ISOPARAMETRIC FUNCTIONS AND SUBMANIFOLDS

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Introduction. The theory of isoparametric functions and a family of isoparametric hypersurfaces began essentially with E. Cartan in 1930's. He defined a real valued function V defined on a Riemannian space form to be *isoparametric* if $\|\text{grad}\, v\|^2 = T \circ V$ and $\Delta V = S \circ V$ for some real valued functions S, T. Then a family of hypersurfaces M_t is called *isoparametric* if $M_t = V^{-1}(t)$ where t is a regular value of V.

Equivalently an isoparametric (family of) hypersurface(s) can be characterized as a (family of parallel) hypersurface(s), (each of) which has constant principal curvature. The isoparametric submanifolds in $\mathbb{R}^N$ and H^N are almost trivial, but in S^N the study of such submanifolds is quite interesting and many mathematicians have contributed to the subject. The generalization of this subject to higher dimensions was done by Carter-West [**1**] and Terng [**5**].

Nomizu [**4**] and Magid and Hahn [**2**] generalized the notion of isoparametric hypersurface and function to semi-Riemannian spaces. In this paper we generalize their work and study the notion of isoparametric submanifolds and maps of codimension at least 2 in semi-Riemannian spaces.

We observe that there are some similarities and many crucial differences between the theory in $\mathbb{R}^N$ and in $\mathbb{R}^N_p$. In fact the tangent bundle TM of each isoparametric submanifold M in $\mathbb{R}^N$ has a decomposition as $TM = \bigoplus_{i=1}^k E_i$ where each E_i is an integrable distribution. The principal curvatures of M are all real with fixed algbraic multiplicities on M. Each E_i is generated by eigenvectors of the principal curvatures of M. Using these facts one gets a finite reflection group (the Coxeter group of M) acting on the normal bundle of M. This is the key to the study of isoparametric submanifolds in $\mathbb{R}^N$. By using this group one can prove many important facts about such submanifolds. For example, the reducibility of each isoparametric into the product of irreducible ones, the fact that each isoparametric submanifold M in $\mathbb{R}^N$ is algebraic, the classification of isoparametric submanifolds.

All these results need serious investigation for an isoparametric submanifold M in $\mathbb{R}^N_p$. Since in sharp contrast to the Riemannian case the principal curvatures of M are not real (in general). The algebraic and geometric multiplicities of a real (if it exists) principal curvature can be different. The geometric multiplicity of a real principal curvature can vary by varying the point x in M, etc. A major consequence of these facts is that isoparametric submanifolds in $\mathbb{R}^N_p$ are not in general algebraic. This is one of the crucial differences between the theory in $\mathbb{R}^N$ and in $\mathbb{R}^N_p$. An example of a nonalgebraic isoparametric submanifold in $\mathbb{R}^N_p$ is given at the end of the paper. Another consequence of the above facts is that in general we do not have a decomposition of TM into integrable distributions, but we just have some partial results in this direction. So the lack of such a decomposition (in general) which results from the lack of any group (similar to the Coxeter group) associated to M makes the study of isoparametric submanifolds in $\mathbb{R}^N_p$ quite different and more difficult than the study of such submanifolds in $\mathbb{R}^N$, and as mentioned above not all the facts which are true for isoparametric submanifolds in $\mathbb{R}^N$ can be generalized to similar facts for such submanifolds in $\mathbb{R}^N_p$. Throughout the paper we have pointed out the differences quite explicitly.

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1. Main Results.

DEFINITION 1. A smooth function $f=(f_{n+1},\ldots,f_{n+m}):\mathbb{R}_p^{n+m}\to\mathbb{R}^m$ is called *isoparametric* if

(i) $\langle \operatorname{gr} f_\alpha, \operatorname{gr} f_\beta\rangle$ and $\Delta f_\alpha=\operatorname{div}(\operatorname{gr} f_\alpha)$ are (continuous) functions of f for each α,β; $n+1\leq\alpha,\beta\leq n+m$,

(ii) $[\operatorname{gr} f_\alpha, \operatorname{gr} f_\beta]$ is a linear combination of $\operatorname{gr} f_{n+1},\ldots,\operatorname{gr} f_{n+m}$ with coefficients being continuous functions of f for each α,β; $n+1\leq\alpha,\beta\leq n+m$.

DEFINITION 2. A connected semi-Riemannian submanifold M^n of $\mathbb{R}_p^{n+m}$ is called *isoparametric* if

(i) the normal bundle NM is flat with trivial holonomy group,

(ii) the characteristic polynomial of the shape operator of M along any parallel normal vector field is the same at all points of M.

THEOREM 3. *Let $f:\mathbb{R}_p^{n+m}\to\mathbb{R}^m$ be an isoparametric function, c a regular value of f such that $\phi\neq f^{-1}(c)$ and $\langle\,,\rangle|_{T(f^{-1}(c))}$ is nondegenerate. Let M be a connected component of $f^{-1}(c)$; then M is an isoparametric submanifold.*

Proof. The proof is very much like the Riemannian case (see **[1]** or **[5]**) and we give just a sketch of it. We prove (i) by finding a global parallel normal orthonormal frame field $\{e_\alpha\}$ on M. In order to prove (ii) we show first that the mean curvature vector field of M, i.e., $H=\frac{1}{n}\sum_{i=1}^{n}\varepsilon_i II(e_i,e_i)=\sum_\alpha H_\alpha e_\alpha$ ($\{e_i\}$ is any (local) orthonormal tangent frame field on M) is parallel on M or, equivalently, each mean curvature H_α is constant on M. Then we get the benefit of parallel surfaces M_t of M. Explicitly by choosing a parallel normal vector field e_α on M and pushing out each point of M in the e_α direction we get the map $\varphi_t:M\to\mathbb{R}_p^{n+m}$ defined by $\varphi_t(x)=x+te_\alpha(x)$. If t is small enough then (locally) $M_t=\varphi_t(M)$ is an n-dimensional immersed submanifold of $\mathbb{R}_p^{n+m}$ which has a global parallel normal orthonormal frame field and its mean curvature with respect to any parallel normal vector field is constant on M_t. Then by using Nomizu's method in **[4]** in our context which says that "a hypersurface N_0 has constant principal curvature if each of its parallel surfaces N_t (for small t) has constant mean curvature" we get (ii).

REMARK 4. We see that each M_t, as defined above, is an isoparametric submanifold. The one parameter family $\{M_t\}$ is called the *isoparametric system* associated with M in the e_α direction; note that $M_0=M$.

The next two propositions deal with the curvature foliation on M.

PROPOSITION 5. *Let M^n be an isoparametric submanifold of $\mathbb{R}_p^{n+m}$, let ξ be a parallel unit ($\langle\xi,\xi\rangle=\pm1$) normal vector field on M and S the shape operator of M along ξ. If K is a real eigenvalue of S, $T_k=\operatorname{Ker}(S\text{-}KI_n)$ and if the dimension of T_k (=geometric multiplicity of K) is a fixed number l, then*

(i) *T_k is integrable,*

(ii) *if T_k is nondegenerate, the integral manifold M_k of T_k is totally geodesic in M,*

(iii) *if the algebraic multiplicity of K is equal to its geometric multiplicity, then T_k is nondegenerate.*

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Proof. (i) if $X, Y \in T_k$, by using the Codazzi equation $((\nabla_v II)(U, W) = (\nabla_u II)(V, W)\ \forall\, U, V, W \in X(M))$ we prove that $\nabla_X Y \in T_k$. Thus $[X, Y] \in T$, i.e., T_k is integrable.

(ii) In this case M is a semi-Riemannian submanifold of M and by part (i) we see that the second fundamental form of M_k in M is identically zero, hence M_k is totally geodesic in M.

(iii) Let the characteristic polynomial of S be $(t-K)^{r_1}(p_2(t))^{r_2}\cdots(p_s(t))^{r_s}$, where each $p_i(t)$ is an irreducible polynomial in $R[t]$. We know that

$$TM = \operatorname{Ker}(S-KI)^{r_1} \oplus \operatorname{Ker}(p_2(S))^{r_2} \oplus \ldots \oplus \operatorname{Ker}(p_s(S))^{r_s}.$$

These kernels are mutually orthogonal, hence each one is nondegenerate. If the algebraic multiplicity of K is equal to its geometric multiplicity, the $\operatorname{Ker}(S-KI) = \operatorname{Ker}(S+KI)^{r_1}$. Thus $\operatorname{Ker}(S-KI)$ is nondegenerate.

Proposition 6. *Let $M^n \subset \mathbb{R}_p^{n+m}$ be a (geodesically) complete isoparametric submanifold, let $\{e_\alpha\}$ be a parallel normal orthonormal frame field on M and S_α the corresponding shape operator of M, let the characteristic polynomial of each S_α be $p_\alpha(t) = (t-K_\alpha)^{r_{\alpha,1}}(p_{\alpha,2}(t))^{r_{\alpha,2}}\ldots(p_{\alpha,s_\alpha}(t))^{r_{\alpha,s_\alpha}}$ and put $T_{K_\alpha} = \{X \in X(M) : S_\alpha X = K_\alpha X\}$. Suppose that $V = \bigcap_\alpha T_{K_\alpha}$ is a nondegenerate $2l$-dimensional subbundle of $TM (l \geq 1)$. Put $v = \sum_\alpha \varepsilon_\alpha K_\alpha e_\alpha \in K^\perp(M)$; then*

(i) V is integrable and its integral manifold L is totally geodesic in M,

(ii) if $v = 0$, the integral manifold L of V through $x \in M$ is the plane $x + \mathbb{R}_s^l$ in $\mathbb{R}_p^{n+m}$, where s is the index of $\langle\, ,\rangle|_V$,

(iii) if $\langle v, v\rangle \neq 0$, L is (a component of) an l-dimensional sphere or pseudosphere if $\langle v, v\rangle > 0$ or pseudohyperbolic space if $\langle v, v\rangle < 0$ with radius $\dfrac{1}{|\langle v,v\rangle|}$ and centre $c = y + \dfrac{v(y)}{\langle v, v\rangle}\ \forall y \in L$.

Proof. (i) By Proposition 5(i), if $X, Y \in V$, then $\nabla_X Y \in V$ so V is integrable and its integral manifold L is totally geodesic in M.

(ii) Since $v = 0$, $K_\alpha = 0$ for all $\alpha, n+1 \leq \alpha \leq n+m$, and the second fundamental form of L in $\mathbb{R}_p^{n+m}$ is $II_L(X, Y) = II_{L,1}(X, Y) + II_{L,2}(X, Y)$ where $II_{L,1}(X, Y) \in TM/V$ and $II_{L,2}(X, Y) \in X^\perp(M)$ for each $X, Y \in V$. In (i) we proved that $II_{L,1}(X, Y) = 0$, we also have $0 = \langle S_\alpha X, Y\rangle = \langle II_{L,2}(X, Y), e_\alpha\rangle$ for each α, and $X, Y \in V$, so $II_{L,2}(X, Y) = 0$, thus $II_L(X, Y) = 0$. Hence L is totally geodesic in $\mathbb{R}_p^{n+m}$, since M is complete, L is isometric to a plane $x + \mathbb{R}_s^l \subset \mathbb{R}_p^{n+m}$.

(iii) We see that the map $\varphi(y) = y + \dfrac{v(y)}{\langle v, v\rangle}$ maps L to a constant vector $c \in \mathbb{R}_p^{n+m}$ and that $II_L(X, Y) = \langle X, Y\rangle v$, $\forall X, Y \in V$, i.e. L is totally umbilic in $\mathbb{R}_p^{n+m}$. The claim is obtained from these two facts.

Definition 7. Let $M^n \subset \mathbb{R}_p^{n+m}$ be a semi-Riemannian submanifold and $\{v_\alpha\}$ be a normal frame field on M, let $\varphi : M \times \mathbb{R}^m \to \mathbb{R}_p^{n+m}$ be defined by $\varphi(x, z) = x + \sum_\alpha z_\alpha v_\alpha(x)$

$\forall x \in M$, $\forall z = (z_{n+1}, \ldots, z_{n+m}) \in \mathbb{R}^m$. A point $e = x + \sum_\alpha z_\alpha v_\alpha(x)$ is called a *focal point* of M if e is a critical value of φ (i.e., the Jacobian of φ at some point $(x, z) \in \varphi^{-1}(e)$ is singular). The set of all focal points of M is called the *focal set* of M. If $z \in \mathbb{R}^m$ is such that for each point $x \in M$, $\varphi(x, z)$ is a critical value of φ, then $\varphi(M \times \{z\})$ is called a *focal manifold* of M associated to z, $\{v_\alpha\}$.

PROPOSITION 8. *Let M^n be an isoparametric submanifold of $\mathbb{R}_p^{n+m}$, ξ be a (local) parallel normal vector field on M, $\langle \xi, \xi \rangle \neq 0$, S the shape operator of M along ξ which has K as a non-zero real eigenvalue.*

(i) *If K has multiplicity l, define $\varphi : M^n \to \mathbb{R}_p^{n+m}$ by $\varphi(x) = x + \frac{1}{K}\xi(x)$ then φ is a submersion of M (at least locally) onto a nondegenerate submanifold of $\mathbb{R}_p^{n+m}$ of codimension $l + m$ that will be denoted by V_K and is the focal manifold of M associated with ξ, K.*

If $(s_K, l - s_K)$ is the signature of T_K, the eigendistribution of S associated with K, then V_K has signature $(s - s_K, n + s_K - l - s)$. The integral submanifold M_K of T_K through $x \in M$ is mapped by φ onto the single point $\varphi(x)$.

If the shape operator of M at x is given by $S = \left[\begin{array}{c|c} KI_l & 0 \\ \hline 0 & A \end{array}\right]$ relative to the orthogonal decomposition $T_z M = T_K(x) \oplus (T_K(x))^\perp$, then the shape operator of V_K associated with $\xi^K = \xi$ (ξ^K is the parallel transport of ξ along the curve $t \mapsto x + t\xi(x)$) is $S_K = A\left(I_{n-l} - \frac{1}{k}A\right)^{-1}$.

(ii) *If the geometric multiplicity of K is constant but different from its algebraic multiplicity, then φ is a submersion onto a submanifold with degenerate metric.*

(iii) *If $\varphi_t : M \to \mathbb{R}_p^{n+m}$ is defined by $\varphi_t(x) = x + \frac{1}{t}\xi(x)$ and $\varphi_t(M) = V_t$ is a submanifold with nondegenerate metric of signature $(s - s_t, n + s_t - l - s)$, $l > 0$, then t is a principal curvature of M of multiplicity l and V_t is the focal manifold of M associated with ξ, t. The eigendistribution T_t has signature $(s_t, l - s_t)$, the shape operator of M along ξ at x is given by*

$$S_\xi = \left[\begin{array}{c|c} tI_l & 0 \\ \hline 0 & A \end{array}\right]$$

relative to the orthogonal decomposition $T_z M = T_K(x) \oplus (T_k(x))^\perp$, where A is $t(tI_{n-l} + S_t)^{-1}S_t$ and S_t is the shape operator of V_t along ξ^t.

Proof. The proofs of (i) and (iii) are almost the same as in the Riemannian case. For (ii) we suppose that the claim is false, i.e., TV_K is nondegenerate, then its normal bundle $T_K \oplus NM$ is nondegenerate, so T_K is nondegenerate and we have $TM = T_K \oplus T_K^\perp$.

By using the fact that algebraic and geometric multiplicities of K are different and representing the shape operator of M with respect to a basis adapted to the decomposition $TM = T_K \oplus T_K^\perp$ we get that T_K is degenerate, which is a contradiction, hence TV_K must be degenerate.

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Here we look at another problem about isoparametric submanifolds. The problem concerns the product of two isoparametric submanifolds. The proof of the following proposition is exactly the same as in the Riemannian case.

PROPOSITION 9. *If $M_i^{n_i}$ is an isoparametric submanifold in $\mathbb{R}_{p_i}^{n_i+l_i}$, $i = 1, 2$, then $M_1 \times M_2$ is isoparametric in $\mathbb{R}_{p_1+p_2}^{n_1+n_2+l_1+l_2}$.*

REMARK 10. The converse of the above proposition is far from being obvious, that is the question of when an isoparametric submanifold $M^n \subset \mathbb{R}_p^{n+m}$ decomposes into two lower dimensional isoparametric submanifolds is open and needs a serious investigation. The solution to this problem is well known in the Riemannian case, in fact if $M^n \subset \mathbb{R}^{n+m}$ (or S^{n+m} or H_0^{n+m}) is an isoparametric submanifold, then it decomposes into "irreducible" isoparametric submanifolds if and only if the Coxeter group of M decomposes into irreducible subgroups. In contrast, in $\mathbb{R}_p^{n+m}$ we have no group associated to the isoparametric submanifold $M^n \subset \mathbb{R}_p^{n+m}$.

REMARK 11. We defined an isoparametric submanifold to be in $\mathbb{R}_p^{n+m}$ and an isoparametric map to have $\mathbb{R}_p^{n+m}$ as its domain. We can define an isoparametric submanifold in a component of S_p^{n+m} or H_p^{n+m}, and an isoparametric map with domain S_p^{n+m} or H_p^{n+m} exactly as we did for $\mathbb{R}_p^{n+m}$. Then almost all the material of the paper goes through with only slight changes.

REMARK 12. It is a well-known fact that, if $M^n \subset \mathbb{R}^{n+m}$ is an isoparametric submanifold, there exists a polynomial map $f : \mathbb{R}^{n+m} \to \mathbb{R}^m$ such that f is isoparametric and M is a regular level of f. The situation in $\mathbb{R}_p^{n+m}$ is far from being obvious and the following example shows that, in contrast to the Riemannian case, in general we fail to have such an f. So the questions in $\mathbb{R}_p^{n+m}$ are as follows.

If $M^n \subset \mathbb{R}_p^{n+m}$ is an isoparametric submanifold, is there any isoparametric map $f : \mathbb{R}_p^{n+m} \to \mathbb{R}^m$ such that M is a component of a nondegenerate regular level $f^{-1}(c)$? A weaker question is under what conditions there is an isoparametric function $f : \mathbb{R}_p^{n+m} \to \mathbb{R}^m$ such that M is a component of $f^{-1}(c)$?

EXAMPLE 13. In this example we illustrate an isoparametric submanifold $M^2 \subset \mathbb{R}_1^4$ which is not algebraic.

Let $A = \frac{1}{\sqrt{2}}(1, -1, 0, 0)$, $B = \frac{1}{\sqrt{2}}(1, 1, 0, 0)$, $C_1 = (0, 0, 1, 0)$, $C_2 = (0, 0, 0, 1)$ and define $M^2 \subset \mathbb{R}_1^4$ by $M = \{\psi(s, t) = tA + C_1 \sin t + tC_2 + sB : s, t \in \mathbb{R}\} = \left\{x = \left(\frac{1}{\sqrt{2}}(t+s), \frac{1}{\sqrt{2}}(-t+s), \sin t, t\right) : t, s \in \mathbb{R}\right\}$. Then $\frac{\partial \psi}{\partial s} = B$, $\frac{\partial \psi}{\partial t} = A + C_2 + C_1 \cos t$ is a frame field for TM and $\{B \cos t + C_1, C_2 + B\}$ is a frame field for NM. Let S_1, S_2 be shape operators of M along $(B \cos t + C_1)$ and $(C_2 + B)$ respectively. Obviously $S_2 \equiv 0$, by a routine calculation we see that $S_1 B = 0$ and $S_1(A + C_2 + C_1 \cos t) = 4B \sin T$. So with respect to the chosen tangent frame field

$$S_1 = \begin{bmatrix} 0 & 4 \sin t \\ 0 & 0 \end{bmatrix}.$$

The calculation also shows that M has a global parallel normal orthonormal frame field. As can be seen, the principal curvature (and its algebraic multiplicity) of M is constant on M in any parallel normal direction. Thus M is isoparametric. The example is interesting from different points of view. The geometric multiplicity of the eigenvalue of S_1 is not constant on M. Thus the minimal polynomial of S_1 is not the same at all points of M. M is not algebraic since if there exists a polynomial map $f:\mathbb{R}_1^4\to\mathbb{R}^2$ such that M is a component of $f^{-1}(c)$, then $\operatorname{gr} f_1$, $\operatorname{gr} f_2$ must be polynomial functions in $x=(x_1,\ldots,x_4)\in M$. Thus $B\cos t+C_1$ must be a polynomial function of x, which certainly it is not, in fact

$$B\cos t+C_1=\left(\sqrt{\frac{1-x_3^2}{2}},\sqrt{\frac{1-x_3^2}{2}},1,0\right).$$

The following is an interesting example of an isoparametric map $f:\mathbb{R}^{3n}_{3p}\to\mathbb{R}^3$. It is the only nontrivial one (except the quadratics) which we have found.

Example 14. Let $\bar{M}$ be the space of $3\times n$ matrices over $\mathbb{R}$, define the scalar product on $\bar{M}$ by $\langle x,y\rangle=\operatorname{tr} xJy'$, where S' means the transpose of the matrix S and

$$J=\left[\begin{array}{c|c}-I_p & 0\\ \hline 0 & I_{n-p}\end{array}\right].$$

Define $f_r;\bar{M}\to\mathbb{R}$ by $f_r(x)=\operatorname{tr}(xJx')^r$ for $r\in Z_+$. Let us check that the map $f:\bar{M}\to\mathbb{R}^3$ defined by $f(x)=(f_1(x),f_2(x),f_3(x))$ is an isoparametric map.

Proof. Easily we get that $\operatorname{gr} f_1(x)=2x$, $\operatorname{gr} f_2(x)=4xJx'x$ and $\operatorname{gr} f_3=6(xJx')^2x$. Thus $\langle\operatorname{gr} f_1,\operatorname{gr} f_1\rangle=4f_1$, $\langle\operatorname{gr} f_1,\operatorname{gr} f_2\rangle=8f_2$, $\langle\operatorname{gr} f_1,\operatorname{gr} f_3\rangle=16f_3$, $\langle\operatorname{gr} f_2,\operatorname{gr} f_2\rangle=16f_3$, $\langle\operatorname{gr} f_2,\operatorname{gr} f_3\rangle=24f_4$, $\langle\operatorname{gr} f_2,\operatorname{gr} f_3\rangle=36f_5$.

We must prove that f_4 and f_5 are functions of f_1,f_2,f_3. Since $(xJx')^r$ is a real symmetric matrix for each $r\in Z_+$, it can be brought into diagonal form. So $(xJx')^r$ is similar to

$$\begin{bmatrix}(\lambda(x))^r & & 0\\ 0 & (\mu(x))^r & 0\\ 0 & & (\nu(x))^r\end{bmatrix}$$

for some $\lambda(x)$, $\mu(x)$, $\nu(x)\in\mathbb{R}$. Thus $f_r=(\lambda(x))^r+(\mu(x))^r+(\nu(x))^r$. By using Newton's formulae for elementary symmetric polynomials in three variables λ,μ,ν we get that $f_4=\frac{1}{6}(f_1^4+3f_2^2-6f_1^2f_2+8f_1f_2)$. Hence $\langle\operatorname{gr} f_2,\operatorname{gr} f_3\rangle=24f_4$ can be expressed in terms of f_1,f_2,f_3. Similarly we obtain that $f_5=\frac{1}{6}(f_1^5-5f_1^3f_2+5f_1^2f_3+5f_2f_3)$, so $\langle\operatorname{gr} f_3,\operatorname{gr} f_3\rangle=36f_5$ can be expressed in terms of f_1,f_2,f_3.

Now we are going to look at $[\operatorname{gr} f_\alpha,\operatorname{gr} f_\beta]$. Let D be the Levi Civita connection in $\bar{M}\cong\mathbb{R}^{3n}_{3p}$ then we have $[\operatorname{gr} f_\alpha,\operatorname{gr} f_\beta]=D_{\operatorname{gr} f_\alpha}\operatorname{gr} f_\beta-D_{\operatorname{gr} f_\beta}\operatorname{gr} f_\alpha$, so $[\operatorname{gr} f_1,\operatorname{gr} f_2]=4\operatorname{gr} f_2$, $[\operatorname{gr} f_1,\operatorname{gr} f_3]=8\operatorname{gr} f_3$, $[\operatorname{gr} f_2,\operatorname{gr} f_3]=48(xJx')^3x=6\operatorname{gr} f_4$. By using the relation $f_4=\frac{1}{6}(f_1^4+3f_2^2-6f_1^2f_2+8f_1f_3)$ we get that $[\operatorname{gr} f_2,\operatorname{gr} f_3]=(4f_1^3\operatorname{gr} f_1+6f_2\operatorname{gr} f_2-12f_1f_2\operatorname{gr} f_1-6f_1^2\operatorname{gr} f_2+8f_3\operatorname{gr} f_1+8f_1\operatorname{gr} f_3)$. Now we begin to calculate Δf_i, $i=1,2,3$. $\Delta f_1=\operatorname{div}(\operatorname{gr} f_1)=6n$, $\Delta f_2=\operatorname{div}(\operatorname{gr} f_2)=\sum_{1\le i\le 3,1\le j\le n}\varepsilon_{ij}\langle D_{z_{ij}}\operatorname{gr} f_2,X_{ij}\rangle$, where $\varepsilon_{ij}=\langle X_{ij},X_{ij}\rangle=-1$, if $1\le i\le 3$, $1\le j\le p$ and $\varepsilon_{ij}=+1$ if $1\le i\le 3$, $p+1\le j\le n$, if we write the points of $\bar{M}$ as

$$x=\begin{bmatrix}x_{11}\ldots x_{1n}\\ x_{21}\ldots x_{2n}\\ x_{31}\ldots x_{3n}\end{bmatrix}.$$

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Then $X_{ij}:\bar{M}\to T\bar{M}$, $T_z\bar{M}\cong\bar{M}\cong\mathbb{R}^{3n}_{3p}$ is defined by $X_{ij}(x)=$ the $3\times n$ matrix with its (i,j) component one and other components zero. Thus $\{X_{ij}:1\le i\le 3, 1\le j\le n\}$ is an orthonormal frame field on $\bar{M}$. So we have

$$\Delta f_2(x)=\sum \varepsilon_{ij}\langle X_{ij}(x)Jx^tx+xJX_{ji}(x)x+xJx^tX_{ij}(x),X_{ij}(x)\rangle=(3n+2)f_1.$$

Similarly we obtain that $\Delta f_3=(3n+4)f_2$. Thus f is an isoparametric map. It is an interesting problem to study the geometry of nondegenerate regular levels of f and also the focal manifolds associated with these regular levels.

The following is an example of an isoparametric submanifold of codimension 2 in $\mathbb{R}^{2n+2}_{n+1}$ which has complex principal curvatures. It is sharply in contrast to the Riemannian case.

EXAMPLE 15. Let $f:\mathbb{R}^{2n+2}_{n+1}\to\mathbb{R}^2$ be defined by

$$f(x)=\left[\sum_{i=1}^{n+1}x_1x_{n+1+i},\frac{1}{2}\left(-\sum_{i=1}^{n+1}x_i^2+\sum_{j=n+2}^{2n+2}x_i^2\right)\right],$$

let us check that f is a (quadratic) isoparametric map and find a nondegenerate regular level of it.

Let

$$A_1=\frac{1}{2}\left[\begin{array}{c|c}0&-I_{n+1}\\\hline I_{n+1}&0\end{array}\right]\quad\text{and}\quad A_2=\tfrac{1}{2}I_{2n+2},$$

then $\{A_1,A_2\}$ is a linearly independent set in Sym$[\mathbb{R}^{2n+2}_{n+1}]$ and we have $A^2_{n+1}=-\frac{1}{4}I_{2n+2}$ so $\{A_1,A_2\}$ generate a 2-dimensional algebra, it is easily seen that f is a (quadratic) isoparametric map [**3**]. Consider the level $f^{-1}((0,\frac{1}{2}b^2))$, $b\neq 0$. Since $b\neq 0$, obviously $\{A_1x,A_2x\}$ is a linearly independent set, so $f^{-1}((0,\frac{1}{2}b^2))$ is a regular level.

For the nondegeneracy of $f^{-1}((0,\frac{1}{2}b^2))$ we examine the system of equations

$$\begin{cases}\langle\alpha A_1x+\beta A_2x,A_1x\rangle=0=\alpha\langle x,x\rangle=\alpha b^2\Rightarrow\alpha=0,\\ \langle\alpha A_1x+\beta A_2x,A_2x\rangle=0=\beta\langle x,x\rangle=\beta b^2\Rightarrow\beta=0.\end{cases}$$

So $f^{-1}((0,\frac{1}{2}b^2))$ is nondegenerate. If M is a component of $f^{-1}((0,\frac{1}{2}b^2))$ then M is a quadratic isoparametric submanifold of codimension 2 and signature (n,n). The shape operator of M along $(-A_2x)$ is $A_2|_{TM}$, which is real and diagonal and the shape operator of M along $-A_1x$ is $A_1|_{TM}$. Note that $A_1^2=-\frac{1}{4}I_{2n+2}$ hence the minimal polynomial of A_1 is $t^2+\frac{1}{4}$. So A_1 has no real eigenvalue, but complex eigenvalues $\pm\frac{1}{2}i$, thus $A_1|_{TM}$ has no real eigenvalue.

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