permutations of the type $(1^{j_1}2^{j_2}6^{j_6}10^{j_{10}}\cdots)$ where $j_1=1$ or 0. The number of self-complementary oriented graphs [5] is

$$Y_n=\frac{1}{n!}\sum_{\alpha\in B_n} 2^{\bar{0}_n(\alpha)} \tag{1}$$

where

$$B_n=\{\alpha\colon\ \alpha\in S_n \text{ and } \alpha \text{ has type } (1^{j_1}2^{j_2}6^{j_6}10^{j_{10}}\cdots) \text{ with } j_1=1 \text{ or } 0\}$$

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and

(2) $$\bar{O}_n(\alpha)=\sum_{p\in N}\frac{p}{2}j_p^2+\sum\sum_{q<r\in N'}(q,r)j_qj_r$$

where

$$N=\{2,6,10,14,\ldots\}$$
$$N'=\{1,2,6,10,14,\ldots\}$$

Case (i) Let $n=2p$. Then

THEOREM 1. $Y_{2p}=(1/p!)2^{p^2-p}(1+[p(p-1)(p-2)/3]2^{8-4p}+0(p^5/2^{(20/3)p}))$.

Proof. The method of proof is the same as in [1]. Let $Y_{2p,k}$ be the contributions from permutations which have $p-(k/2)$ cycles of length 2. Then

$$Y_{2p,0}=\frac{1}{p!}2^{p^2-p}$$

$$Y_{2p,6}=Y_{2p,0}\frac{p(p-1)(p-2)}{3}2^{8-4p}$$

and

$$Y_{2p}=\sum_{t=0}^{[(p-1)/2]}Y_{2p,4t+2}$$

Let $\ell_0=4p/3$ and $n_o=2t_0+1$. We prove that for any $t_0\geq 1$

(3) $$Y_{2p}=Y_{2p,0}\left(1+\sum_{t=2}^{t_0-1}\frac{Y_{2p,4t+2}}{Y_{2p,0}}+O\left(\frac{p^{n_0}}{2^{l_0n_0}}\right)\right)$$

For $t_0=1$, we have

$$Y_{2p}\simeq Y_{2p,0}=\frac{1}{p!}2^{p^2-p}$$

To prove (3), it is enough if we show that

$$\sum_{t=t_0}^{[p-1/2]}Y_{2p,4t+2}=\frac{1}{p!}2^{p^2-p}O\left(\frac{p^{n_0}}{2^{\ell_0n_0}}\right)$$

We first obtain the upper bounds for each $Y_{2p,k}$. The number of permutations in S_{2p} with j_2 cycles of length 2 is bounded by

$$\frac{(2p)!}{2^{j_2}j_2!}\frac{(2p)!}{\left(p-\frac{k}{2}\right)!\,2^{p-k/2}}$$

The contribution $\bar{O}_{2p}(\alpha)$ is largest when α has the type $(2^{p-(k/2)}6^{k/6})$. Therefore,

$$\bar{O}_{2p}(\alpha)\leq\frac{6p^2-4kp+k^2}{6}.$$

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With these bounds, we have, for each k

$$Y_{2p,k} \leq \frac{1}{\left(p-\frac{k}{2}\right)! \, 2^{p-k/2}} 2^{6p^2-4kp+k^{2/6}} \leq p\frac{k}{2} \cdot y_{2p,0} \cdot 2^{(-4kp+\cdots)/6}$$

Let

$$\ell_1 = \frac{4p-3-k}{3}$$

$$\ell_2 = \frac{2p-3}{3}$$

$$\ell_3 = \frac{4p-3-4k_0-2}{3}$$

$$\ell_4 = \frac{4p}{2}$$

and $n_1 = 4t_0+3$. Then,

$$Y_{2p,k} \leq Y_{2p,0}\left(\frac{p}{2\ell_1}\right)^{k/2} \tag{4}$$

Since $k \leq 2p$

$$Y_{2p,k} \leq Y_{2p,0}\left(\frac{p}{2\ell_2}\right)^{k/2} \tag{5}$$

Summing from $t-t_0$ to $[(p-1)/2]$ we have

$$\sum_{t=t_0}^{[(p-1)/2]} Y_{2p,4t+2} \leq Y_{2p,0} \sum_{t=t_0}^{[(p-1)/2]} \left(\frac{p}{2\ell_2}\right)^{2t+1}$$

$$\text{i.e.} \sum_{t=t_0}^{[(p-1)/2]} Y_{2p,4t+2} = cY_{2p,0}\left(\frac{p}{2\ell_2}\right)^{n_0} \tag{6}$$

where c is a constant close to 1. Put $k=4t_0+2$ in (4), then,

$$Y_{2p,k} \leq Y_{2p,0}\left(\frac{p}{2\ell_3}\right)^{n_0} - Y_{2p,0} O\left(\frac{p^{n_0}}{2\ell_0 n_0}\right) \tag{7}$$

and hence

$$\sum_{t=t_0}^{2t_0} Y_{2p,4t+2} = Y_{2p,0} O\left(\frac{p^{n_0}}{2\ell_0 n_0}\right)$$

From (6) it follows that

$$\sum_{t=2t_0+1}^{[(p-1)/2]} Y_{2p,4t+2} = Y_{2p,0} O\left(\frac{p}{2\ell_2}\right)^{n_1}$$

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But

$$O\left(\frac{p}{{}_2\ell_2}\right)^{n_1} = O\left(\frac{p}{{}_2\ell_4}\right)^{n_0}$$

Hence

$$Y_{2p} = Y_{2p,0}O\left(\frac{p^{n_0}}{{}_2\ell_0 n_0}\right)$$

The statement of the theorem follows by setting $t_0 = 2$. Case (ii) $n = 2p + 1$.
Case (ii) $n = 2p + 1$.

THEOREM 2

$$Y_{2p+1} = \frac{2^{p^2}}{p!}\left(1 + \frac{p(p-1)(p-2)}{3}2^{6-4p} + O(p^5/2^{(20/3)p})\right)$$

Proof. Same as in Theorem 1.

The following table gives the numbers of self-complementary oriented graphs up to the second approximation.

n	Y_n	First approximation	Second approximation
3	2	2	2
4	2	2	2
5	8	8	8
6	12	10.67	12
7	88	85.33	87.89
8	176	170.67	175.79
9	2752	2730.67	2757.98
10	8784	8738.13	8825.51

Since a self-complementary oriented graph is also a self-complementary tournament, the first approximation of this can be found in [1, page 215].

A similar attempt to obtain asymptotic formulae for self-converse digraphs and oriented self-converse graphs does not lead to a satisfactory result.

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Department of Mathematics,
Indian Institute of Technology, Kanpur
India, 208016.