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INTEGRAL REPRESENTATION OF SMOOTH FUNCTIONS IN WEIGHT CLASSES AND ITS APPLICATION

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§ 1. Introduction

Let R^n be the n-dimensional Euclidean space, and for each point $x = (x_1, \cdots, x_n)$ we write $|x| = (x_1^2 + \cdots + x_n^2)^{1/2}$. For a multi-index $\alpha = (\alpha_1, \cdots, \alpha_n)$, we denote by x^α the monomial $x_1^{\alpha_1} \cdots x_n^{\alpha_n}$, which has degree $|\alpha| = \sum_{j=1}^n \alpha_j$. Similarly, if $D_j = \partial/\partial x_j$ for $1 \leqq j \leqq n$, then

$$D^\alpha = D_1^{\alpha_1} \cdots D_n^{\alpha_n}$$

denotes a differential operator of order $|\alpha|$. We also write $\alpha! = \alpha_1! \cdots \alpha_n!$. Throughout this paper, let $1 < p < \infty$ and $(1/p) + (1/p') = 1$. For a real number r, we denote by $L^{p,r}$ the class of all measurable functions f for which

$$\|f\|_{p,r} = \left(\int |f(x)(1 + |x|)^r|^p dx\right)^{1/p} < \infty .$$

The notation $\mathscr{D}$ denotes the LF-space consisting of all C^∞-functions with compact support. The symbol $\mathscr{D}'$ stands for the topological dual of $\mathscr{D}$. Let m be a positive integer. We denote by $L_m^{p,r}$ the space of all $u \in \mathscr{D}'$ such that $D^\alpha u \in L^{p,r}$ for any α with $|\alpha| = m$. We set

$$|u|_{m;p,r} = \sum_{|\alpha|=m} \|D^\alpha u\|_{p,r} .$$

If u belongs to $\mathscr{D}$, then u can be represented by its partial derivatives of m-th order as follows (Yu.G. Reshetnyak [4]):

$$u(x) = \sum_{|\alpha|=m} \frac{m}{\sigma_n \alpha!} \int \frac{(x-y)^\alpha}{|x-y|^n} D^\alpha u(y) dy ,$$

where σ_n denotes the surface area of the unit sphere. In this paper, we are concerned with integral representation of $u \in C^\infty \cap L_m^{p,r}$ and its integral

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estimates. As an application we give an improvement of T.S. Pigolkina's result ([3]).

Throughout this paper, we use the symbol C for generic positive constant whose value may be different at each occurrence, even on the same line.

§ 2. Integral representation of smooth functions and its integral estimates

The following lemma is due to G.O. Okikiolu ([2]).

LEMMA 2.1. *Let* (S, m_S) *and* (T, m_T) *be measure spaces, and let* $K(s, t)$ *be a measurable function on* $S \times T$. *Suppose that there are positive measurable functions* ϕ_1 *on* S, ϕ_2 *on* T *and positive constants* M_1, M_2 *such that*

$$\int_T \phi_2(t)^{p'} |K(s, t)| dm_T(t) \leqq M_1^{p'} \phi_1(s)^{p'}, \tag{2.1}$$

$$\int_S \phi_1(s)^p |K(s, t)| dm_S(s) \leqq M_2^p \phi_2(t)^p. \tag{2.2}$$

If the operator Kf *is defined by*

$$Kf(s) = \int_T K(s, t) f(t) dm_T(t),$$

then

$$\left(\int_S |Kf(s)|^p dm_S(s)\right)^{1/p} \leqq M_1 M_2 \left(\int_T |f(t)|^p dm_T(t)\right)^{1/p}.$$

The following lemma is proved by applications of Lemma 2.1.

LEMMA 2.2. *Let* f *be a measurable function on* $(1, \infty)$. *Then*:

(i) *If* $q < 1$ *and* $\ell > 0$, *then*

$$\int_1^\infty \left|\int_1^s (s-t)^{\ell-1} f(t) dt\right|^p s^{(q-\ell)p-1} ds \leqq C \int_1^\infty |f(t)|^p t^{qp-1} dt.$$

(ii) *If* $\ell > 0$, *then*

$$\int_1^\infty \left|\int_1^s (s-t)^{\ell-1} f(t) dt\right|^p (1 + \log s)^{-p} s^{(1-\ell)p-1} ds \leqq C \int_1^\infty |f(t)|^p t^{p-1} dt.$$

(iii) *If* $q > \ell > 0$, *then*

$$\int_1^\infty \left|\int_s^\infty (t-s)^{\ell-1} f(t) dt\right|^p s^{(q-\ell)p-1} ds \leqq C \int_1^\infty |f(t)|^p t^{qp-1} dt.$$

Proof. (i) It suffices to show

$$\int_1^\infty \left| \int_1^s s^{-\ell}(s-t)^{\ell-1} t^{-qp+1} f(t) t^{qp-1} dt \right|^p s^{qp-1} ds \leqq C \int_1^\infty |f(t)|^p t^{qp-1} dt .$$

We take $(S, m_S) = ((1, \infty), s^{qp-1}ds)$, $(T, m_T) = ((1, \infty), t^{qp-1}dt)$ and

$$K(s, t) = \begin{cases} s^{-\ell}(s-t)^{\ell-1} t^{-qp+1}, & 1 < t < s, \\ 0, & 1 < s \leqq t. \end{cases}$$

Since $q < 1$, we can choose a number a such that $-1/p' < a < -q + (1/p)$. For $\phi_1(s) = \phi_2(s) = s^a$, we can show (2.1) and (2.2). Hence we obtain (i) by application of Lemma 2.1.

(ii) It is enough to show

$$\int_1^\infty \left| \int_1^s s^{-\ell}(s-t)^{\ell-1} t^{-p+1} f(t) t^{p-1} dt \right|^p (1 + \log s)^{-p} s^{p-1} ds \leqq C \int_1^\infty |f(t)|^p t^{p-1} dt .$$

We take $(S, m_S) = ((1, \infty), (1 + \log s)^{-p} s^{p-1} ds)$, $(T, m_T) = ((1, \infty), t^{p-1}dt)$ and

$$K(s, t) = \begin{cases} s^{-\ell}(s-t)^{\ell-1} t^{-p+1}, & 1 < t < s, \\ 0, & 1 < s \leqq t. \end{cases}$$

We can show (2.1) and (2.2) for $\phi_1(s) = s^{-1/p'}(1 + \log s)^{(1-\varepsilon)/p'}$ and $\phi_2(t) = t^{-1/p'}(1 + \log t)^{-\varepsilon/p'}$ with $0 < \varepsilon < 1$.

(iii) It is sufficient to show

$$\int_1^\infty \left| \int_s^\infty (t-s)^{\ell-1} t^{-q+(1/p)} f(t) dt \right|^p s^{(q-\ell)p-1} ds \leqq C \int_1^\infty |f(t)|^p dt .$$

We take $(S, m_S) = ((1, \infty), s^{(q-\ell)p-1}ds)$, $(T, m_T) = ((1, \infty), dt)$ and

$$K(s, t) = \begin{cases} (t-s)^{\ell-1} t^{-q+(1/p)}, & 1 < s < t, \\ 0, & 1 < t \leqq s. \end{cases}$$

For $\phi_1(s) = s^{(\ell-q)/p'}$ and $\phi_2(t) = t^{-1/pp'}$, we can show (2.1) and (2.2). We complete the proof of the lemma.

Let $u \in C^\infty$. For a nonnegative integer k, by Taylor's formula u can be represented as follows:

$$u(x) = \sum_{|\gamma| \leqq k} \frac{(|x|-1)^{|\gamma|}}{\gamma!} (x')^\gamma D^\gamma u(x') + (k+1) \sum_{|\gamma| = k+1} \int_1^{|x|} \frac{(|x|-t)^k}{\gamma!} (x')^\gamma D^\gamma u(tx') dt$$

where $x' = x/|x|$. The remainder term in Taylor's formula can be considered as integral representation of a function by its partial derivatives. Using spherical coordinates, Taylor's formula and Lemma 2.2 (i), (ii), we obtain

PROPOSITION 2.3. *Let ℓ and m be positive integers such that $\ell \geqq m$, and $u \in C^\infty$.*

(i) *If $m-(n/p)-r>0$, then*

$$\left(\int_{|x|\geqq 1}|u(x)|^p|x|^{(r-\ell)p}dx\right)^{1/p}$$
$$\leqq C\sum_{|\beta|\leqq \ell-m}\left(\int_{|\xi|=1}|D^\beta u(\xi)|^p d\sigma(\xi)\right)^{1/p}+C\sum_{|\alpha|=\ell-m+1}\left(\int_{|x|\geqq 1}|D^\alpha u(x)|^p|x|^{(r-m+1)p}dx\right)^{1/p}.$$

(ii) *If $m-(n/p)-r=0$, then*

$$\left(\int_{|x|\geqq 1}|u(x)|^p(1+\log|x|)^{-p}|x|^{(r-\ell)p}dx\right)^{1/p}$$
$$\leqq C\sum_{|\beta|\leqq \ell-m}\left(\int_{|\xi|=1}|D^\beta u(\xi)|^p d\sigma(\xi)\right)^{1/p}+C\sum_{|\alpha|=\ell-m+1}\left(\int_{|x|\geqq 1}|D^\alpha u(x)|^p|x|^{(r-m+1)p}dx\right)^{1/p}.$$

Now we shall deal with integral representation of $u \in C^\infty \cap L_m^{p,r}$. For this purpose we prepare two lemmas. We denote by Σ_n the unit sphere $\{\xi \in R^n;\ |\xi|=1\}$, and let $m_\sigma(E)$ represent the surface area of $E \subset \Sigma_n$.

LEMMA 2.4. *Let $1-(n/p)-r<0$. If $\Phi=\{\phi_i\}_{i=1,\cdots,n} \subset C^\infty \cap L^{p,r}$ is a family of functions such that $D_i\phi_j=D_j\phi_i$ for all $i,j=1,\cdots,n$, then for each point x there exists a set $E^\Phi(x) \subset \Sigma_n$ with $m_\sigma(\Sigma_n-E^\Phi(x))=0$, which satisfies the following conditions*:

(i) *For $\xi \in E^\Phi(x)$*

$$\int_0^\infty |\phi_i(x-s\xi)|ds<\infty\,, \qquad i=1,\cdots,n\,.$$

(ii) *If we put*

$$\psi(x)=\sum_{i=1}^n \xi_i\int_0^\infty \phi_i(x-s\xi)ds\,,$$

then $\psi(x)$ is independent of $\xi \in E^\Phi(x)$, and $D_j\psi=\phi_j$ $(j=1,\cdots,n)$.

(iii) *For $\xi \in E^\Phi(x)$ and $t>0$*

$$\psi(x-t\xi)=\sum_{i=1}^n \xi_i\int_t^\infty \phi_i(x-s\xi)ds\,.$$

Proof. By the assumption $\phi_i \in C^\infty \cap L^{p,r}$, we see that

$$\infty>\int_{|y|\geqq 1}|\phi_i(x-y)|^p|y|^{pr}dy=\int_{|\xi|=1}\int_1^\infty |\phi_i(x-s\xi)|^p s^{pr+n-1}ds\,d\sigma(\xi)\,.$$

Hence, if we put $E_x^1 = \cup_{i=1}^n \left\{\xi; \int_1^\infty |\phi_i(x - s\xi)|^p s^{pr+n-1} ds = \infty\right\}$, then $m_\sigma(E_x^1) = 0$. By Hölder's inequality and the condition $1 - (n/p) - r < 0$, we see that for $\xi \in \Sigma_n - E_x^1$

$$\int_0^\infty |\phi_i(x - s\xi)| ds < \infty, \qquad i = 1, \cdots, n. \tag{2.3}$$

Furthermore, by the assumption $\phi_i \in C^\infty \cap L^{p,r}$, we have

$$\begin{aligned} \infty &> \int_{|y|\geqq 1} |\phi_i(x - y)|^p |y|^{pr} dy \\ &= \int_1^\infty \int_0^{2\pi} \cdots \int_0^\pi |\phi_i(x - s\xi)|^p s^{pr+n-1} (\sin\theta_1)^{n-2} \cdots (\sin\theta_{n-2}) d\theta_1 \cdots d\theta_{n-1} ds \end{aligned}$$

where $\xi_j = \cos\theta_j \prod_{k=1}^{j-1} \sin\theta_k\,(1 \leqq j \leqq n-1)$ and $\xi_n = \prod_{k=1}^{n-1} \sin\theta_k$. Hence there exist sets $D_{x,j}^i \subset D_j = [0, \pi] \times \cdots \times \overset{j}{\widehat{[0, \pi]}} \times \cdots \times [0, \pi] \times [0, 2\pi] \subset R^{n-2}$ $(j = 1, \cdots, n-1)$ such that $m_{n-2}(D_{x,j}^i) = 0$ and for $(\theta_1, \cdots, \hat{\theta}_j, \cdots, \theta_{n-1}) \in D_j - D_{x,j}^i$

$$\begin{aligned} &\int_1^\infty \int_0^\pi |\phi_i(x - s\xi)|^p s^{pr+n-1} (\sin\theta_j)^{n-1-j} d\theta_j ds < \infty, \qquad j = 1, \cdots, n-2, \\ &\int_1^\infty \int_0^{2\pi} |\phi_i(x - s\xi)|^p s^{pr+n-1} d\theta_{n-1} ds < \infty \end{aligned} \tag{2.4}$$

where m_{n-2} stands for the $(n-2)$-dimensional Lebesgue measure and the symbol $\wedge$ denotes that the j-th element is deleted. For each positive number $\varepsilon < \pi/2$, we put $C_{s,j,\varepsilon}(\theta_j) = x - s\xi$, $\varepsilon \leqq \theta_j \leqq \pi - \varepsilon$ $(j = 1, \cdots, n-2)$, and $C_{s,n-1}(\theta_{n-1}) = x - s\xi$, $0 \leqq \theta_{n-1} \leqq 2\pi$. We shall prove that for $(\theta_1, \cdots, \hat{\theta}_j, \cdots, \theta_{n-1}) \in D_j - D_{x,j}^i$

$$\liminf_{s\to\infty} \int_\varepsilon^{\pi-\varepsilon} |\phi_i(x - s\xi)| \left|\frac{dC_{s,j,\varepsilon}(\theta_j)}{d\theta_j}\right| d\theta_j = 0, \qquad j = 1, \cdots, n-2, \tag{2.5}$$

$$\liminf_{s\to\infty} \int_0^{2\pi} |\phi_i(x - s\xi)| \left|\frac{dC_{s,n-1}(\theta_{n-1})}{d\theta_{n-1}}\right| d\theta_{n-1} = 0. \tag{2.6}$$

We give the proof of (2.5). We note that

$$\left|\frac{dC_{s,j,\varepsilon}(\theta_j)}{d\theta_j}\right| = s(\sin\theta_1)\cdots(\sin\theta_{j-1}).$$

We may assume that $\sin\theta_1 \cdots \sin\theta_{j-1} \neq 0$. Suppose

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$$\liminf_{s\to\infty}\int_{\varepsilon}^{\pi-\varepsilon}|\phi_i(x-s\xi)|\left|\frac{dC_{s,j,\varepsilon}(\theta_j)}{d\theta_j}\right|d\theta_j = a > 0\,.$$

Then there exists a number s_0 such that for $s \geqq s_0$

$$a/2 < \int_{\varepsilon}^{\pi-\varepsilon}|\phi_i(x-s\xi)|s(\sin\theta_1)\cdots(\sin\theta_{j-1})d\theta_j\,.$$

By Hölder's inequality we have

$$\int_{s_0}^{\infty}\int_{\varepsilon}^{\pi-\varepsilon}|\phi_i(x-s\xi)|^p s^{pr+n-1}(\sin\theta_j)^{n-1-j}d\theta_j ds \geqq Ca^p\int_{s_0}^{\infty}s^{pr+n-1-p}ds = \infty$$

since $1-(n/p)-r<0$. However this contradicts (2.4). Hence we obtain (2.5). We put $E_x^{2,i} = \bigcup_{j=1}^{n-2}\{\xi;\,(\theta_1,\,\cdots,\,\hat{\theta}_j,\,\cdots,\,\theta_{n-1})\in D_{x,j}^i,\ 0<\theta_j<\pi\}\cup\{\xi;\,(\theta_1,\,\cdots,\,\theta_{n-2})\in D_{x,n-1}^i,\ 0\leqq\theta_{n-1}\leqq 2\pi\}$, and $E^{\Phi}(x) = \Sigma_n - (E_x^1\cup\bigcup_{i=1}^n E_x^{2,i})$. Then $m_\sigma(\Sigma_n - E^{\Phi}(x)) = 0$. By (2.3), (2.5), (2.6) and Stokes' theorem, we see that for $\xi,\ \eta\in E^{\Phi}(x)$

$$\sum_{i=1}^n \xi_i\int_0^\infty \phi_i(x-s\xi)ds = \sum_{i=1}^n \eta_i\int_0^\infty \phi_i(x-s\eta)ds\,.$$

The formulas $D_j\psi = \phi_j$ $(j=1,\cdots,n)$ follow from Stokes' theorem. Thus we obtain (i) and (ii). The assertion (iii) follows from (ii) and the fundamental theorem of calculus. We complete the proof of the lemma.

By repeating use of Lemmas 2.2 (iii) and 2.4, we have

LEMMA 2.5. *Let $m-(n/p)-r<0$ and $u\in C^\infty\cap L_m^{p,r}$. Then:*

(i) *For a multi-index α with $|\alpha| = m$*

$$\int|x-y|^{m-n}|D^\alpha u(y)|dy < \infty$$

for every $x\in R^n$.

(ii) *There exists a set $E_x\subset\Sigma_n$ with $m_\sigma(\Sigma_n - E_x) = 0$ such that for $\xi\in E_x$*

$$\sum_{|\alpha|=m}\int_0^\infty s^{m-1}|D^\alpha u(x-s\xi)|ds < \infty$$

and

$$\sum_{|\alpha|=m}(m/\alpha!)\xi^\alpha\int_0^\infty s^{m-1}D^\alpha u(x-s\xi)ds$$

is independent of $\zeta\in E_x$.

(iii) *If we set*

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$$v(x) = \sum_{|\alpha|=m} (m/\alpha!)\xi^\alpha \int_0^\infty s^{m-1} D^\alpha u(x - s\xi)ds\,, \qquad \xi \in E_x\,,$$

then for $\xi \in E_x$ *and* $t > 0$

$$v(x - t\xi) = \sum_{|\alpha|=m} (m/\alpha!)\xi^\alpha \int_t^\infty (s - t)^{m-1} D^\alpha u(x - s\xi)ds\,.$$

(iv)

$$\left(\int_{|x|\geqq 1} |v(x)|^p |x|^{p(r-m)} dx\right)^{1/p} \leqq C \sum_{|\alpha|=m} \left(\int_{|x|\geqq 1} |D^\alpha u(x)|^p |x|^{pr} dx\right)^{1/p}.$$

Now we shall prove

THEOREM 2.6. *Let* $u \in C^\infty \cap L_m^{p,r}$ *and we suppose that the integral part* $[m - (n/p) - r] = k$ *of* $m - (n/p) - r$ *is not greater than* $m - 2$. *Then there exists a polynomial* $P(x) = \sum_{k+1 \leqq |\beta| \leqq m-1} c_\beta x^\beta$ *such that, if we set* $v(x) = u(x) - P(x)$, *then for* $k + 1 \leqq |\gamma| \leqq m - 1$

$$\begin{aligned} D^\gamma v(x) &= \sum_{|\alpha|=m-|\gamma|} \frac{m - |\gamma|}{\sigma_n \alpha!} \int \frac{(x - y)^\alpha}{|x - y|^n} D^{\alpha+\gamma} u(y)dy \\ &= \sum_{|\alpha|=m-|\gamma|} \frac{m - |\gamma|}{\alpha!} \xi^\alpha \int_0^\infty s^{m-|\gamma|-1} D^{\alpha+\gamma} u(x - s\xi)ds \end{aligned}$$

for almost every $\xi \in \Sigma_n$.

Proof. For each β with $|\beta| = m - 1$ we see that

$$D^\beta u(-t_1\xi) - D^\beta u(-t\xi) = -\sum_{j=1}^n \xi_j \int_t^{t_1} D^{e_j+\beta} u(-s\xi)ds \tag{2.7}$$

where the symbol e_j stands for the multi-index $(0, \cdots, \overset{j}{1}, \cdots, 0)$. Since $k \leqq m - 2$ implies $1 - (n/p) - r < 0$, by the condition $u \in C^\infty \cap L_m^{p,r}$ and Hölder's inequality we have

$$\int_0^\infty |D^{e_j+\beta} u(-s\xi)|ds < \infty$$

for almost every $\xi \in \Sigma_n$. Hence $D^\beta u(-t_1\xi)$ converges to $C_\beta(\xi)$ as $t_1 \to \infty$, and by (2.7) we have

$$C_\beta(\xi) - D^\beta u(0) = -\sum_{j=1}^n \xi_j \int_0^\infty D^{e_j+\beta} u(-s\xi)ds \tag{2.8}$$

for almost every $\xi \in \Sigma_n$. It follows from Lemma 2.5 and (2.8) that $C_\beta(\xi)$ are the same for almost all $\xi \in \Sigma_n$ and we write $C_\beta(\xi) = C_\beta$. Moreover we have

$$D^\beta u(x - t\xi) - D^\beta u(x) = -\sum_{j=1}^{n} \xi_j \int_0^t D^{e_j+\beta} u(x - s\xi) ds .$$

By an argument similar to the proof of Lemma 2.4 (ii), we see that $D^\beta u(x - t\xi) - D^\beta u(-t\xi) \to 0$ as $t \to \infty$ for almost every $\xi \in \Sigma_n$. Hence for almost every $\xi \in \Sigma_n$ we have

$$D^\beta u(x) = C_\beta + \sum_{j=1}^{n} \xi_j \int_0^\infty D^{e_j+\beta} u(x - s\xi) ds .$$

We put $u_1(x) = u(x) - \sum_{|\beta|=m-1} (C_\beta/\beta!) x^\beta$. For β with $|\beta| = m - 1$ we see that

$$D^\beta u_1(x) = D^\beta u(x) - C_\beta = \sum_{j=1}^{n} \xi_j \int_0^\infty D^{e_j+\beta} u(x - s\xi) ds$$

for almost every $\xi \in \Sigma_n$. Next let $2 \leqq \ell \leqq m - k - 2$. Suppose that $u_\ell(x) = u(x) - \sum_{m-\ell \leqq |\beta| \leqq m-1} (C_\beta/\beta!) x^\beta$ and for each multi-index γ with $m - \ell \leqq |\gamma| \leqq m - 1$, $D^\gamma u_\ell$ can be represented as follows:

$$D^\gamma u_\ell(x) = \sum_{|\delta|=m-|\gamma|} \frac{m - |\gamma|}{\delta!} \xi^\delta \int_0^\infty s^{m-|\gamma|-1} D^{\delta+\gamma} u(x - s\xi) ds$$

for almost every $\xi \in \Sigma_n$. If $|\gamma| \geqq m - \ell \geqq k + 2$, then $m - |\gamma| - (n/p) - r < 0$, so that by Lemma 2.5 (iv) we have

$$\int_{|x| \geqq 1} |D^\gamma u_\ell(x)|^p |x|^{p(r-(m-|\gamma|))} dx < \infty ,$$

and hence, since $m - |\gamma| - (n/p) - r < -1$,

$$\int_0^\infty |D^\gamma u_\ell(-s\xi)| ds < \infty$$

for almost every $\xi \in \Sigma_n$. Therefore for $|\zeta| = m - \ell - 1$, $D^\zeta u_\ell(-s\xi)$ converges to C_ζ as $s \to \infty$, and

$$D^\zeta u_\ell(x) = C_\zeta + \sum_{j=1}^{n} \xi_j \int_0^\infty D^{e_j+\zeta} u_\ell(x - s\xi) ds$$

for almost every $\xi \in \Sigma_n$. We put $u_{\ell+1} = u_\ell(x) - \sum_{|\zeta|=m-\ell-1} (C_\zeta/\zeta!) x^\zeta = u(x) - \sum_{m-\ell-1 \leqq |\beta| \leqq m-1} (C_\beta/\beta!) x^\beta$. For γ with $m - \ell \leqq |\gamma| \leqq m - 1$ we see that

$$D^\gamma u_{\ell+1}(x) = D^\gamma u_\ell(x) = \sum_{|\delta|=m-|\gamma|} \frac{m - |\gamma|}{\delta!} \xi^\delta \int_0^\infty s^{m-|\gamma|-1} D^{\delta+\gamma} u(x - s\xi) ds .$$

Let $|\gamma| = m - \ell - 1$. Since $D^{e_j+\gamma} u \in C^\infty \cap L^{p,r}$ and $\ell - (n/p) - r < 0$, by Lemma 2.8(iii) and Fubini's theorem we have

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$$D^{\gamma}u_{\ell+1}(x) = D^{\gamma}u_{\ell}(x) - C_{\gamma} = \sum_{j=1}^{n}\xi_j\int_0^{\infty}D^{e_j+\gamma}u_{\ell}(x-s\xi)ds$$
$$= \sum_{j=1}^{n}\xi_j\int_0^{\infty}\sum_{|\delta|=m-|\gamma|-1}\frac{m-|\gamma|-1}{\delta!}\xi^{\delta}\int_t^{\infty}(s-t)^{m-|\gamma|-2}D^{e_j+\delta+\gamma}u(x-s\xi)dsdt$$
$$= \sum_{|\delta|=m-|\gamma|}\frac{m-|\gamma|}{\delta!}\xi^{\delta}\int_0^{\infty}s^{m-|\gamma|-1}D^{\delta+\gamma}u(x-s\xi)ds$$

for almost every $\xi\in\Sigma_n$. Thus we obtain the function $v = u_{m-k-1}$ which possesses the following properties: $v(x) = u(x) - \sum_{k+1\leqq|\beta|\leqq m-1}(C_{\beta}/\beta!)x^{\beta}$ and for γ with $k+1\leqq|\gamma|\leqq m-1$

$$D^{\gamma}v(x) = \sum_{|\alpha|=m-|\gamma|}\frac{m-|\gamma|}{\alpha!}\xi^{\alpha}\int_0^{\infty}s^{m-|\gamma|-1}D^{\alpha+\gamma}u(x-s\xi)ds$$

for almost every $\xi\in\Sigma_n$. Therefore we also have

$$D^{\gamma}v(x) = (1/\sigma_n)\int_{|\xi|=1}D^{\alpha}v(x)d\sigma(\xi)$$
$$= \sum_{|\alpha|=m-|\gamma|}\frac{m-|\gamma|}{\sigma_n\alpha!}\int_{|\xi|=1}\int_0^{\infty}\xi^{\alpha}s^{m-|\gamma|-1}D^{\alpha+\gamma}u(x-s\xi)ds\,d\sigma(\xi)$$
$$= \sum_{|\alpha|=m-|\gamma|}\frac{m-|\gamma|}{\sigma_n\alpha!}\int\frac{(x-y)^{\alpha}}{|x-y|^n}D^{\alpha+\gamma}u(y)dy\,.$$

we complete the proof of the theorem.

The following corollary is a consequence of Proposition 2.3, Lemma 2.5 and Theorem 2.6.

Corollary 2.7. *Let* $k = [m-(n/p)-r]$ *and* $u\in C^{\infty}\cap L_m^{p,r}$. *Then there exists a polynomial* $P(x) = \sum_{k+1\leqq|\beta|\leqq m-1}C_{\beta}x^{\beta}$ *such that, if we set* $v(x) = u(x) - P(x)$, *then for* γ *with* $k+1\leqq|\gamma|\leqq m-1$

$$\left(\int_{|x|\geqq 1}|D^{\gamma}v(x)|^p|x|^{p(r-(m-|\gamma|))}dx\right)^{1/p}\leqq C\sum_{|\alpha|=m}\left(\int_{|x|\geqq 1}|D^{\alpha}u(x)|^p|x|^{pr}dx\right)^{1/p}$$

and for γ *with* $|\gamma|\leqq k$

$$\left(\int_{|x|\geqq 1}|D^{\gamma}v(x)|^p|x|^{p(r-(m-|\gamma|))}dx\right)^{1/p}$$
$$\leqq C\sum_{|\alpha|=m}\left(\int_{|x|\geqq 1}|D^{\alpha}u(x)|^p|x|^{pr}dx\right)^{1/p}+C\sum_{|\gamma|\leqq|\delta|\leqq k}\left(\int_{|\xi|=1}|D^{\delta}v(\xi)|^p d\sigma(\xi)\right)^{1/p}$$
$$(m-(n/p)-r\neq 0,1,\cdots,m-1)\,,$$

$$\left(\int_{|x|\geqq 1}|D^{\gamma}v(x)|^p(1+\log|x|)^{-p}|x|^{p(r-(m-|\gamma|))}dx\right)^{1/p}$$

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$$\leqq C \sum_{|\alpha|=m} \left(\int_{|x| \geqq 1} |D^\alpha u(x)|^p |x|^{pr} dx \right)^{1/p} + C \sum_{|\gamma| \leqq |\delta| \leqq k} \left(\int_{|\xi|=1} |D^\delta v(\xi)|^p d\sigma(\xi) \right)^{1/p}$$
$$(m - (n/p) - r = 0, 1, \cdots, m-1).$$

§ 3. Density of finite functions

T.S. Pigolkina [3] proved that, if $r \leqq 0$, then for $u \leqq L_m^{p,r}$ there exists a sequence $\{\phi_N\}_{N=1,2,\dots} \subset \mathscr{D}$ such that $|u - \phi_N|_{m;p,r}$ tends to 0 as $N \to \infty$. In this section we show that the above assertion holds for all real number r.

First we establish an analogue of Theorem 3.1 in [1]. By [1; p. 13], for each positive integer m, there exists a function ω_m which possesses the following properties:

(i) $\omega_m \in C^\infty$.

(ii) $\omega_m(x) = 0$ for x with $|x| \geqq 1$.

(iii) $\displaystyle\int \omega_m(x) dx = 1$.

(iv) $\displaystyle\int \omega_m(x) x^\gamma dx = 0$ for γ with $1 \leqq |\gamma| \leqq m-1$.

We put $\Omega_0 = \{|x| < 5/4\}$ and $\Omega_j = \{3(2^{j-3}) < |x| < 9(2^{j-3})\}$, $j = 1, 2, \cdots$. As in [1; Lemma 1.2], for $\{\Omega_j\}_{j=0,1,\dots}$ there exist functions $\{\rho_j\}_{j=0,1,\dots} \subset C^\infty$ which satisfy the following conditions:

(i) $\rho_j \geqq 0$ *and* $\sum_{j=0}^\infty \rho_j(x) = 1$ *for all* $x \in R^n$.

(ii) supp $\rho_j \subset \Omega_j$, $j = 0, 1, \cdots$.

(iii) $|D^\alpha \rho_j(x)| \leqq C 2^{-j|\alpha|}$ *for all* $x \in R^n$.

For a locally integrable function u, we set

$$E_\varepsilon^m u(x) = \sum_{j=0}^\infty \rho_j(x) \int u(x - \varepsilon 2^j y) \omega_m(y) dy, \qquad \varepsilon > 0.$$

By an argument similar to the proof of Theorem 3.1 in [1], we obtain

Proposition 3.1. *If* $u \in L_m^{p,r}$, *then* $E_\varepsilon^m u \in C^\infty \cap L_m^{p,r}$ *and* $|E_\varepsilon^m u - u|_{m;p,r}$ *tends to* 0 *as* $\varepsilon \to 0$.

Now we shall prove

Theorem 3.2. *If* $u \in L_m^{p,r}$, *then there exists a sequence* $\{\phi_N\}_{N=1,2,\dots} \subset \mathscr{D}$ *such that* $|u - \phi_N|_{m;p,r}$ *tends to* 0 *as* $N \to \infty$.

Proof. By Proposition 3.1, it suffices to show the theorem for $u \in C^\infty \cap L_m^{p,r}$. By Corollary 2.7 there exists a polynomial P of degree $m-1$ such that, if we put $v = u - P$, then v has the properties in Corollary

2.7. We take a function $h \in C^\infty(R^1)$ such that $0 \leqq h \leqq 1$, $h(t) = 1$ for $t \leqq 1$ and $h(t) = 0$ for $t \geqq 2$. For $N = 2, 3, \cdots$, we set

$$g_N(x) = \begin{cases} h((\log|x|)/\log N), & \text{for } x \neq 0, \\ 1, & \text{for } x = 0. \end{cases}$$

Then $g_N \in \mathcal{D}$. For a multi-index β with $|\beta| \geqq 1$ we have

$$D^\beta g_N(x) = 0 \quad \text{if } |x| < N \text{ or if } |x| > N^2, \tag{3.1}$$

$$|D^\beta g_N(x)| \leqq C(\log N)^{-1}|x|^{-|\beta|} \text{ for all } x \in R^n. \tag{3.2}$$

We put $\phi_N = g_N v$. For α with $|\alpha| = m$ we see that

$$\begin{aligned} &\|D^\alpha(u - \phi_N)\|_{p,r} = \|D^\alpha(v - g_N v)\|_{p,r} \\ &\leqq \left(\int_{|x| \geqq N} |D^\alpha v(x)|^p (1 + |x|)^{pr} dx\right)^{1/p} + C\sum_{\beta < \alpha} \left(\int |D^{\alpha-\beta} g_N(x) D^\beta v(x)|^p (1 + |x|)^{pr}\, dx\right)^{1/p} \\ &= I_N^1 + I_N^2. \end{aligned}$$

Since $D^\alpha v = D^\alpha u \in L^{p,r}$, I_N^1 tends to 0 as $N \to \infty$. By (3.1) and (3.2) we have

$$I_N^2 \leqq C\sum_{\beta < \alpha} \left(\int_{N \leqq |x| \leqq N^2} |D^\beta v(x)|^p (1 + \log|x|)^{-p} |x|^{p(r - (m - |\beta|))}\, dx\right)^{1/p}.$$

Hence by Corollary 2.7, I_N^2 tends to 0 as $N \to \infty$. Since $\phi_N \in \mathcal{D}$, we obtain the theorem.

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