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DEVELOPABILITY AND SOME NEW REGULARITY AXIOMS

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1. Introduction. In a recent publication H. Brandenburg [**5**] introduced D-completely regular topological spaces as a natural extension of completely regular (not necessarily T_1) spaces: Whereas every closed subset A of a completely regular space X and every $x \in X \backslash A$ can be separated by a continuous function into a pseudometrizable space (namely into the unit interval), D-completely regular spaces admit such a separation into developable spaces. In analogy to the work of O. Frink [**16**], J. M. Aarts and J. de Groot [**19**] and others ([**38**], [**46**]), Brandenburg derived a base characterization of D-completely regular spaces, which gives rise in a natural way to two new regularity conditions, D-regularity and weak regularity.

It is the purpose of this paper to derive fundamental structural properties of these classes of spaces, to elaborate their relationships, and, what will prove to be quite laborious, to give examples. The most important among these are modifications of a regular, but not completely regular, T_1-space due to J. Thomas [**39**].

Further results concerning D-complete regularity and related properties may be found in [**4**], [**6**], [**7**], [**9**], [**20**], and [**32**].

2. Fundamental definitions. Subcategories of Top are always assumed to be full and isomorphism-closed. "Space" means throughout "topological space". All properties discussed, in particular regularity and complete regularity, are not assumed to be automatically T_1. *R_0-spaces* [**13**], [**36**] are defined by the property that every open set is a union of closed sets.

Let X be a space. We then call a family $\mathscr{B}$ of subsets of X an *F_σ-base* if $\mathscr{B}$ is a base for the open sets and for every $B \in \mathscr{B}$ there exists a countable subfamily $(B_i)_{i \in \mathbf{N}}$ of $\mathscr{B}$ such that

$$B = \bigcup \{X \backslash B_i | i \in \mathbf{N}\}.$$

Similarly, we call a family $\mathscr{G}$ a *G_δ-base* if $\mathscr{G}$ is a base for the closed sets and if for every $G \in \mathscr{G}$ there exists a countable subfamily $(G_i)_{i \in \mathbf{N}}$ of $\mathscr{G}$ such that

$$G = \bigcap \{X \backslash G_i | i \in \mathbf{N}\}.$$

Clearly, $\{X \backslash G | G \in \mathscr{G}\}$ is an F_σ-base if $\mathscr{G}$ is a G_δ-base, and conversely.

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In [20] the existence of a developable T_1-space $\mathbf{D}$ was proved, such that the equivalences stated in the following definition hold. In [32] a space was constructed with similar properties.

Recall that a space X is called *developable* [2] if there exists a countable collection $(\mathscr{B}_i)_{i \in \mathbf{N}}$ of open covers of X such that for every $x \in X$ $\{\mathrm{St}(x, \mathscr{B}_i) | i \in \mathbf{N}\}$ is a neighborhood base at x.

2.1. *Definition.* A space X (not necessarily T_1) is called *D-completely regular* if it satisfies the following equivalent conditions:

(1) X has an F_σ-base;

(2) whenever A is a neighborhood of some point $x \in X$ then there exists an open subset B of $\mathbf{D}$ and a continuous function $f: X \to \mathbf{D}$ such that

$$x \in f^{-1}[B] \subset A;$$

(3) whenever A is a neighborhood of some point $x \in X$ then there exist an open subset B of $\mathbf{D}$ and finitely many continuous functions $f_i: X \to \mathbf{D}$, $i = 1, \ldots, n$, such that

$$x \in \bigcap_{i=1}^{n} f_i^{-1}[B] \subset A.$$

The category of D-completely regular spaces is denoted by D CompReg.

Note that D-completely regular spaces were called D-regular in [4], [5], [7], [10]. This notation, however, does not reflect the fact that D-completely regular spaces are derived in analogy to completely regular spaces, which is clearly expressed in our terminology. Moreover, our terminology is widespread (though not common usage) in papers on categorical topology (see e.g. [15], [30] [31], [41]). We are thus free to use the notion D-regular for a property motivated by investigations on developable spaces [20], which embodies, similar to regularity, information on the local character of a topology.

2.2. *Definition.* A space X is called

(1) *D-regular*, if every point of X has a neighborhood base consisting of open F_σ-sets. The subcategory of D-regular spaces is denoted D Reg.

(2) *weakly regular*, if every point of X has a neighborhood base consisting of F_σ-sets. The subcategory of weakly regular spaces is denoted WReg.

2.3. Recall that spaces in which every closed set is a G_δ-set are called *perfect*. The implications contained in the diagram (see top of next page) are easily verified. In this diagram no further implications are valid (see also 7.1).

$$\begin{tikzcd}
\text{developable} \arrow[d, Rightarrow] & & \text{pseudometrizable} \arrow[ll, Rightarrow] \arrow[dd, Rightarrow] \\
\text{perfect} \arrow[d, Rightarrow] & & \\
D\text{-completely regular} \arrow[d, Rightarrow] & & \text{completely regular} \arrow[ll, Rightarrow] \arrow[d, Rightarrow] \\
D\text{-regular} \arrow[dr, Rightarrow] & & \text{regular} \arrow[dl, Rightarrow] \\
& \text{weakly regular} \arrow[d, Rightarrow] & \\
& R_0 &
\end{tikzcd}$$

3. D-completely regular spaces. D-completely regular spaces were introduced in [**4**] and further studied in [**5**], [**6**], [**7**], [**20**], and [**32**]. As is expressed by condition 2.1(3) DCompReg is a bireflective subcategory of Top, i.e., DCompReg is closed under the formation of initial sources (for an explicit proof see [**20**], Theorem 3.5 or [**5**]). We assume the reader to be familiar with the basic notions of categorical topology, which can be found in [**23**].

The following theorem, partially contained in [**4**] and [**5**], follows directly from the property mentioned above.

3.1. THEOREM. DCompReg *is closed under the formation of inverse images, subspaces, products and inverse limits.*

3.2. For any bireflective subcategory $\mathscr{B}$ of Top, i.e., a subcategory closed under the formation of arbitrary initial structures, there exists a functor $B:\text{Top} \rightarrow \mathscr{B}$, called the associated bireflector, which assigns to every space $(X, \mathscr{X})$ a space $(X, \mathscr{X}_b)$ such that the following conditions are fulfilled:

(1) $(X, \mathscr{X}_b) \in \text{ob}(\mathscr{B})$;

(2) $1_X:(X, \mathscr{X}) \rightarrow (X, \mathscr{X}_b)$ is continuous;

(3) whenever $(Y, \mathscr{Y})$ is a $\mathscr{B}$-object and $f:(X, \mathscr{X}) \rightarrow (Y, \mathscr{Y})$ a continuous function, then $f:X \rightarrow Y$, viewed as a function from $(X, \mathscr{X}_b)$ to $(Y, \mathscr{Y})$, is continuous.

3.3. A first attempt to describe the bireflector $DCR:\text{Top} \rightarrow D\text{CompReg}$ in [**5**] remained unsatisfactory because it involved transfinite constructions relying on subsets defined externally. We derive here a characterization of DCR that uses only intrinsic properties.

3.4. We call a collection $\mathscr{B}$ of subsets of a space X an *open complemen-*

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tary system if $\mathscr{B}$ consists of open sets such that for every $B \in \mathscr{B}$ there exist $B_1, B_2, \ldots \in \mathscr{B}$ with $B = \cup \{X \backslash B_i | i \in \mathbf{N}\}$. Recall that a collection of subsets of a set X is called a *ring* if it is closed under the formation of finite intersections and finite unions. Finally, a subset A of a space X is called a *strongly open F_σ-set* if there exists a countable open complementary system $\mathscr{B}(A)$ with $A \in \mathscr{B}(A)$.

3.5. LEMMA. (1) *Every open complementary system $\mathscr{B}$ of a space X with cardinality less than $\kappa (\omega < \kappa)$ is contained in an open complementary ring $\mathscr{C}$ with cardinality less than κ.*

(2) *If $\mathscr{B}_1, \ldots, \mathscr{B}_n$ are finitely many open complementary systems, then $(\mathscr{B}_1 \wedge \ldots \wedge \mathscr{B}_n) \cup (\mathscr{B}_1 \vee \ldots \vee \mathscr{B}_n)$ is an open complementary system.*

Proof. (1) One obtains $\mathscr{C}$ by taking all finite unions of finite intersections of elements of $\mathscr{B}$ (see also [**4**; 3.1.17]). (2) follows from a straightforward computation.

3.6. COROLLARY. *Finite intersections of strongly open F_σ-sets are strongly open F_σ-sets.*

3.7. THEOREM. *The bireflector*

$$DCR: \mathrm{Top} \rightarrow D\mathrm{CompReg}$$

associated to DCompReg *assigns to every space $(X, \mathscr{X})$ the D-completely regular space $(X, \mathscr{X}_{dcr})$ which has the set of all strongly open F_σ-sets of $\mathscr{X}$ as a base.*

Proof. It follows from 3.6 that

$$\mathscr{F} = \{A | A \text{ is a strongly open } F_\sigma\text{-set in } \mathscr{X}\}$$

forms a base for a topology $\mathscr{X}_{dcr}$ on X. Since for every strongly open F_σ-set A in X $\mathscr{B}(A) \subset \mathscr{F}$ clearly holds, $\mathscr{F}$ is an F_σ-base. Hence, $(X, \mathscr{X}_{dcr})$ is D-completely regular. Let now $f: (X, \mathscr{X}) \rightarrow (Y, \mathscr{Y})$ be a continuous map into a D-completely regular space, and let $A \in \mathscr{G}$ be given, where $\mathscr{G}$ is an F_σ-base for $\mathscr{Y}$. It is easy to prove that $f^{-1}\mathscr{G} = \{f^{-1}[G] | G \in \mathscr{G}\}$ is an open complementary system in $\mathscr{X}$ containing $f^{-1}(A)$. Thus, $f^{-1}(A)$ is a strongly open F_σ-set, proving that $f: (X, \mathscr{X}_{dcr}) \rightarrow (Y, \mathscr{Y})$ is continuous.

3.8. COROLLARY. *A space X is D-completely regular if and only if every point $x \in X$ has a neighborhood base consisting of strongly open F_σ-sets.*

Let us now turn to the investigation of final structures of D-completely regular spaces.

3.9. PROPOSITION. *Coproducts of D-completely regular spaces are D-completely regular.*

Proof. Let $(X_\iota)_{\iota \in I}$ be a family of D-completely regular spaces whose coproduct is X. We consider X as the disjoint union of the X_ι's and refer to each X_i as a subspace of X. For every $i \in I$ there is an F_σ-base $\mathscr{F}_i$ for X_i. Define

$$\mathscr{F} := \bigcup \{\{F \cup (X \backslash X_i) | F \in \mathscr{F}_i\} | i \in I\} \cup \bigcup \{\mathscr{F}_i | i \in I\}.$$

Clearly, $\mathscr{F}$ is a base for X. Now let $G \in \mathscr{F}$ be given. In case $G = F \cup (X \backslash X_i)$ for some $F \in \mathscr{F}_i$, there exists a countable family

$$(F_n)_{n \in \mathbf{N}} \subset \mathscr{F}_i \text{ with } F = \bigcup \{X_i \backslash F_n | n \in \mathbf{N}\}.$$

This implies $G = \bigcup \{X \backslash F_n | n \in \mathbf{N}\}$, and each F_n belongs to $\mathscr{F}$. In case $G = F$ for some $F \in \mathscr{F}_i$ one proceeds analogously.

3.10. Quotients of D-completely regular spaces, however, need not be D-completely regular. The Sierpinski space, for example, is a quotient of the unit interval with its natural topology without being weakly regular, not even R_0. This poor behaviour with respect to quotients which D-completely regular spaces share with almost all other classes of spaces determined by separation axioms can be remedied for regular and completely regular T_1-spaces by imposing stronger assumptions on the quotient map, namely open-closed [**11**], or by assuming that the identification process is of a special nature [**3**]. We ignore whether an analogous result holds for D-completely regular spaces in the first case, but in the latter, the results partially carry over.

3.11. Lemma. *If a compact subspace A of a D-completely regular space X is contained in an open set C, then there exists a strongly open F_σ-set F with $A \subset F \subset C$.*

Proof. We may assume that X has an F_σ-base $\mathscr{F}$ closed with respect to finite unions (3.5(1)). By virtue of the compactness of A finitely many elements of $\mathscr{F}$, disjoint from $X \backslash C$, suffice to cover A. Their union F belongs to $\mathscr{F}$ and is therefore a strongly open F_σ-set.

3.12. Proposition. *Let X be a D-completely regular space and $(A_i)_{i \in I}$ a pairwise disjoint closure-preserving family of closed compact subspaces of X. The space Y is obtained from X by identifying each A_i to a point. Then Y is D-completely regular.*

Proof. Let $y \in Y$ and a neighborhood V of y be given. In case $y = q(x)$ for some $x \in X \backslash \bigcup \{A_i | i \in I\}$, where q is the quotient map, a strongly open F_σ-set G with

$$x \in G \subset q^{-1}[V] \cap (X \backslash \bigcup \{A_i | \mathrm{i} \in I\})$$

exists. It is easy to see that $g[G]$ is a strongly open F_σ-set with $y \in q[G] \subset V$. In case $y = q[A_j]$ for some $j \in I$ Lemma 3.11 enables one to proceed analogously.

Since $q:X \to Y$ in 3.12 is *perfect*, (i.e., continuous, closed, onto, and $f^{-1}(y)$ is compact for all $y \in Y$) one might be led to believe that D-complete regularity is invariant with respect to perfect maps. Although we do not know a counterexample we hold this assumption more likely to be invalid, even between Hausdorff spaces. A related question is whether D-complete regularity is invariant under open-closed surjective mappings between T_1 or Hausdorff spaces (like completely regular spaces [**11**]) or at least under open perfect mappings.

Concerning inverse invariance we have the following example.

3.13. *Example.* D-complete regularity is not inversely preserved under open perfect mappings. Let X denote the disjoint union of $\mathbf{R}$ and $[0, 1)$ equipped with the following topology: Subsets of $[0, 1)$ are open if and only if they are open in the natural topology on $[0, 1)$; neighborhoods of points $r \in \mathbf{R}$ contain almost all points of $\mathbf{R}$ and a set of the form $(1 - \epsilon, 1)$ for some ϵ, $0 < \epsilon < 1$. As the space Y we take the unit interval $[0, 1)$. Then $q:X \to Y$, which identifies all the points of $\mathbf{R}$ to 1 is open perfect. X contains the reals with cofinite topology as a subspace. Since the reals with this topology are not even weakly regular (see also 7.12) and since subspaces of weakly regular spaces are again weakly regular (see 5.2) X cannot be weakly regular. This implies in particular that X is not D-regular.

From the remarks preceding 3.1 we know that the lattice of D-completely regular topologies on a fixed set X is closed under the formation of suprema (initial structures). In analogy to results in [**27**] concerning (among others) regular and completely regular topologies, we obtain a negative answer for infima:

3.14. *Example.* The infimum of two D-completely regular topologies needs not to be D-completely regular.

Consider an uncountable set X and let p and q be distinct points of X. We define a tology $\mathscr{X}_p$ on X by declaring each point distinct from p open, and every complement of a finite subset of X. The topology $\mathscr{X}_q$ is defined analogously. Clearly, $\mathscr{X}_p$ (and for reasons of symmetry, $\mathscr{X}_q$) is a compact Hausdorff space and hence D-completely regular. This can also directly be derived by observing that

$$\mathscr{F} = \{A \subset X\backslash\{p\} | A \text{ finite}\} \cup \{B \subset X | p \in B, X\backslash B \text{ finite}\}$$

is an F_σ-base for $\mathscr{X}_p$. The infimum $\mathscr{X}$ of $\mathscr{X}_p$ and $\mathscr{X}_q$ is, however, not even weakly regular. This can be seen by indirect argument. Assume $\mathscr{X}$ were weakly regular. Since $\{p\}$ and $\{q\}$ are closed in $\mathscr{X}$ there exists an open subset A of $\mathscr{X}$ and countably many closed subsets $B_1, B_2, \ldots$ of $(X, \mathscr{X})$ such that

$$p \in A \subset \cup \{B_i | i \in \mathbf{N}\} \subset X\backslash\{q\}.$$

This, however, is impossible since each B_i is finite so that the uncountable set A cannot be contained in $\cup \{B_i | i \in \mathbf{N}\}$.

4. D-regular spaces. The following lemma establishes that the class DReg of D-regular spaces is bireflectively contained in Top.

4.1. LEMMA. *The class of D-regular spaces is closed with respect to initial sources.*

Proof. Let X be a space equipped with the initial topology with respect to a class $(f_i:X \to Y_i)_{i \in I}$ of maps into D-regular spaces Y_i. Let $x \in X$ and an open neighborhood A of x be given. Then there are finitely many indices $i_1, \ldots, i_n \in I$ and open sets $A_j \subset Y_j$ for every $j \in N := \{1, \ldots, n\}$ such that

$$f_{i_1}{}^{-1}[A_1] \cap \ldots \cap f_{i_n}{}^{-1}[A_n] \subset A.$$

For every $j \in N$ there exists an open F_σ-set F_j with $f_{i_j}(x) \in F_j \subset A_j$ since Y_j is D-regular. It is easy to see that $f_{i_j}{}^{-1}[F_j]$ is an open F_σ-set containing x for every j, and if we define

$$F := f_{i_1}{}^{-1}[F_1] \cap \ldots \cap f_{i_n}{}^{-1}[F_n],$$

then F is the desired open F_σ-neighborhood of x.

A direct consequence of 4.1 is:

4.2. THEOREM. *DReg is closed under the formation of inverse images, subspaces, suprema, products and inverse limits.*

4.3. Finite intersections of open F_σ-sets are open F_σ-sets. This implies that the class mapping

$$P:\text{Top} \to \text{Top}$$

which leaves every function unchanged and assigns to every space $(X, \mathscr{X})$ the space $(X, \mathscr{X}_p)$ where the open F_σ-sets of $\mathscr{X}$ are a base for $\mathscr{X}_p$, is a functor. Since $(X, \mathscr{X})$ is D-regular if and only if $\mathscr{X} = \mathscr{X}_p$, P is called a *prereflector* for DReg. The bireflector associated to DReg can now be obtained in a transfinite procedure described in [**28**]: Define $P_0 := P$ and for every ordinal number α

$$P_{\alpha+1} := P \circ P_\alpha.$$

If β is a limit ordinal, P_β is obtained as the functor which assigns to every space $(X, \mathscr{X})$ the final topology on X with respect to the sink

$$(1_X:P_\alpha(X, \mathscr{X}) \to X)_{\alpha<\beta}.$$

Likewise, the functor $Q:\text{Top} \to \text{Top}$ is defined as the class mapping which assigns to every space $(X, \mathscr{X})$ the infimum of all spaces $P_\alpha(X, \mathscr{X})$,

$\alpha \in \mathrm{Ord}$. Since $Q(X, \mathscr{X})$ is always D-regular, we may restrict the image of Q and obtain:

4.4. THEOREM. *The bireflector* $DR:\mathrm{Top} \to D\mathrm{Reg}$ *associated to* $D\mathrm{Reg}$ *is obtained as the image restriction of* Q.

4.5. In [**40**] the bireflector associated to the category Reg of regular spaces was constructed by means of a transfinite iteration of an "ultra-closure" operator. We may similarly define for subsets A of a space X

$$u\,\mathrm{cl}\,A := \bigcap \{B \subset X | A \subset B,\ B \text{ closed } G_\delta\}.$$

Then $u\,\mathrm{cl}$ defines the same topology as the functor P described above and the transfinite procedure above corresponds to the one in [**40**].

4.6. PROPOSITION. *Coproducts of D-regular spaces are D-regular.*

Proof. Let X be the coproduct of D-regular spaces X_i, $i \in I$, and let $j_i:X_i \to X$ denote the injections. If A is a neighborhood of some $x \in X$, then $x \in j_i[X_i]$ for some $i \in I$, and since there exists an open F_σ-set F with

$$j_i^{-1}(x) \subset F \subset j_i^{-1}[A],$$

$j_i[F]$ is an open F_σ-set in X satisfying $x \in j_i[F] \subset A$.

4.7. Quotients or infima of D-regular spaces need not be D-regular (see 3.10, 3.14). However, the invariance of regularity under open, closed, continuous and surjective maps known from [**11**] carries over to D-regularity.

4.8. THEOREM. *Whenever* X *is D-regular,* $f:X \to Y$ *is an open, closed and continuous mapping onto some space* Y*, then* Y *is D-regular.*

Proof. Let $y \in Y$ and a neighborhood V of y be given. Then, for $x \in f^{-1}(y)$, $U := f^{-1}[V]$ is a neighborhood of x. Hence there exists an open F_σ-set F satisfying $x \in F \subset U$. Clearly, $f[F]$ is an open F_σ-set in Y satisfying $y \in f[F] \subset V$.

Certain identifications, similar to those in [**3**], preserve D-regularity. To demonstrate this we start with a lemma.

4.9. LEMMA. *Let* A *and* B *be two disjoint subsets of a space* X *with* B *closed. If* A *is a Lindelöf space or, in case* A *is closed, the boundary of* A *is Lindelöf, the following statements hold:*

(1) *If* X *is weakly regular, then there exists an* F_σ*-set* F *such that*

$$A \subset \mathrm{int}\, F \subset F \subset X \backslash B.$$

(2) *If* X *is D-regular, then there exists an open* F_σ*-set* F *such that*

$$A \subset F \subset X \backslash B.$$

Proof. (1) We assume that A is closed and has a boundary $K = A\backslash \operatorname{int} A$ which is a Lindelöf space. This implies that there exist countably many points $x_1, x_2, \ldots$ in K and F_σ-sets $F_1, F_2, \ldots$ such that $x_i \in \operatorname{int} F_i$ and

$$K \subset \operatorname{int} G \subset G = \bigcup \{F_i | i \in \mathbf{N}\} \subset X\backslash B.$$

Clearly, $F := A \cup G$ is an F_σ-set with $A \subset \operatorname{int} F \subset F \subset X\backslash B$. The remaining assertions are proved analogously.

4.10. PROPOSITION. *Let X be a D-regular space and $(A_\iota)_{\iota \in I}$ a pairwise disjoint closure preserving family of closed sets with Lindelöf boundaries. Then the space Y, which is obtained from X by identifying each A_ι to a point, is D-regular.*

Proof. Using 4.9, the proof can be modelled as in 3.12.

4.11. If $f:X \to Y$ is a perfect map, and Y is D-regular, then X need not be D-regular (3.13). What happens in case X is assumed to be Hausdorff? Under what assumptions does X D-regular imply that Y is D-regular? We have so far no answers to these questions.

4.12. PROPOSITION. *Let X be a space with the property that every intersection of countably many open sets is open. Then the following conditions are equivalent:*

(1) *X is weakly regular;*
(2) *X is D-regular;*
(3) *X is D-completely regular;*
(4) *X is regular;*
(5) *X is completely regular;*
(6) *X is zerodimensional, i.e., has a base consisting of clopen sets.*

The proof is elementary and hence omitted. Consequently, all these regularity concepts coincide for P-spaces in the sense of [**29**] or in the sense of [**18**]. In particular, finite R_0-spaces, discrete and indiscrete spaces fulfill all the properties listed above.

4.13. PROPOSITION. *For an R_0-space X the following conditions are equivalent:*

(1) *X is D-regular;*
(2) *Every open cover $\mathscr{A}$ of X has a refinement consisting of open F_σ-sets.*

Proof. (1) implies (2) trivially. To show the converse let an open neighborhood A of $x \in X$ be given. Then $\{A, X\backslash \operatorname{cl}\{x\}\}$ is an open cover of X which has a refinement $\mathscr{F}$ consisting of open F_σ-sets. Choose some $F \in \mathscr{F}$ with $x \in F$, then F is an open F_σ-set with $F \subset A$.

5. Weakly regular spaces. Most of the results of the preceding section can analogously be derived for weakly regular spaces. The proofs given there need not essentially be altered and are therefore omitted.

The class WReg of weakly regular spaces is bireflectively contained in Top. This is expressed by the following lemma.

5.1. LEMMA. WReg *is closed with respect to initial sources.*

5.2. THEOREM. WReg *is closed under the formation of inverse images, subspaces, suprema, products, coproducts and inverse limits. Quotients or infima of weakly regular spaces, however, need not be weakly regular* (*see* 3.10, 3.14).

5.3. Consider the following property P for open sets O of a topological space $(X, \mathscr{X})$:

(P) For every point $x \in O$ there exists an F_σ-set F such that

$$x \in \operatorname{int} F \subset F \subset O.$$

Since the intersection of any two sets satisfying (P) is itself a set satisfying (P), a topology $\mathscr{X}_p$ can be derived from $(X, \mathscr{X})$ by taking as a base for $\mathscr{X}_p$ all sets satisfying (P). This defines a functor $P:\mathrm{Top} \to \mathrm{Top}$, $P(X, \mathscr{X}) := (X, \mathscr{X}_p)$, $P(f) := f$. Clearly, a space X is weakly regular if and only if $X = PX$. Hence, P is a prereflection for WReg, and the bireflector $WR:\mathrm{Top} \to \mathrm{WReg}$ associated to WReg is obtained from P in the way described in 4.3 and 4.4.

5.4. In analogy to 4.5 and the procedure in [**40**] the prereflection P can be described alternatively by means of an ultraclosure operator.

$$u \operatorname{cl} A := \{B \subset X \mid A \subset B,\ B \text{ closed and for every } x \in X\backslash B \text{ there exists an } F_\sigma\text{-set } F_x \text{ with } x \in \operatorname{int} F_x \subset F_x \subset X\backslash B\}.$$

5.5. WReg is not simply generated, i.e., there is no weakly regular space X such that every weakly regular space is homeomorphic to a subspace of a power of X. This follows from a theorem of [**22**], stating that for every T_1-space R there exists a regular T_1-space S such that all continuous functions from S to R must be constant. See also [**28**; 4.4.5], [**31**].

5.6. THEOREM. *Whenever X is weakly regular, and $f:X \to Y$ is an open, closed and continuous mapping onto some space Y, then Y is weakly regular.*

5.7. We call a mapping $f:X \to Y$ *Lindelöf-perfect* if f is continuous, onto, closed and every set of the form $f^{-1}(y)$ for $y \in Y$ is a Lindelöf space. Perfect maps are clearly Lindelöf-perfect. It is well known [**25**; p. 148] that regularity is preserved under perfect mappings. This can be generalized for weakly regular spaces to Lindelöf-perfect mappings.

5.8. THEOREM. *Whenever X is weakly regular, and $f:X \to Y$ is a Lindelöf-perfect mapping onto some space Y, then Y is weakly regular.*

Proof. Let an open neighborhood V of some $y \in Y$ be given. Then $f^{-1}(y)$ is a Lindelöf subspace of the open set $f^{-1}[V]$. By virtue of 4.9(1) there exists an F_σ-set F such that

$$f^{-1}(y) \subset \text{int } F \subset F \subset f^{-1}[V].$$

Since $U := Y \backslash f[X \backslash \text{int } F]$ is open and satisfies $y \in U \subset f[F]$, we conclude that the F_σ-set $f[F]$ fulfills $y \in \text{int} f[F) \subset f[F] \subset V$.

Obviously, the preceding theorem remains true if we assume X to be a T_1-space and, instead of each $f^{-1}(y)$ being Lindelöf, each $f^{-1}(y)$ having a Lindelöf boundary.

5.9. THEOREM. *Let X be a weakly regular space and $(A_i)_{i \in I}$ a pairwise disjoint closure preserving family of closed sets with Lindelöf boundaries. Then the space Y, obtained from X by identifying each A_i to a point, is weakly regular.*

Regularity is an inverse invariant between Hausdorff spaces under perfect mappings [**21**]. As the following result shows, this relationship can be generalized to weakly regular spaces. The condition that X is Hausdorff is necessary and cannot be weakened to T_1, even if f is additionally assumed to be open, as was shown in 3.13.

5.10. THEOREM. *Suppose f is a perfect mapping from a Hausdorff space X onto a weakly regular space Y. Then X is weakly regular.*

Proof. Let U be an open neighborhood of $x \in X$. If $f^{-1}(f(x)) \subset U$, there exists an F_σ-set F in Y with

$$f(x) \in \text{int } F \subset F \subset Y \backslash f[X \backslash U],$$

which implies that $f^{-1}[F]$ is an F_σ-set satisfying

$$x \in \text{int} f^{-1}[F] \subset f^{-1}[F] \subset U.$$

Now suppose $f^{-1}(f(x)) \cap (X \backslash U) \neq \emptyset$. Since this set is compact and X is Hausdorff, there exists an open set V with

$$f^{-1}(f(x)) \cap (X \backslash U) \subset V \subset \text{cl } V \subset X \backslash \{x\}.$$

Hence $Y \backslash f[X \backslash (U \cup V)]$ is an open neighborhood of $f(x)$ which implies the existence of an F_σ-set F in Y with

$$f(x) \in \text{int } F \subset F \subset Y \backslash f[X \backslash (U \cup V)].$$

It is easily verified that $f^{-1}[F] \backslash V$ is an F_σ-set in X with

$$x \in \text{int } (f^{-1}[F] \backslash V) \subset f^{-1}[F] \backslash V \subset U.$$

Finally, we give a characterization of weak regularity by means of covers. Whenever $\mathscr{A}$ is a family of subsets of a space X we define

$$\operatorname{int}\mathscr{A} := \{\operatorname{int} A | A \in \mathscr{A}\}.$$

5.11. PROPOSITION. *For an R_0-space X the following conditions are equivalent:*

(1) *X is weakly regular;*

(2) *every open cover $\mathscr{A}$ of X has a refinement $\mathscr{B}$ consisting of F_σ-sets, such that* int B *covers X.*

6. Relations between regularity conditions and developability. In analogy to Urysohn's classical metrization theorem, H. Brandenburg [**7**] established the equivalence of the conditions (2) and (3) of the following proposition. It is easily seen that this equivalence can further be extended to weakly regular spaces.

6.1. PROPOSITION. *For a topological space X the following conditions are equivalent:*

(1) *X is weakly regular and second countable;*

(2) *X is D-regular and second countable;*

(3) *X is developable and second countable.*

The assumption of second countability in the implication (1) $\Rightarrow$ (3) is unnecessarily strong. A base $\mathscr{B}$ for the open sets of a topological space X is called *locally countable* if every point of X has a neighborhood intersecting at most countably many members of $\mathscr{B}$.

6.2. PROPOSITION. *Every weakly regular, open hereditary Lindelöf space with locally countable base is developable.*

Proof. For every open subset A of X and $x \in A$ there exists a neighborhood F_x, contained in A, which is an F_σ-set. Since X is open hereditary Lindelöf there are countably many points $x_1, x_2, \ldots$ of A such that

$$A = \bigcup \{F_{x_i} | i \in \mathbf{N}\}.$$

Hence A is itself F_σ. By virtue of [**8**; 2.5] this implies that X is developable.

6.3. A space X was called *subnormal* in [**12**] if every pair of closed and disjoint subsets of X can be separated by G_δ-sets. This notion should not be confused with subnormality as defined in [**26**] and what is usually referred to as finitely subparacompact. X was called *D-normal* in [**7**], [**20**] if every pair of closed and disjoint subsets of X can be separated by closed G_δ-sets. Perfect spaces are D-normal, and every D-normal R_0-space is D-completely regular. The following results are modelled after a classical theorem essentially due to A. Tychonoff [**43**] which states that every regular Lindelöf space is normal.

6.4. THEOREM. *Let X be a Lindelöf space.*
(1) *If X is weakly regular, then X is subnormal.*
(2) [**7**] *If X is D-regular, then X is D-normal.*

Proof. (1) Let A and B be two disjoint closed sets. For every $x \in A$ there exists an F_σ-set F_x with $x \in \operatorname{int} F_x \subset F_x \subset X \backslash B$. Since X is Lindelöf, countably many $\operatorname{int} F_{x_i}$ suffice to cover A. Hence

$$G: = \bigcap_{i \in \mathbf{N}} X \backslash F_{x_i}$$

is a G_δ-set with $B \subset G \subset \bar{G} \subset X \backslash A$. By applying the same procedure again a second G_δ-set L can be obtained, satisfying $A \subset L$ and $L \cap G = \emptyset$.

7. Examples. The implications of 2.3 are illustrated in the figure below. The inserted numbers refer to examples described in what follows.

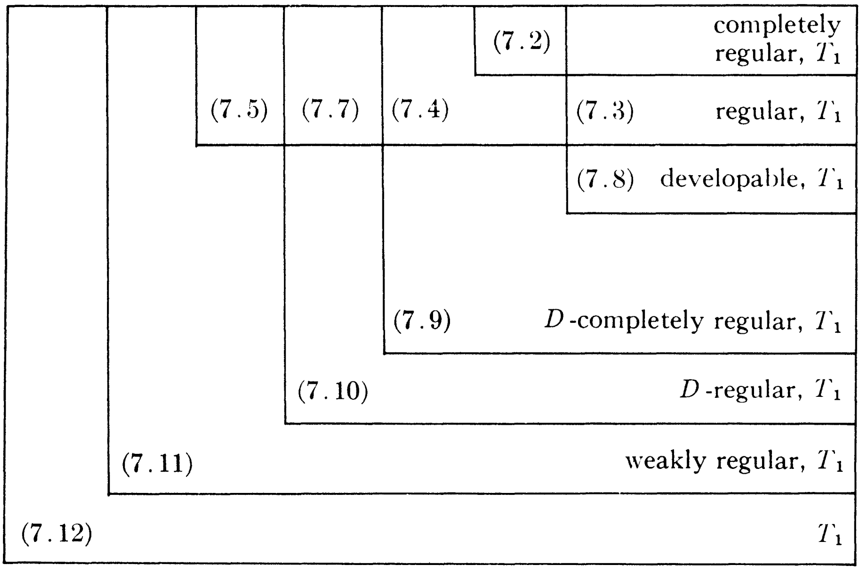

FIGURE 7.1

In cases where we had to choose among a variety of candidates we tried to find the simplest. Note also that we only specify properties relevant to our considerations here.

7.2. The open ordinal space (cf. [**37**], p. 175) is a completely regular perfect, but not developable T_1-space. Further examples are the Sorgenfrey line and the bow-tie space (loc. cit.). The closed ordinal space (loc. cit.) is a completely regular, but not perfect T_1-space, since $\{\omega\}$, the singleton having the first uncountable ordinal as unique element, is not a G_δ-set. This shows also that for D-completely regular spaces the system of all closed sets need not to be a G_δ-base.

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In a recent paper [**34**] it was shown that the power $\mathbf{N}^{\aleph_1}$ of the discrete space of natural numbers is not perfect, not even subnormal, but, obviously, completely regular.

We turn now to regular, but not completely regular, T_1-spaces. Examples of such spaces are legion but exclusively nontrivial. However, since the first example was found by A. Tychonoff [**42**], the complexity of such spaces decreased continuously, and with Thomas' space [**39**] an example of acceptable simplicity is available. It will play a crucial role in what follows.

7.3. Examples of regular, developable T_1-spaces (= Moore spaces), not being completely regular are contained, for example, in [**1**], [**45**], and announced in [**35**]. These examples, however, are all modifications of a space due to F. B. Jones [**24**] whereas a space of A. Mysior [**33**] seems to be different in nature.

7.4. We show further properties of Thomas' space X [**39**] which is known to be a regular, not completely regular T_1-space. To make the investigations self-contained we start by giving an outline of this space. Consider the following subsets of the Cartesian plane. For a fixed even integer n, $L(n)$ is the set of points (n, y) with $0 \leqq y < 1/2$. S_1 is the union of the sets $L(n)$. For a fixed odd integer n, and $k \geqq 2$, $p(n, k) = (n, 1 - (1/k))$, and $T(n, k)$ is the set of points of the form $(n \pm t, 1 - t - (1/k))$ as t ranges over the interval $(0, 1 - (1/k)]$. These are the points on the legs of an isosceles right triangle with hypotenuse lying along the x-axis, and $p(n, k)$ the vertex at the right angle. S_2 is the set of all $p(n, k)$ and S_3 is the union of the sets $T(n, k)$. The underlying set of Thomas' space X is the union of S_1, S_2, and S_3, plus two additional points p^- and p^+. Its topology is defined by specifying the neighborhoods of each point. The topology is discrete at each point of S_3. A neighborhood of the point $p(n, k)$ must contain all but finitely many points of $T(n, k)$. A neighborhood of a point (n, y) of $L(n)$ contains all but finitely many of the points in X which have the same ordinate as y, and an abscissa which differs from n by less than 1. A neighborhood of p^- contains a set of the form

$$U(c) = \{(u, v) \in S_1 \cup S_2 \cup S_3 | u < c\} \cup \{p^-\}$$

for some real number c, and a neighborhood of p^+ contains a set of the form

$$V(c) = \{(u, v) \in S_1 \cup S_2 \cup S_3 | c < u\} \cup \{p^+\}$$

for some real number c.

We assert that

(a) X is D-completely regular;

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(b) X is subnormal (X is even subparacompact, which is stronger than subnormal. For the definition of subparacompact see [**10**]);

(c) X is not D-normal (and hence not perfect).

ad(a). Let $N: = \{2, 3, \ldots\}$. For every even integer n and every finite set $K \subset N$ we define

$$F(n, K) = U(n) \cup L(n) \cup \bigcup \{T(n, k)|k \in N, k \notin K\} \cup \{p(n, k)|k \in N\backslash K\},$$

$$G(n, K) = V(n) \cup L(n) \cup \bigcup \{T(n, k)|k \in N, k \notin K\} \cup \{p(n, k)|k \in N\backslash K\},$$

and

$$\mathscr{B} = \{F(n, K)|n \text{ an even integer}, K \subset N \text{ finite}\} \cup \{G(n, K)|n \text{ an even integer}, K \subset N \text{ finite}\} \cup \{B \subset X|B \text{ clopen}\}.$$

Clearly, $\mathscr{B}$ is a base for the open sets of X. $\mathscr{B}$ is an F_σ-base, since for every clopen set $B \in \mathscr{B}$ we have $X\backslash B \in \mathscr{B}$, and because

$$F(n, K) = \bigcup \{X\backslash G(n + 2, \{k\})|k \in N\backslash K\},$$

$$G(n, K) = \bigcup \{X\backslash F(n - 2, \{k\})|k \in N\backslash K\}.$$

ad(b). Let two disjoint sets A and B of X be given. Whenever $p(n, k) \in A$ (resp. B) then there exists an open neighborhood

$$W(n, k) \subset (T(n, k) \cup \{p(n, k)\})$$

of $p(n, k)$ such that $B \cap W(n, k) = \emptyset$ (resp. $A \cap W(n, k) = \emptyset$). Consider

$$A': = A \cup \bigcup \{W(n, k)|p(n, k) \in A\}$$

and

$$B': = B \cup \bigcup \{W(n, k)|p(n, k) \in B\}.$$

Suppose there exists $x \in A' \cap B'$. Then $x \in W(n, k) \cap W(n', k')$ for some $p(n, k) \in A$ and $p(n', k') \in B$. This, however, is impossible. Let

$$x \in A\backslash \bigcup \{W(n, k)|p(n, k) \in A\}$$

be given. Then $x \in T(n, k)$ for some n and k, or $x \in L(n)$ for some n, or $x = p^-$ or $x = p^+$. In any case $\{x\}$ is a G_δ-set, i.e.,

$$\{x\} = \bigcap \{R_i^x|i \in \mathbf{N}\}$$

where each R_i^x is an open set containing x: if $x \in T(n, k)$ choose $R_i^x = \{x\}$ for all i, if $x \in L(n)$ choose R_i^x as a decreasing chain of basic neighborhoods, and if $x = p^-$ (resp. p^+) choose $R_i^x = U(2i)$ (resp.

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$V(2i)$). This implies

$$A' = \bigcap \{ \bigcup \{W(n, k) | p(n, k) \in A\} \cup \bigcup\{R_i^x | x \in A \setminus \bigcup \{W(n, k)|p(n, k) \in A\}\} | i \in \mathbf{N}\},$$

i.e., A' is a G_δ-set. For reasons of symmetry B' is also a G_δ-set, and therefore A and B are separated by disjoint G_δ-sets.

ad(c). Let an even integer n be given. Clearly, $A = X \setminus V(n)$ and $B = X \setminus U(n + 1)$ are disjoint and closed sets. Suppose there were disjoint and closed G_δ-sets C and D with $A \subset C$ and $B \subset D$. Whenever F is an open set containing B, then F contains almost all points of $T(n + 1, k)$ for all $k \in N$. Hence D contains $\bigcup \{T(n + 1, k)|k \in N\}$ except for at most countably many points. Consider the set R of all $y \in L(n)$ with the property that there exists $x \in T(n + 1, k)$ such that $x \notin D$. Obviously, R is countable, and there exists $z \in L(n) \setminus R$. Since D is closed and $z \notin D$, z is contained in an open set which is disjoint from D. This, however, is a contradiction to $z \notin R$ and X is therefore not D-normal.

It was proved in [**22**] that for every T_1-space Y a regular T_1-space X exists such that every continuous function from X to Y is necessarily constant. By virtue of 2.1(2) this provides us immediately with an example of a regular, not D-completely regular space. We prefer, however, to use in what follows modifications of Thomas' space to construct examples, mostly because we think it advantageous to avoid cardinal arithmetic as employed in [**22**].

7.5. The "Mountain Chain Space" is obtained from Thomas' space in the following way. Let p^- and p^+ be two distinct points not belonging to the Cartesian plane. Consider the following subsets of the plane enlarged by $\{p^-, p^+\}$:

$$(\forall n \in \mathbf{Z}) \quad P(n) := \bigcup \{(n, r)|1/2 < r < 1\}$$

$$S_1 := \bigcup \{P(n)|n \in \mathbf{Z}\}$$

$$(\forall n \in \mathbf{Z}; \forall r, 1/2 < r < 1) \quad T(n, r) := \{(n + 1, r - t)|0 < t \leqq r\} \cup \{(n - t, r - t)|0 < t \leqq t\}.$$

$$S_2 := \bigcup \{T(n, r)|n \in \mathbf{Z}, 1/2 < r < 1\},$$

$$(\forall c \in \mathbf{R}) \quad U(c) := \{(x, y) \in S_1 \cup S_2 | x < c\} \cup \{p^-\},$$

$$(\forall c \in \mathbf{R}) \quad V(c) := \{(x, y) \in S_1 \cup S_2 | x > c\} \cup \{p^+\},$$

and finally

$$X := S_1 \cup S_2 \cup \{p^-, p^+\}.$$

X has the shape of a mountain chain:

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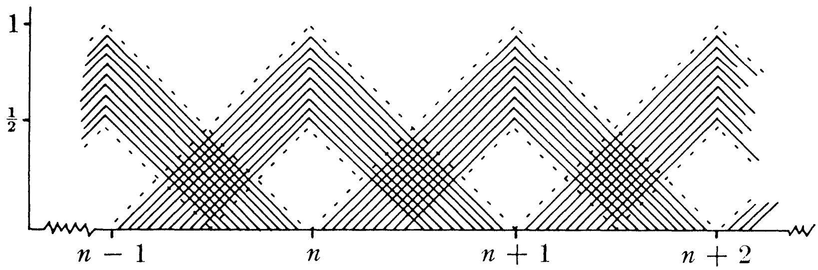


The topology on X is determined by the following neighborhood systems. Each point of S_2 is declared open. A neighborhood of a point (n, r) of S_1 contains all but finitely many points of $T(n, r)$. A neighborhood of p^- contains some $U(c)$, and a neighborhood of p^+ contains some $V(c)$. It is trivial to check the neighborhood system at each point has a basis of closed sets and every point is closed. Hence, X is a regular T_1-space. Consider an open F_σ-set A. We assert that, if A contains almost all points of $P(n)$, then A contains almost all points of $P(n - 1)$ and $P(n + 1)$.

Suppose there is an infinite subset B of $P(n + 1)$ disjoint from A. Since A is an F_σ-set there are countably many closed sets $A(k)$, $k \in \mathbf{N}$, such that $A = \cup A(k)$. Hence, for every $k \in \mathbf{N}$ and $(n + 1, r) \in B$ there is a neighborhood of $(n + 1, r)$ disjoint from $A(k)$. We may assume that this neighborhood is $T(n + 1, r)$ reduced by finitely many elements. This implies that for every $(n + 1, r) \in B$ there exists a set R_r which is obtained from $T(n + 1, r)$ by deleting countably many points disjoint from $A = \cup A(k)$. Consider $(n, s) \in A \cap P(n)$. Since A is open there is a neighborhood W_s of (n, s) contained in A. We may think of this neighborhood as $T(n, s)$ reduced by finitely many points. Since $W_s \cap R_r = \emptyset$ for every $(n + 1, r) \in B$, the singletons $T(n, s) \cap T(n + 1, r)$ must have been deleted from $T(n + 1, r)$ for almost all $(n + 1, \mathrm{r}) \in B$. This implies that at most countably many elements of $P(n)$ belong to A, a contradiction. Hence, A contains almost all points of $P(n + 1)$ and also, for reasons of symmetry, of $P(n - 1)$.

Assume now that X is D-regular. We would then find for $U(c)$, $c \in \mathbf{R}$, an open F_σ-set A satisfying $p^- \in A \subset U(c)$. But for some $d \in \mathbf{R}$ $U(d) \subset A$ clearly must hold, which implies by virtue of the argument above that A contains almost all points of $P(n)$ for every $n \in \mathbf{Z}$. Thus, $A \not\subset U(c)$, a contradiction. Therefore, X is not D-regular.

7.6. Since Thomas' space [**39**] is a subspace of the mountain chain space X it is clear that every real-valued function of X takes the same values at p^- and p^+. This is even true for continuous functions into first countable Hausdorff spaces. Using techniques due to [**14**] or [**17**] X may be used to obtain a regular space Y such that every real-valued function of Y is constant.

7.7. The "Skyline Space" is obtained by combining properties of Thomas' original space and the "Mountain Chain Space" introduced in 7.5. Again, the underlying set X is a subset of the Cartesian plane enlarged by two distinct points p^- and p^+. Consider the following sets:

$$(\forall n \in \mathbf{Z} \text{ even}) \qquad L(n) = \{(n, y) | 0 \leqq y < 1/2\},$$

$$(\forall n \in \mathbf{Z} \text{ odd}, k \in \mathbf{N} \text{ even}) \qquad R(n, k) = \{(x, k) | n + 1 < x < (2n + 1)/2\}$$

$$(\forall n \in \mathbf{Z} \text{ odd}, k \in \mathbf{N} \text{ odd}) \qquad R(n, k) = \{(x, k) | (2n - 1)/2 < x < n + 1\}$$

$$(\forall n \in \mathbf{Z} \text{ odd}, k \in \mathbf{N}) \qquad P(n,k) = \{(n, y) | k - (1/2)^{2k} < y < k\}.$$

For $(n, s) \in P(n, k)$ we define furthermore

$$T(n, s) = \{(x, s) | (n - 1) + (1/2)^{2k} + (k - s) < x < (n + 1) - (1/2)^{2k} - (k - s)\} \cup$$
$$\cup \{(x, y) | x = (n - 1) + (1/2)^{2k} + (k - s) \quad \text{or} \quad x = (n + 1) - (1/2)^{2k} - (k - s), 0 < y < s\},$$

and

$$(\forall n \in \mathbf{Z} \text{ odd}, k \in \mathbf{N}) \qquad Q(n,k) := \cup \{T(n, s) | (n, s) \in P(n,k)\}.$$

If $(p, k) \in R(n, k)$ we define

$$S(p, k) := \{(p, y) | k - (1/2) < y < k + 1\}.$$

The underlying set of the Skyline Space is then

$$X := Y \cup \{p^-, p^+\}, \quad \text{where}$$
$$Y := \cup \{L(n) | n \in \mathbf{Z}, \text{even}\} \cup \cup \{Q(n, k) | n \in \mathbf{Z} \text{ odd}, k \in \mathbf{N}\} \cup$$
$$\cup \{S(p, k) | 0 < |n - p| < 1/2, n \in \mathbf{Z} \text{ odd}, k \in \mathbf{N}\}.$$

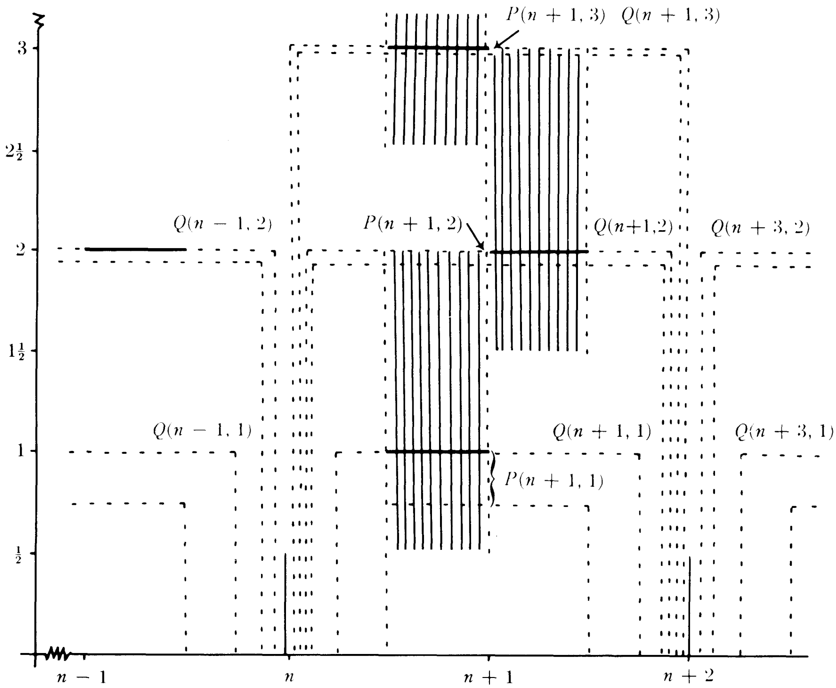
X has the shape in the figure following. We topologize X by specifying neighborhood systems for its points: A neighborhood of a point $(n, y) \in L(n)$ contains a set of the form

$$W(n, y) \backslash \cup \{Q(n + 1, k) | k \in I\},$$

where

$$W(n, y) = \{(x, y) \in Y | n - \tfrac{1}{2} < x < n + \tfrac{1}{2}\}$$

and I is some finite subset of $\mathbf{N}$. If $(n, y) \in P(n, k)$, then a neighborhood of (n, y) contains all but finitely many points of $T(n, y)$. A neighborhood of a point $(x, k) \in R(n, k)$ contains all but finitely many points of $S(x, k)$.



Neighborhoods of p^- contain a set of the form

$$U(c) = \{(x, y) \in Y | x < c\} \cup \{p^-\}$$

for some $c \in \mathbf{R}$, and symmetrically, neighborhoods of p^+ contain a set of the form

$$V(c) = \{(x, y) \in Y | c < x\} \cup \{p^+\}$$

for some $c \in \mathbf{R}$. All other points of X are declared open. It is easily checked that the neighborhood systems just specified induce a topology such that every point is closed and has a neighborhood base consisting of closed sets. Hence X is a regular T_1-space.

We show that X is D-regular. Points belonging to Y even have neighborhood systems consisting of clopen sets. If suffices to prove that p^- has a neighborhood base consisting of open F_σ-sets since the assertion follows for p^+ for reasons of symmetry. Consider $U(n + 2)$ for an even integer n. Clearly, $U(n + 2)$ is open. For every $i \in \mathbf{N}$ we define

$$A(i) := U(n) \cup L(n) \cup \bigcup \{Q(n + 1, k) | k \leqq i\} \cup$$
$$\cup \{S(p, k) | (p, k) \in R(n + 1, k), k < i\}.$$

Then every $A(i)$ is closed and $U(n + 2) = \bigcup \{A(i) | i \in \mathbf{N}\}$. Hence, X is D-regular.

Suppose now that $A \subset X$ is an open F_σ-set such that for some even integer n an uncountable subset $B(n)$ of $L(n)$ belongs to A. We assert that then for every $k \in \mathbf{N}$ almost all points of $P(n+1, k)$ and $P(n-1, k)$ belong to A.

We assume first that for every $k \in \mathbf{N}$ an infinite subset $P'(k)$ of $P(n+1, k)$ is disjoint from A. There exist closed sets $A(i)$, $i \in \mathbf{N}$, such that $A = \cup A(i)$. Let $i \in \mathbf{N}$ be fixed. We then conclude that for every $k \in \mathbf{N}$ and $(n+1, s) \in P'(\mathrm{k})$ there exists a neighborhood of $(n+1, s)$ disjoint from $A(i)$. One may think of this neighborhood as $T(n+1, s)$ with at most finitely many points deleted. Hence, $A(i)$ contains for every $(n+1, s) \in P'(k)$ and $k \in \mathbf{N}$ at most finitely many elements of $T(n+1, s)$, which implies by virtue of $A = \cup A(i)$ that A contains for every $(n+1, s) \in P'(k)$ and $k \in \mathbf{N}$ at most countably many elements of $T(n+1, s)$. Clearly, for every $k \in \mathbf{N}$ there is at most a countable set $C(k)$ of points $(n, y) \in A$ such that $W(n, y) \cap Q(n+1, k) \subset A$. Since A is uncountable there exists

$$(n, y) \in A \setminus \cup \{C(k) | k \in \mathbf{N}\},$$

and since A is open, there exists a finite subset I of $\mathbf{N}$ such that

$$W(n, y) \setminus \cup \{Q(n+1, l) | l \in I\} \subset A.$$

Consequently, $(n, y) \in \cup \{C(k) | k \in \mathbf{N}\}$, a contradiction. We conclude that there exists $k \in \mathbf{N}$ such that almost all points of $P(n+1, k)$ belong to A. The same argument as employed in 7.5 can now be used to show that almost all points of $P(n+1, k)$ belong to the open F_σ-set A for all $k \in \mathbf{N}$. For reasons of symmetry the assertion is also true for $P(n-1, k)$, $k \in \mathbf{N}$.

Now suppose X is D-completely regular. Then X has an F_σ-base $\mathscr{F}$. Consider the neighborhood $U(c)$ of p^- for some $c \in \mathbf{R}$. There exists $F \in \mathscr{F}$ with $p^- \in F \subset U(c)$, and, since $\mathscr{F}$ is an F_σ-base, there exist moreover a countable subfamily $(F(n))_{n \in \mathbf{N}}$ of $\mathscr{F}$ such that

$$F = \cup \{X \setminus F(n) | n \in \mathbf{N}\}.$$

Since $U(d) \subset F \subset U(c)$ for some $d \in \mathbf{R}$,

$$m := \max \{i \in \mathbf{Z} | i \text{ is even and uncountably many points of } L(i) \text{ belong to } F\}$$

exists. It follows from the preceding argumentation that almost all points of $P(m+1, k)$ for every $k \in \mathbf{N}$ belong to F. Note that every $F(n)$ is also an open F_σ-set. Since uncountably many points of $L(m+2)$ are contained in $F(n)$ for every $n \in \mathbf{N}$, it follows that for all $n \in \mathbf{N}$ almost all points of $P(m+1, \mathrm{k})$ are contained in $F(n)$ for every $k \in \mathbf{N}$. Hence $\cap F(n)$ contains for every $k \in \mathbf{N}$ all but at most countably many points of $P(m+1, k)$. This implies

$$F \cap \cap \{F(n) | n \in \mathbf{N}\} \neq \emptyset$$

in contradiction to $F = \bigcup \{X\backslash F(n) | n \in \mathbf{N}\}$. Hence, X is not D-completely regular.

7.8. The set $\mathbf{N}$ of positive integers equipped with the cofinite topology is a developable, not regular T_1-space. If for $k, l \in \mathbf{N}$ $A(k, l) = \{1, 2, \ldots, k\}\backslash\{l\}$, then $(\mathscr{A}(k, l))_{(k,l)\in \mathbf{N}\times\mathbf{N}}$ is a development for $\mathbf{N}$, where

$$\mathscr{A}(k, l) = \{\mathbf{N}\backslash A(k, l), \mathbf{N}\backslash\{l\}\}.$$

Another example is the real line X with neighborhoods of any nonzero point being as in the usual topology, while neighborhoods of 0 will have the form $U\backslash A$, where U is a neighborhood of 0 in the usual topology and $A = \{1/n | n \in \mathbf{N}\}$. Then X is Hausdorff but not regular [**44**; 14.2]. X is second countable since all finite intersections of elements of a countable base of the usual topology enlarged by $\{S(0, 1/n)\backslash A | n \in \mathbf{N}\}$ is a base for X. X is weakly regular: The only critical point is 0. Consider $S(0, \epsilon)\backslash A$. For every $n \in \mathbf{N}$ let $U(n, k)$ denote the open neighborhood

$$S(1/n, ((1/n) - 1/(n + 1))/2^k)$$

of $1/n$. Then

$$S(0, \epsilon/2)\backslash A \subset \bigcup \{[-\epsilon/2, \epsilon/2] \cap (X\backslash \bigcup \{U(n, k) | n \in \mathbf{N}\}) \\ | k \in \mathbf{N}\} \subset S(0, \epsilon)\backslash A.$$

By virtue of 6.1 X is developable.

In the following considerations X may be used instead of $\mathbf{N}$ to yield Hausdorff spaces with all other properties unchanged.

7.9. The product X of the closed ordinal space (7.2) and $\mathbf{N}$ with cofinite topology (7.8) is a D-completely regular T_1-space, not being regular or perfect, since (a) the class of D-completely regular spaces is product-closed, (b) perfectness and regularity are hereditary, and (c) the components of X are subspaces of X.

7.10. A similar argument as in 7.9 yields that the product of the Skyline Space (7.7) and $\mathbf{N}$ with cofine topology (7.8) is a D-regular, not regular, not D-completely regular T_1-space.

7.11. The same reasoning yields that the product of the Mountain Chain Space (7.5) and $\mathbf{N}$ with cofinite topology is a weakly regular, not regular, not D-regular T_1-space.

7.12. The real line equipped with the cofinite topology is a simple example of a T_1-space, not being weakly regular.

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