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ASYMPTOTIC BOUNDS FOR THE DISTRIBUTION OF THE SUM OF DEPENDENT RANDOM VARIABLES

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Abstract

Suppose that $X_1, \dots, X_n$ are random variables with the same known marginal distribution F but unknown dependence structure. In this paper we study the smallest possible value of $\mathbb{P}(X_1 + \cdots + X_n < s)$ over all possible dependence structures, denoted by $m_{n,F}(s)$. We show that $m_{n,F}(ns) \to 0$ for s no more than the mean of F under weak assumptions. We also derive a limit of $m_{n,F}(ns)$ for any $s \in \mathbb{R}$ with an error of at most $n^{-1/6}$ for general continuous distributions. An application of our result to risk management confirms that the worst-case value at risk is asymptotically equivalent to the worst-case expected shortfall for risk aggregation with dependence uncertainty. In the last part of this paper we present a dual presentation of the theory of complete mixability and give dual proofs of theorems in the literature on this concept.

Keywords: Dependence bound; complete mixability; value at risk; modeling uncertainty

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1. Introduction

Let $\boldsymbol{X} = (X_1, \dots, X_n)$ be a random vector with the same known marginal distributions F, denoted as $X_i \sim F$, $i = 1, \dots, n$. When F is known but the joint distribution of $(X_1, \dots, X_n)$ is unknown, the distribution of $\boldsymbol{X}$ is undetermined with some marginal constraints. For any $s \in \mathbb{R}$ and $\psi : \mathbb{R}^n \to \mathbb{R}$, let

$$m_{\psi,F}(s) = \inf\{\mathbb{P}(\psi(\boldsymbol{X}) < s) \colon X_i \sim F,\ i = 1, \dots, n\}$$

and

$$w_{\psi,F}(s) = \inf\{\mathbb{P}(\psi(\boldsymbol{X}) \neq s) \colon X_i \sim F,\ i = 1, \dots, n\}.$$

The $\mathbb{P}(\psi(\boldsymbol{X}) \leq s)$ and $\mathbb{P}(\psi(\boldsymbol{X}) = s)$ cases, and the cases concerning the largest, instead of the smallest, possible values, are technically similar; we focus on the $\mathbb{P}(\psi(\boldsymbol{X}) < s)$ case in this paper. The study of $m_{\psi,F}(s)$ originated from a question raised by A. N. Kolmogorov, and partially answered in Makarov (1982) as the first result for $n = 2$ and $\psi(x, y) = x + y$. There has been extensive research on this topic over the past few decades. Admittedly, most of the recent research on $m_{\psi,F}(s)$ has been motivated by the rapidly growing applications in financial risk management in the past several years. Roughly speaking, finding $m_{\psi,F}(s)$ is equivalent to finding the worst-case value at risk with dependence uncertainty, which plays an important role in the study of risk aggregation. We refer the reader to Embrechts and Puccetti (2010) for an overview on this topic, where the connection between $m_{\psi,F}(s)$ and risk management is

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explained in detail. Numerical calculation of $m_{\psi,F}(s)$ and its importance in quantifying model uncertainty are discussed in the more recent paper Embrechts *et al.* (2013).

Unfortunately, when $n \geq 3$, the quantity $m_{\psi,F}(s)$ is not solved except for a few special cases of F and ψ. The most studied and most interesting choice of ψ is the sum function $\psi_n(\boldsymbol{X}) = X_1 + \cdots + X_n$ due to its mathematical tractability and financial interpretation as the aggregate risk. Equivalent forms of ψ_n includes the product function $\Pi_n(\boldsymbol{X}) = X_1 \times \cdots \times X_n$, noting that $m_{\Pi_n,F}(s) = m_{\psi_n,G}(\log s)$, where G is the distribution of $\log X$, $X \sim F$. In this paper we will focus on $\psi_n(\boldsymbol{X})$. For simplicity, throughout, we define $m_{n,F} = m_{\psi_n,F}$ and $w_{n,F} = w_{\psi_n,F}$ for the sum functions ψ_n, $n = 1, 2, \ldots$.

A duality theorem for $m_{\psi,F}$ was given in Gaffke and Rüschendorf (1981) and used in Rüschendorf (1982) to find $m_{n,F}$ for uniform and binomial distributions. Besides the uniform and binomial cases, explicit values of $m_{n,F}$ were not found until Wang and Wang (2011) revealed the connection between $m_{n,F}$ and the class of *completely mixable distributions*, introduced in the same paper. A distribution F is said to be n-completely mixable if there exist (dependent) random variables $X_1, \ldots, X_n$, identically distributed as F, such that $X_1 + \cdots + X_n$ is a constant. Based on complete mixability, Wang *et al.* (2013) gave explicit values of $m_{n,F}$ for F with tail-monotone densities. The reader is also referred to Denuit *et al.* (1999) for a study of $m_{n,F}$ using the method of copulas, to Embrechts and Puccetti (2006) for a lower bound using the duality, and to Puccetti and Rüschendorf (2013) for the connection between the sharpness of the duality bounds and complete mixability. A history of the study of $m_{n,F}$ and its connection to mass-transportation theory can be found in the book Rüschendorf (2013).

Recent developments in complete mixability has raised increasing attention in quantitative risk management, not limited to the problems related to $m_{n,F}$. The concept is of importance in variance minimization and convex ordering with constraints, and was studied prior to the formal introduction of complete mixability; see, for example, Rüschendorf and Uckelmann (2002). The concept of complete mixability was later studied and used in the research of risk aggregation with dependence uncertainty; see, for example, Puccetti *et al.* (2012), Wang *et al.* (2013), Puccetti and Rüschendorf (2013), Embrechts *et al.* (2013), and Bernard *et al.* (2013). It turns out that the concept has a dual representation based on the quantities $m_{n,F}(s)$ and $w_{n,F}(s)$, which will be given in this paper.

In this paper we study the asymptotic limit of the probability $m_{n,F}$ as $n \to \infty$ based on the duality theorem in Gaffke and Rüschendorf (1981). We will show that, for any continuous distribution F with a bounded density,

$$m_{n,F}(ns) \to F(a_0) \quad \text{as } n \to \infty,$$

where $a_0 = \inf\{a \in \mathbb{R} \colon \mathbb{E}[X \mid X \geq a] \geq s,\ X \sim F\}$. The convergence rate will also be obtained. Our result has a clear interpretation in risk management. It suggests that, for general continuous distributions with bounded density, the worst-case value at risk (VaR) and worst-case expected shortfall (ES) are asymptotically equivalent, and that the superadditivity ratio of VaR is asymptotically equal to the value of ES/VaR for F. This phenomenon, from a risk management point of view, was first noted in the recent paper Puccetti and Rüschendorf (2014) and later in the paper Puccetti *et al.* (2013) with assumptions and technical approaches completely different from this paper. In the last part of this paper we will construct a bridge that connects $m_{n,F}(s)$, $w_{n,F}(s)$, and the theory of complete mixability.

The rest of the paper is organized as follows. In Section 2 we give the dual representation for the quantities $m_{n,F}(s)$ and $w_{n,F}(s)$. Two admissible sets will be introduced and their properties will be studied. In Section 3 we will present our main results on the asymptotic bounds for

$m_{n,F}(s)$, and discuss their applications in risk management. In Section 4 we give the dual representation of the complete mixability. We conclude in Section 5. Throughout the paper, we identify probability measures with the corresponding cumulative distribution functions.

2. Dual representation and admissible sets

In this section we associate the probabilities $m_{\psi_n,F}$ and $w_{\psi_n,F}$ with an optimization problem over some functional sets, called *admissible sets*, and study the properties of the admissible sets. Throughout the paper, we use the notation $x \vee y = \max\{x, y\}$, $x \wedge y = \min\{x, y\}$, and $(x)_+ = \max\{x, 0\}$ for x and y either numbers, functions, or random variables.

2.1. Dual representation of the infimum distribution of the sum

A duality for $m_{\psi,F}$ was given in Gaffke and Rüschendorf (1981) and Rüschendorf (1982):

$$m_{\psi,F}(s) = 1 - \inf\biggl\{n\int f\,\mathrm{d}F;\ f\colon \mathbb{R} \to \mathbb{R} \text{ is bounded and measurable such that } \sum_{i=1}^{n} f(x_i) \geq \mathbf{1}_{[s,+\infty)}(\psi(x_1, \dots, x_n)) \text{ for all } x_i \in \mathbb{R},\ i = 1, \dots, n\biggr\}. \tag{2.1}$$

For simplicity, we define $m_{n,F} = m_{\psi_n,F}$ and $w_{n,F} = w_{\psi_n,F}$ for the sum functions ψ_n, $n = 1, 2, \dots$. To better study the values of $m_{n,F}$ and $w_{n,F}$ using the duality, for $\mu \in \mathbb{R}$, we define the *admissible sets*

$$A_n(\mu) = \biggl\{f\colon \mathbb{R} \to \mathbb{R},\ \text{measurable},\ \frac{1}{n}\sum_{i=1}^{n} f(x_i) \geq \mathbf{1}_{\{[n\mu,\infty)\}}(x_1 + \cdots + x_n) \text{ for all } x_1, \dots, x_n \in \mathbb{R}\biggr\}$$

and

$$B_n(\mu) = \biggl\{f\colon \mathbb{R} \to \mathbb{R},\ \text{measurable},\ \frac{1}{n}\sum_{i=1}^{n} f(x_i) \geq \mathbf{1}_{\{n\mu\}}(x_1 + \cdots + x_n) \text{ for all } x_1, \dots, x_n \in \mathbb{R}\}.$$

It is obvious that $A_n(\mu) \subset B_n(\mu)$. Note that here μ is any real number and in the later sections it is often chosen as the mean of a distribution F. The following lemma states the relationship between the probabilities $m_{n,F}$ and $w_{n,F}$, and the admissible sets A_n and B_n.

Lemma 2.1. *For any $\mu \in \mathbb{R}$ and any distribution F, we have*

$$m_{n,F}(n\mu) = 1 - \inf\biggl\{\int f\,\mathrm{d}F\colon f \in A_n(\mu)\biggr\}$$

and

$$w_{n,F}(n\mu) = 1 - \inf\biggl\{\int f\,\mathrm{d}F\colon f \in B_n(\mu)\biggr\}.$$

Proof. To be more specific, by taking $\psi(\boldsymbol{X}) = X_1 + \cdots + X_n$ in (2.1), we obtain

$$m_{n,F}(n\mu) = 1 - \inf\biggl\{n \int f \,\mathrm{d}F;\ f\colon \mathbb{R} \to \mathbb{R} \text{ is bounded and measurable such that} \\ \sum_{i=1}^{n} f(x_i) \geq \mathbf{1}_{[n\mu,+\infty)}(x_1 + \cdots + x_n) \text{ for all } x_i \in \mathbb{R},\ i = 1, \ldots, n\biggr\}. \tag{2.2}$$

Since any function f is the limit of bounded functions, the boundedness in (2.2) can be dropped. Thus, simply replacing nf in (2.2) by f, we have the first equality $m_{n,F}(n\mu) = 1 - \inf\{\int f \,\mathrm{d}F\colon f \in A_n(\mu)\}$.

For the second equality, take $\psi(x_1, \ldots, x_n) = \mathbf{1}_{\{n\mu\}}(x_1 + \cdots + x_n)$ in (2.1). We have

$$\begin{aligned} m_{\psi,F}(1) &= 1 - \inf\biggl\{n \int f \,\mathrm{d}F;\ f\colon \mathbb{R} \to \mathbb{R} \text{ is bounded and measurable such that} \\ &\qquad \sum_{i=1}^{n} f(x_i) \geq \mathbf{1}_{[1,+\infty)}(\mathbf{1}_{\{n\mu\}}(x_1 + \cdots + x_n)) \text{ for all } x_i \in \mathbb{R},\ i = 1, \ldots, n\biggr\} \\ &= 1 - \inf\biggl\{\int f \,\mathrm{d}F\colon f \in B_n(\mu)\biggr\}. \end{aligned}$$

Note that $m_{\psi,F}(1) = \inf\{\mathbb{P}(\mathbf{1}_{n\mu}(X_1 + \cdots + X_n) < 1)\colon X_i \sim F,\ i = 1, \ldots, n\} = w_{n,F}(n\mu)$. Thus, $w_{n,F}(n\mu) = 1 - \inf\{\int f \,\mathrm{d}F\colon f \in B_n(\mu)\}$.

The quantities $m_{n,F}(n\mu)$ and $w_{n,F}(n\mu)$, when μ is chosen as the mean of F, turn out to be closely related to the concept of complete mixability. We will use them to formulate the theory of complete mixability in Section 4. Before that, we first study the properties of the two sets $A_n(\mu)$ and $B_n(\mu)$.

2.2. Properties of the admissible sets

Using the duality in Lemma 2.1, we can examine the probabilities $m_{n,F}(n\mu)$ and $w_{n,F}(n\mu)$ by investigating the sets $A_n(\mu)$ and $B_n(\mu)$. Hence, it would be of interest to derive some relevant properties of the admissible sets. Throughout the rest of the paper, we will use a class of functions f_a for $a, \mu \in \mathbb{R}$ defined as (for simplicity, μ is dropped in the notation)

$$f_a(x) = (1 + a(x - \mu))_+.$$

Note that $(f_a \wedge n)/n$ is exactly the admissible functions used in Section 4 of Embrechts and Puccetti (2006). For technical reasons, at this moment we do not truncate f_a by n as in the above paper.

In the following, we introduce a few propositions concerning some properties of the admissible sets. These properties will be used to derive the asymptotic behavior of the admissible sets, and later they contribute to the proof of our main result in Section 3. We first introduce some elements in $A_n(\mu)$ and $B_n(\mu)$. The following proposition gives important forms of the elements in $A_n(\mu)$ and $B_n(\mu)$; later we will see that the functions f_a are fundamental in the asymptotic sense for the sets $A_n(\mu)$ and $B_n(\mu)$. The proof is quite straightforward and is thus omitted.

Proposition 2.1. *Let $n \in \mathbb{N}$ and $\mu \in \mathbb{R}$.*

(a) $f_a \in B_n(\mu)$ *for* $a \in \mathbb{R}$ *and* $f_a \in A_n(\mu)$ *for* $a \geq 0$. *In particular,*

(i) *if* $\mu \neq 0$ *then* $f_{1/\mu}(x) = (x/\mu)_+ \in B_n(\mu)$*;*

(ii) *if* $\mu > 0$ *then* $f_{1/\mu}(x) = (x/\mu)_+ \in A_n(\mu)$*;*

(iii) $f_0(x) = 1 \in A_n(\mu) \subset B_n(\mu)$.

(b) $n\,\mathbf{1}_{[\mu,\infty)}(\cdot) \in A_n(\mu) \subset B_n(\mu)$.

In the next result we list some properties of the admissible sets. In summary, the sets $A_n(\mu)$ and $B_n(\mu)$ are convex, and a dominating or truncated function of an element in $A_n(\mu)$ or $B_n(\mu)$ is still in $A_n(\mu)$ or $B_n(\mu)$. These simple properties provide analytical convenience and will be used later. Their proofs are also quite straightforward and are thus omitted.

Proposition 2.2. *Let $n \in \mathbb{N}$ and $\mu \in \mathbb{R}$.*

(a) $A_n(\mu)$ *is a convex set, i.e. for any* $\lambda \in [0, 1]$ *and* $f, g \in A_n(\mu)$*, we have* $\lambda f + (1-\lambda)g \in A_n(\mu)$.

(b) *If* $f \in A_n(\mu)$ *then* $f \geq 0$.

(c) *If* $f \in A_n(\mu)$, $g\colon \mathbb{R} \to \mathbb{R}$, *and* $g \geq f$, *then* $g \in A_n(\mu)$.

(d) *If* $f \in A_n(\mu)$ *then* $f \wedge n \in A_n(\mu)$.

(e) *The above holds if* $A_n(\mu)$ *is replaced by* $B_n(\mu)$.

One may wonder the effect of n on the sets $A_n(\mu)$ and $B_n(\mu)$. The next proposition states the connection between the sets $A_n(\mu)$ (and also $B_n(\mu)$) for different values of n.

Proposition 2.3. *Let $n, k \in \mathbb{N}$ and $\mu \in \mathbb{R}$.*

(a) $A_{n+k}(\mu) \subset A_n(\mu) \cup A_k(\mu)$. *In particular,* $A_{dn}(\mu) \subset A_n(\mu)$ *for all* $d \in \mathbb{N}$.

(b) $B_{n+k}(\mu) \subset B_n(\mu) \cup B_k(\mu)$. *In particular,* $B_{dn}(\mu) \subset B_n(\mu)$ *for all* $d \in \mathbb{N}$.

Proof. For any $f \in A_{n+k}(\mu)$ and $f \notin A_k(\mu)$, there exist $y_1, \ldots, y_k \in \mathbb{R}$ such that $y_1 + \cdots + y_k \geq k\mu$ and $\sum_{j=1}^{k} f(y_j) < k$. Note that, for any $x_1, \ldots, x_n \in \mathbb{R}$ such that $x_1 + \cdots + x_n \geq n\mu$, we have

$$\sum_{i=1}^{n} f(x_i) + \sum_{j=1}^{k} f(y_j) \geq n + k,$$

since $\sum_{i=1}^{n} x_i + \sum_{j=1}^{k} y_j \geq (n+k)\mu$. This implies that $\sum_{i=1}^{n} f(x_i) > n$ and $f \in A_n(\mu)$. Thus, $A_{n+k}(\mu) \subset A_n(\mu) \cup A_k(\mu)$. The proof for $B_n(\mu)$ is similar.

The fact that $A_{dn}(\mu) \subset A_n(\mu)$ tells us that, roughly speaking (although not strictly), the set $A_n(\mu)$ gets smaller as n gets larger. It motivates us to study the asymptotic behavior of $A_n(\mu)$ as $n \to \infty$. Fortunately, we are able to characterize the limit of $A_n(\mu)$. Before presenting this result, we give a lemma whose proof is trivial by definitions.

Lemma 2.2. *Let $n, k \in \mathbb{N}$ and $\mu \in \mathbb{R}$.*

(a) *If* $f \in A_n(\mu)$ *then* $(n-k)f(\mu - ks) + kf(\mu + (n-k)t) \geq n$ *for all* $t, s \in \mathbb{R}$, $t \geq s$, *and* $k = 0, \ldots, n$. *In particular,* $f(t) \geq 1$ *for all* $t \geq \mu$.

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(b) *If* $f \in B_n(\mu)$ *then* $(n-k)f(\mu - ks) + kf(\mu + (n-k)s) \geq n$ *for all* $s \in \mathbb{R}$ *and* $k = 0, \ldots, n$. *In particular,* $f(\mu) \geq 1$.

The following theorem characterizes the limit of $A_n(\mu)$ as $n \to \infty$. It is clear from the theorem that f_a plays a fundamental role in the limit of $A_n(\mu)$.

Theorem 2.1. *Let* $A(\mu) = \bigcap_{n=1}^{\infty} A_n(\mu)$. *Then*

(a) $A(\mu) = \{f : \mathbb{R} \to \mathbb{R},\ f \geq f_a \text{ for some } a \geq 0\}$*;*

(b) $\lim_{n\to\infty} A_n(\mu)$ *exists and equals* $A(\mu)$.

Proof. (a) If $f \geq f_a$ then by Proposition 2.1(a) and Proposition 2.2(c) we have $f \in A_n(\mu)$ for all $n \in \mathbb{N}$. In the following we will show that, for any $f \in A(\mu)$, we have $f \geq f_a$ for some $a \geq 0$. For any $f \in A(\mu)$, it is obvious that $f \geq 0$. Let $d_1 = \sup\{(1 - f(\mu - c))/c : c > 0\}$ and $d_2 = \inf\{(f(\mu + c) - 1)/c : c > 0\}$. By Lemma 2.2(a) we know that $d_2 \geq 0$. If $d_1 \leq d_2$ then we have $f(x) \geq f_{d_2}(x)$.

Now suppose that $d_1 > d_2$. Then there exist $c_1 > 0$ and $c_2 > 0$ such that

$$\frac{f(\mu + c_2) - 1}{c_2} < \frac{1 - f(\mu + c_1)}{c_1}. \tag{2.3}$$

On the other hand, by Lemma 2.2(a) we know that

$$f(\mu + (n-k)t) - 1 \geq \frac{n-k}{k}(1 - f(\mu - ks))$$

for all $n \in \mathbb{N}$, $t, s \in \mathbb{R}$, $t \geq s$, and $k = 1, \ldots, n$. We take $k_n = \lceil c_1 n/(c_1 + c_2) \rceil$. It is easy to see that $(n - k_n)/k_n \leq c_2/c_1$ and $(n - k_n)/k_n \to c_2/c_1$ as $n \to \infty$. Furthermore, take $s_n = c_1/k_n$ and $t_n = c_2/(n - k_n)$. Then $s_n \leq t_n$ and

$$f(\mu + c_2) - 1 \geq \frac{n - k_n}{k_n}(1 - f(\mu - c_1)).$$

By taking $n \to \infty$, we obtain that (2.3) is violated. Thus, $d_1 \leq d_2$ holds true and $f(x) \geq f_{d_2}(x)$.

(b) Recall that $\liminf_{n\to\infty} A_n(\mu) = \lim_{m\to\infty} \bigcap_{n=m}^{\infty} A_n(\mu)$ and $\limsup_{n\to\infty} A_n(\mu) = \lim_{m\to\infty} \bigcup_{n=m}^{\infty} A_n(\mu)$. It is obvious that

$$A(\mu) \subset \liminf_{n\to\infty} A_n(\mu) \subset \limsup_{n\to\infty} A_n(\mu).$$

We use the same argument as in (a) for any $f \in A_k(\mu)$ for some $k \in \mathbb{N}$. Assume that $d_1 > d_2$. Note that, for all $\varepsilon > 0$, there exist $N \in \mathbb{N}$ such that, for all $n > N$, $(n - k_n)/k_n \geq c_2/c_1 - \varepsilon$. We rewrite (2.3) as

$$f(\mu + c_2) - 1 = \frac{c_2}{c_1}(1 - f(\mu + c_1)) - \delta, \qquad \delta > 0. \tag{2.4}$$

Thus, by taking ε which violates (2.4), we find that if $d_1 > d_2$ for f then $f \notin A_n(\mu)$ for all $n > N$. This implies that

$$\liminf_{n\to\infty} A_n(\mu) \subset \limsup_{n\to\infty} A_n(\mu) \subset A(\mu).$$

Finally, we conclude that $A(\mu) = \liminf_{n\to\infty} A_n(\mu) = \limsup_{n\to\infty} A_n(\mu)$; thus, $A(\mu) = \lim_{n\to\infty} A_n(\mu)$.

Remark 2.1. A similar asymptotic result for the limit of $B_n(\mu)$ is not available using a similar method, owing to the fact that the elements in $B_n(\mu)$ are less regulated than in $A_n(\mu)$.

3. Asymptotic bounds on the distribution function of the sum

Motivated by the analysis on $A_n(\mu)$, we first provide a new result on the bound for $m_{n,F}(n\mu)$, where μ is the mean of F, which implies that $m_{n,F}(n\mu) \to 0$ as $n \to \infty$ under a weak condition on F. Then we extend the result to $m_{n,F}(s)$ for any $s \in \mathbb{R}$. Finally, we will present applications of our results to risk management. All the distributions F discussed in this section are continuous since we will always assume a bounded density.

3.1. Asymptotic result of $m_{n,F}(n\mu)$, where μ is the mean of F

In Section 2.2 we found that $\lim_{n\to\infty} A_n(\mu) = A(\mu) = \{f : \mathbb{R} \to \mathbb{R},\ f \geq f_a \text{ for some } a \geq 0\}$. One may immediately note that $\int f_a \,\mathrm{d}F \geq \int 1 + a(x-\mu)\,\mathrm{d}F = 1$ for all $a \geq 0$. This suggests, not directly implies, the possibility that, when n is large, $m_{n,F}(n\mu) = 1 - \inf\{\int f \,\mathrm{d}F : f \in A_n(\mu)\}$ may be close to 0 since the set $A_n(\mu)$ contains mostly functions greater than f_a for some a. This motivates us to use the duality to investigate the asymptotic behavior of $m_{n,F}(n\mu)$. Before providing the main result, we first present a lemma.

Lemma 3.1. *Define* $k_n(x, y) = \lceil xn/(x + y)\rceil$ *for* $x, y \in \mathbb{R}$.

(a) *For any* $f \in A_n(\mu)$ *and* $a \geq 0$, *we have*

$$f(x) - f_a(x) \geq a(\mu - x) - \frac{k_n(\mu - x, c)}{n - k_n(\mu - x, c)}(f(\mu + c) - 1)$$

for any $x < \mu$ *and* $c \geq 0$. *Here by convention we use* $1/0 = +\infty$.

(b) *Let* $a = \inf\{(f(\mu + c) - 1)/c : p \leq c \leq q\}$. *Then* $f(x) - f_a(x) \geq 0$ *for any* $x \in [\mu + p, \mu + q]$.

Proof. We prove only part (a) as part (b) is trivial. By Lemma 2.2(a) we know that

$$f(\mu + (n - k)t) - 1 \geq \frac{n - k}{k}(1 - f(\mu - ks))$$

for all $n \in \mathbb{N}$, $t, s \in \mathbb{R}$, $t \geq s$, and $k = 1, \dots, n$. For any $x < \mu$ and $c \geq p$, it is easy to see that $(n - k_n(\mu - x, c))/k_n(\mu - x, c) \leq c/(\mu - x)$. Take $s = (\mu - x)/k_n(\mu - x, c)$ and $t = c/(n - k_n(\mu - x, c))$. Then $s \leq t$ and

$$f(\mu + c) - 1 \geq \frac{n - k_n(\mu - x, c)}{k_n(\mu - x, c)}(1 - f(x)).$$

Hence (by setting $1/0 = +\infty$ when $k_n = n$),

$$f(x) \geq 1 - \frac{k_n(\mu - x, c)}{n - k_n(\mu - x, c)}(f(\mu + c) - 1).$$

Finally,

$$f(x) - f_a(x) \geq a(\mu - x) - \frac{k_n(\mu - x, c)}{n - k_n(\mu - x, c)}(f(\mu + c) - 1).$$

Theorem 3.1. *Let* F *be a distribution on* $[0, 1]$ *with mean* μ *and a bounded density* $F' \leq m_0$. *Then* $m_{n,F}(n\mu) \leq 2n^{-1/3}m_0$ *for* $n \geq 3^3$.

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Proof. First, without loss of generality, we assume that $\mu = \frac{1}{2}$. We will comment on the case $\mu \neq \frac{1}{2}$ at the end of the proof. To avoid displaying too many fractions in equations, we still use the notation μ for $\frac{1}{2}$.

It is obvious that, when $n \geq 3$, $p := n^{-2/3} < \mu$. Take any $g \in A_n(\mu)$, and let $f = g \wedge n$. Then $f \in A_n(\mu)$ by Proposition 2.2(d). We will show that $\int f \,\mathrm{d}F \geq 1 - n^{-1/3} m_0$.

We assume that $a := \inf\{(f(\mu+c)-1)/c\colon p \leq c \leq \mu\}$ is attained at a point $c_0 \in [p, \mu]$ such that $a = (f(\mu + c_0) - 1)/c_0$. By definition, It is obvious that $0 \leq a \leq (n-1)/\mu = 2(n-1)$. The case when this infimum is not attained is similar and will be explained later.

Next we calculate $\int (f - f_a)\,\mathrm{d}F$. Note that $f_a(x) = 0$ for $x \leq \mu - 1/a$. By Lemma 3.1(b) we have $f(x) - f_a(x) \geq 0$ for $x \in [\mu + p, 1]$. We first consider the case $a < 1/p$. We can write

$$\int_0^1 (f - f_a)\,\mathrm{d}F \geq \int_{0\vee(\mu-1/a)}^{\mu-p} (f - f_a)\,\mathrm{d}F + \int_{\mu-p}^{\mu+p} (f - f_a)\,\mathrm{d}F. \tag{3.1}$$

By taking $c = c_0$ in Lemma 3.1(a) we have

$$\int_{0\vee(\mu-1/a)}^{\mu-p} (f - f_a)\,\mathrm{d}F \geq \int_{0\vee(\mu-1/a)}^{\mu-p} a(\mu - x)\left(1 - \frac{k_n(\mu - x, c_0)}{n - k_n(\mu - x, c_0)}\frac{c_0}{\mu - x}\right)\mathrm{d}F. \tag{3.2}$$

Note that in the integral of (3.2), $\mu - x \in [p, \mu]$ and $c_0 \in [p, 1-\mu]$. Let $b = (\mu - x)/(\mu - x + c_0)$. Then $p/(1/2 + p) = p/(1 - \mu + p) \leq b \leq \mu/(\mu + p) = 1/2/(1/2 + p)$ and, hence, $b(1-b) \geq p/2(1/2+p)^2$. It is easy to see that

$$\begin{aligned}
\frac{k_n(\mu - x, c_0)}{n - k_n(\mu - x, c_0)}\frac{c_0}{\mu - x} &\leq \frac{bn + 1}{(1-b)n - 1}\frac{1-b}{b} \\
&= 1 + \frac{1}{b(1-b)n - b} \\
&\leq 1 + \frac{2(1/2 + p)^2}{pn - (1/2 + p)}.
\end{aligned}$$

Also, note that, since the mean of F is $\frac{1}{2}$ and F is supported in [0, 1], we have

$$\frac{1}{2} = \int x\,\mathrm{d}F \leq \left(1 - F\left(\frac{1}{2} - p\right)\right) + \left(\frac{1}{2} - p\right)F\left(\frac{1}{2} - p\right).$$

Therefore, $F(\frac{1}{2} - p) \leq 1/(1 + 2p)$. By (3.2) we have

$$\begin{aligned}
\int_{0\vee(\mu-1/a)}^{\mu-p} (f - f_a)\,\mathrm{d}F &\geq \int_{0\vee(\mu-1/a)}^{\mu-p} a(\mu - x)\left(-\frac{2(1/2+p)^2}{pn - (1/2+p)}\right)\mathrm{d}F \\
&\geq -a\left(\mu - \mu + \frac{1}{a}\right)\frac{2(1/2+p)^2}{pn - (1/2+p)}F\left(\frac{1}{2} - p\right) \\
&\geq -\frac{1/2 + p}{pn - (1/2+p)} \\
&= \frac{1}{n^{1/3}}\frac{1 + 2n^{-2/3}}{2 - n^{-1/3} - 2n^{-1}}.
\end{aligned}$$

Some straightforward algebra shows that

$$\frac{1 + 2n^{-2/3}}{2 - n^{-1/3} - 2n^{-1}} \leq \frac{2}{3}$$

for $n \geq 3^3$. In the following we also assume that $n \geq 3^3$. Thus,

$$\int_{0\vee(\mu-1/a)}^{\mu-p} (f - f_a)\,\mathrm{d}F \geq -\frac{2}{3}n^{-1/3}. \tag{3.3}$$

On the other hand, since $f(x) \geq 0$ for $x < \mu$ and $f(x) \geq 1$ for $x \geq \mu$, we have

$$\begin{aligned}\int_{\mu-p}^{\mu+p} (f - f_a)\,\mathrm{d}F &\geq -\int_{\mu-p}^{\mu} f_a\,\mathrm{d}F + \int_{\mu}^{\mu+p} (1 - f_a)\,\mathrm{d}F \\ &\geq -m_0\bigg(\int_{\mu-p}^{\mu} (1 + a(x-\mu))\,\mathrm{d}x + \int_{\mu}^{\mu+p} a(x-\mu)\,\mathrm{d}x\bigg) \\ &= -m_0 p \\ &= -n^{-2/3}m_0. \end{aligned} \tag{3.4}$$

Finally, by (3.1), (3.3), and (3.4), we conclude that

$$\int_0^1 (f - f_a)\,\mathrm{d}F \geq -n^{-1/3}\bigg(\frac{2}{3} + n^{-1/3}m_0\bigg).$$

Also, note that m_0 is the maximum density of a distribution on [0, 1]; hence, $m_0 \geq 1$. Thus,

$$\int_0^1 (f - f_a)\,\mathrm{d}F \geq n^{-1/3}\bigg(\frac{2}{3} + n^{-1/3}m_0\bigg) \geq n^{-1/3}\bigg(\frac{2}{3}m_0 + n^{-1/3}m_0\bigg) \geq -n^{-1/3}m_0. \tag{3.5}$$

Now we consider the case $1/p \leq a \leq 2(n-1)$. In this case, we have

$$\begin{aligned}\int_0^1 (f - f_a)\,\mathrm{d}F &\geq \int_{\mu-1/a}^{\mu+p} (f - f_a)\,\mathrm{d}F \\ &= -\int_{\mu-1/a}^{\mu} f_a\,\mathrm{d}F + \int_{\mu}^{\mu+p} (1 - f_a)\,\mathrm{d}F \\ &\geq -m_0\bigg(\int_{\mu-1/a}^{\mu} (1 + a(x-\mu))\,\mathrm{d}x + \int_{\mu}^{\mu+p} a(x-\mu)\,\mathrm{d}x\bigg) \\ &= -m_0\bigg(\frac{1}{2a} + \frac{ap^2}{2}\bigg) \\ &\geq -n^{-1/3}m_0. \end{aligned} \tag{3.6}$$

Combining (3.5) and (3.6), we have

$$\int_0^1 (f - f_a)\,\mathrm{d}F \geq -n^{-1/3}m_0$$

for both a and $n \geq 3^3$.

We can easily verify that $\int f_a\,\mathrm{d}F \geq \int (1 + a(x-\mu))\,\mathrm{d}F = 1$. Thus,

$$\int_0^1 f\,\mathrm{d}F \geq 1 - n^{-1/3}m_0.$$

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Now we comment on the case when $a = \inf\{(f(\mu+c)-1)/c\colon p \le c \le \mu\}$ is not attained at any point $c_0 \in [p, \mu]$. In this case, for each $\delta > 0$, there exist $0 < \varepsilon < \delta$ such that we can find $c_\varepsilon \in [p, \mu]$ where $(f(\mu+c_\varepsilon)-1)/c_\varepsilon = a+\varepsilon$. Every argument in the above proof is still true if a is replaced by $a+\varepsilon$ and c_0 is replaced by c_ε, except for $f \ge f_{a+\varepsilon}$ no longer holds for $x \in [u+p, 1]$ (Lemma 3.1(b) is not satisfied). Thus, using the same argument, we have

$$\int_0^1 (f - f_{a+\varepsilon})\,\mathrm{d}F \ge -n^{-1/3}m_0 - \int_{u+p}^1 (f_{a+\varepsilon} - f)\,\mathrm{d}F.$$

Note that $f_{a+\varepsilon} - f \le f_{a+\varepsilon} - f_a$ since $f \ge f_a$ for $x \in [u+p, 1]$; thus,

$$\int_{u+p}^1 (f_{a+\varepsilon} - f)\,\mathrm{d}F \le \int_{u+p}^1 (f_{a+\varepsilon} - f_a)\,\mathrm{d}F = \varepsilon \int_p^{1-\mu} x\,\mathrm{d}F \le \delta.$$

It follows that

$$\int_0^1 f\,\mathrm{d}F \ge 1 - n^{-1/3}m_0 - \delta.$$

Since $\delta > 0$ is arbitrary, we have $\int_0^1 f\,\mathrm{d}F \ge 1 - n^{-1/3}m_0$.

In summary, for any $g \in A_n(\mu)$ and $f = g \wedge n$, we have $\int_0^1 f\,\mathrm{d}F \ge 1 - n^{-1/3}m_0$ and, therefore, $\int_0^1 g\,\mathrm{d}F \ge 1 - n^{-1/3}m_0$ since $g \ge f$. As g is chosen arbitrarily, we conclude that

$$\inf\left\{\int g\,\mathrm{d}F\colon g \in A_n(\mu)\right\} \ge 1 - n^{-1/3}m_0.$$

Equivalently, $m_{n,F}(n\mu) \le n^{-1/3}m_0$.

Finally, we consider the general case $\mu \ne \frac{1}{2}$. If $\mu > \frac{1}{2}$, let $X \sim F$ and G be the distribution of $X/2\mu$. Note that G has mean $\frac{1}{2}$ and it is easy to see $m_{n,G}(n/2) = m_{n,F}(n\mu)$. The maximum density of G is $2\mu m_0 \le 2m_0$. The case $\mu < \frac{1}{2}$ is similar. Thus, for any distribution F with maximum density m_0, we can conclude that $m_{n,F}(n\mu) \le 2n^{-1/3}m_0$.

Remark 3.1. Our result is only meaningful when n is large. Note that only when $n \ge (2m_0)^3 \ge 2^3$ is our bound less than 1, so it is reasonable to assume that $n \ge 3^3$. In this paper we are more interested in the asymptotic results; hence, the case for small n is not our focus. Also, from the proof, we can see that the bound can be improved to $m_{n,F}(n\mu) \le \max\{2\mu, 2(1-\mu)\}n^{-1/3}m_0$.

We conclude this section with the following immediate corollary.

Corollary 3.1. *Let F be a distribution on $[a, b]$ with mean μ and a bounded density $F' \le m_0$. Then $m_{n,F}(ns) \le 2n^{-1/3}(b-a)m_0$ for $n \ge 3^3$ and all $s \le \mu$. In particular, we have $m_{n,F}(ns) \to 0$ as $n \to \infty$ for all F supported in a finite interval with mean μ and a bounded density, and $s \le \mu$.*

3.2. Asymptotic result of $m_{n,F}(ns)$, $s \in \mathbb{R}$

We will use the results obtained in Section 3.1 to give an upper bound on $m_{n,F}(ns)$ for any $s \in \mathbb{R}$. Here we use the notation ns for any real number instead of s to allow asymptotic analysis. Note that the existing results in the literature usually concern lower bounds on $m_{n,F}(ns)$; see, for example, Embrechts and Puccetti (2006) and Wang *et al.* (2013). A lower bound on $m_{n,F}(ns)$ can be obtained by taking the supremum of $1 - \int f\,\mathrm{d}F$ over a collection of candidate functions $f \in A_n(s)$, such as $f_a \wedge n$ used in Embrechts and Puccetti (2006). An upper bound on $m_{n,F}(ns)$, on the other hand, is more challenging to obtain. It also gives approximations for $m_{n,F}(ns)$

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since lower bounds on $m_{n,F}(ns)$ are well documented. In this paper we give an upper bound on $m_{n,F}(ns)$ for a continuous distribution F with a finite mean. The case in which $F(s) = 0$ or $F(s) = 1$ is trivial, so we only consider $0 < F(s) < 1$.

Theorem 3.2. *Suppose that a distribution F has a bounded density $F' \le m_0$ and a finite mean μ, and that $0 < F(s) < 1$. We define $a_0 = \inf\{a \in \mathbb{R} \colon \mathbb{E}[X \mid X \ge a] = s,\ X \sim F\}$ for $s \ge \mu$.*

(a) *We have*

$$m_{n,F}(ns) \le 2n^{-1/3} m_0 (b-a)(F(b) - F(a)) + F(a)$$

for $n \ge 3^3$ and any $a < b$ such that $(1/(F(b) - F(a))) \int_a^b x \, \mathrm{d}F(x) = s$.

(b) *For $s < \mu$, there exists $N \in \mathbb{N}$ such that $m_{n,F}(ns) \le n^{-1/6}$ for any $n \ge N$.*

(c) *For $s \ge \mu$, $m_{n,F}(ns) \le F(a_0) + o(1)$ as $n \to \infty$.*

(d) *For $s \ge \mu$, if F has a finite variance then there exists $N \in \mathbb{N}$ such that $m_{n,F}(ns) \le n^{-1/6} + F(a_0)$ for any $n \ge N$.*

(e) *Suppose that the support of F is in $[c, d]$, $-\infty < c < d < \infty$. Then $m_{n,F}(ns) \le 2n^{-1/3} m_0 (d - c) + F(a_0)$ for $n \ge 3^3$.*

Proof. (a) Let F_1, F_2, and F_3 be the conditional distributions of F on $(-\infty, a)$, $[a, b)$, and $[b, \infty)$, respectively, and let $p_1 = F(a)$, $p_2 = F(b) - F(a)$, and $p_3 = 1 - F(b)$. Note that $F = p_1 F_1 + p_2 F_2 + p_3 F_3$ and that the mean of F_2 is s. Let A, B, and C be disjoint sets with probabilities p_1, p_2, and p_3, respectively. Then

$$\begin{aligned} m_{n,F}(ns) &= \inf\{\mathbb{P}(X_1 + \cdots + X_n < ns) \colon X_i \sim F,\ i = 1, \dots, n\} \\ &\le \inf\{\mathbb{P}(X_1 + \cdots + X_n < ns) \colon X_i = \mathbf{1}_A X_{i,1} + \mathbf{1}_B X_{i,2} + \mathbf{1}_C X_{i,3}, \\ &\qquad X_{i,j} \sim F_j,\ i = 1, \dots, n,\ j = 1, 2, 3\} \\ &= \sum_{j=1}^{3} p_j \inf\{\mathbb{P}(X_{1,j} + \cdots + X_{n,j} < ns) \colon X_{i,j} \sim F_j,\ i = 1, \dots, n\}. \end{aligned}$$

Since $a < s < b$, we have

$$\begin{aligned} m_{n,F}(ns) &\le \sum_{j=1}^{3} p_j \inf\{\mathbb{P}(X_1 + \cdots + X_n < ns) \colon X_i \sim F_j,\ i = 1, \dots, n\} \\ &= p_1 + p_2 \inf\{\mathbb{P}(X_1 + \cdots + X_n < ns) \colon X_i \sim F_j,\ i = 1, \dots, n\} \\ &\le F(a) + (F(b) - F(a)) 2n^{-1/3} m_0 (b - a). \end{aligned} \tag{3.7}$$

This completes the proof of (a).

(b) Suppose that $s < \mu$. We take $a_n = s - n^{1/6}/3m_0$ and b_n such that $(1/(F(b_n) - F(a_n))) \int_{a_n}^{b_n} x \, \mathrm{d}F(x) = s$. Such b_n is always possible since $a_n < s < \mu$. It is easy to see that $b_n \le b_0$, where $s \le b_0 < \infty$ is such that $(1/F(b_0)) \int_{-\infty}^{b_0} x \, \mathrm{d}F(x) = s$. We can see that (3.7) becomes

$$m_{n,F}(ns) \le F(a_n) + F(b_n) 2n^{-1/3} m_0 \left(b_0 - s + \frac{1}{3m_0} n^{1/6} \right) \le F(a_n) + n^{-1/6}$$

for large n. We also noted that $F(a_n)|a_n| \to 0$ since F has a finite mean. Thus, $F(a_n) = o(n^{-1/6})$ and $m_{n,F}(ns) \le n^{-1/6}$ for large n.

(c) Suppose that $s > \mu$. We take $b_n = s + n^{1/6}/3m_0$ and a_n such that $(1/(F(b_n) - F(a_n))) \int_{a_n}^{b_n} x \, \mathrm{d}F(x) = s$. It is easy to see that $a_n \ge a_0$, where $-\infty < a_0 < s$ is such that $(1/(1 - F(a_0))) \int_{a_0}^{\infty} x \, \mathrm{d}F(x) = s$. We can see that (3.7) becomes

$$m_{n,F}(ns) \le F(a_n) + F(b_n) 2n^{-1/3} m_0 \biggl(s + \frac{1}{3m_0} n^{1/6} - a_0 \biggr) \le F(a_n) + n^{-1/6} \qquad (3.8)$$

for large n. Thus, by noting that $a_n \to a_0$ as $n \to \infty$ and $F(a_n) - F(a_0) \le m_0(a_n - a_0)$, we have $m_{n,F}(ns) \le F(a_0) + o(1)$.

For $m_{n,F}(n\mu)$, write $a_0(s)$ such that $(1/(1 - F(a_0(s)))) \int_{a_0(s)}^{\infty} x \, \mathrm{d}F(x) = s$ for $s > \mu$. We have $m_{n,F}(n\mu) \le m_{n,F}(ns) \le F(a_0(s)) + o(1)$ for $s > \mu$. By taking the limit as $s \to \mu$ and noting that $a_0(s) \to a_0(\mu)$, we find that the result holds for $m_{n,F}(n\mu)$.

(d) Suppose that $s > \mu$. Again, we take $b_n = s + n^{1/6}/3m_0$ and a_n such that $\int_{a_n}^{b_n} x \, \mathrm{d}F(x) = s$. As in part (c), (3.8) holds. We will show that $F(a_n) - F(a_0) = o(1/b_n)$. Note that $\int_{a_0}^{\infty} (s - x) \, \mathrm{d}F(x) = \int_{a_n}^{b_n} (s - x) \, \mathrm{d}F(x)$. This implies that

$$(s - a_n)(F(a_n) - F(a_0)) \le \int_{a_0}^{a_n} (s - x) \, \mathrm{d}F(x) = \int_{b_n}^{\infty} (x - s) \, \mathrm{d}F(x). \qquad (3.9)$$

Note that F has a finite variance; hence, $\int_{b_n}^{\infty} (x - s) \, \mathrm{d}F(x) = o(1/b_n)$. Since $s - a_n \to s - a_0 > 0$, it follows from (3.9) that $F(a_n) - F(a_0) = o(1/b_n) = o(n^{-1/6})$. By (3.8) we have $m_{n,F}(ns) \le F(a_0) + n^{-1/6}$. The case in which $s = \mu$ is similar to part (c).

(e) The proof of this part can be directly obtained from (3.7) by letting $a = c$ and $b = b_0$ in part (b) for $s \le \mu$, and $a = a_0$ and $b = d$ for $s > \mu$.

Remark 3.2. We may directly use Lemma 3.1 for $\mu = s$ and apply the proof of Theorem 3.1 to obtain the same asymptotic result for $m_{n,F}(ns)$. That is, to show that $\int (f - f_a) \, \mathrm{d}F \to 0$ for all $f \in A_n(s)$, where $f_a = (1 + a(x - s))_+$ as in Section 2.2 with μ replaced by s. The two methods are equivalent.

Remark 3.3. Our assumption on the distribution F is very weak. Note that our asymptotic results do not require F to have a bounded support. For $s < \mu$, we only need F to have a finite mean and a bounded density. For $s \ge \mu$, we also need F to have a variance to obtain a convergence rate of $n^{-1/6}$. The asymmetry between the two cases is due to the fact that the convergence of $F(a_n) \to F(a)$ and the convergence of $n^{-1/3} b_n \to 0$ are different in nature. Also, note that our bound is only meaningful for large values of n.

Wang *et al.* (2013) obtained $m_{n,F}(ns) \ge F(a_0)$ for $s \ge \mu$ for any distribution F with a finite mean (see Corollary 2.4 in their paper). Hence, the upper bound on $m_{n,F}(ns)$ obtained above and $m_{n,F}(ns)$ converge to the same limit $F(a_0)$ or 0, and, for a distribution F with finite variance, $|m_{n,F}(ns) - F(a_0)| \le n^{-1/6}$ for $s \ge \mu$. We combine this result in the following corollary.

Corollary 3.2. *For any distribution F with finite mean, we have $m_{n,F}(ns) \to F(a_0)$ for all $s \ge \mu$, where $a_0 = \inf\{a \in \mathbb{R} \colon \mathbb{E}[X \mid X \ge a] \ge s,\ X \sim F\}$. Moreover, if F has a finite variance then $F(a_0) \le m_{n,F}(ns) \le F(a_0) + n^{-1/6}$ for large n.*

Remark 3.4. When the support of F is in $\mathbb{R}_+$, we can also combine the upper bound in Corollary 3.2 with the dual bound given in Embrechts and Puccetti (2006). That is, for F with a finite variance, we have

$$1 - \inf_{a \geq 0} \int (f_a \wedge n) \,\mathrm{d}F \leq m_{n,F}(ns) \leq F(a_0) + n^{-1/6}, \tag{3.10}$$

where $f_a = (1 + a(x - s))_+$ as in Section 2.2 with μ replaced by s. It was pointed out in Wang *et al.* (2013) that $F(a_0) \leq 1 - \inf_{a \geq 0} \int (f_a \wedge n) \,\mathrm{d}F$; hence, (3.10) gives a possibly better estimation of $m_{n,F}(ns)$ if F is supported in $\mathbb{R}_+$.

3.3. Applications in risk management

One of the strongest motivations to study the bound function $m_{n,F}(s)$ is to induce the sharp bounds on quantile-based risk measures of the aggregate risk $S = X_1 + \cdots + X_n$, when the marginal distributions of $X_1, \ldots, X_n$ are given but the dependence structure among them is unknown. This is a typical setting of dependence uncertainty in risk management and has been studied extensively in the literature; a history and recent developments on dependence uncertainty can be found in Bernard *et al.* (2013). A widely used risk measure is the so-called VaR at level α, defined as

$$\mathrm{VaR}_\alpha(F) = \inf\{s \in \mathbb{R} \colon F(s) \geq \alpha\} =: F^{-1}(s), \qquad \alpha \in (0, 1).$$

An upper bound on the above VaR, called the worst-case VaR, is defined as

$$\overline{\mathrm{VaR}}_\alpha(n, F) = \sup\{\mathrm{VaR}_\alpha(X_1 + \cdots + X_n) \colon X_i \sim F,\ i = 1, \ldots, n\}.$$

Computing the worst VaR is of great interest in the recent research of quantitative risk management; the reader is referred to Embrechts and Puccetti (2006), Embrechts and Puccetti (2010), Puccetti and Rüschendorf (2013), and Wang *et al.* (2013) for the study of this problem and applications in practice. It is well known that, for a continuous distribution F, $m_{n,F}$ is strictly increasing, invertible, and $\overline{\mathrm{VaR}}_\alpha(n, F) = m_{n,F}^{-1}(\alpha)$; see, for example, Embrechts and Puccetti (2006) and Wang *et al.* (2013). The following corollary states the asymptotic behavior of $\overline{\mathrm{VaR}}_\alpha(n, F)$. The result is, with no surprise, related to the other popular risk measure ES (sometimes also called TVaR), defined as

$$\mathrm{ES}_\alpha(F) = \frac{1}{1 - \alpha} \int_\alpha^1 F^{-1}(p) \,\mathrm{d}p, \qquad \alpha \in [0, 1),$$

for F with a finite mean.

Corollary 3.3. *For F with a finite mean and a bounded density, $\overline{\mathrm{VaR}}_\alpha(n, F)/n \to \mathrm{ES}_\alpha(F)$ as $n \to \infty$ for $\alpha \in (0, 1)$.*

Proof. Note that $\mathrm{ES}_{F(a_0)}(F) = s$ and $\overline{\mathrm{VaR}}_{F(a_0)}(n, F)/n = m_{n,F}^{-1}(F(a_0))/n \to s = \mathrm{ES}_{F(a_0)}(F)$ for any $a_0 \in \mathbb{R}$ by Corollary 3.2 and the asymptotic continuity of $m_{n,F}$.

Remark 3.5. Wang *et al.* (2013) pointed out that $m_{n,F}(ns) \geq F(a_0)$ is equivalent to $\overline{\mathrm{VaR}}_\alpha(n, F) \leq n\mathrm{ES}_\alpha(F)$. This result can also be explained from the risk management perspective. By the coherence of the ES (see Artzner *et al.* (1999)), the worst-case ES is

$$\overline{\mathrm{ES}}_\alpha(n, F) := \sup\{\mathrm{ES}_\alpha(X_1 + \cdots + X_n) \colon X_i \sim F,\ i = 1, \ldots, n\} = n\mathrm{ES}_\alpha(F).$$

By definition, it is clear that $\mathrm{VaR}_\alpha(F) \le \mathrm{ES}_\alpha(F)$ for any distribution F; thus, we have $\overline{\mathrm{VaR}}_\alpha(n, F) \le \overline{\mathrm{ES}}_\alpha(n, F) = n\mathrm{ES}_\alpha(F)$. Corollary 3.3 suggests that, for large n, $\overline{\mathrm{VaR}}$ and $\overline{\mathrm{ES}}$ are asymptotically equivalent. Thus, when n is large, using the worst-case VaR or the worst-case ES for risk regulation will not lead to much difference. From the risk management perspective, this phenomenon was mentioned in Puccetti and Rüschendorf (2014) under a strong mixable assumption on the distribution which requires an equivalence of $m_{n,F}(ns) = \int (f_a \wedge n)\,\mathrm{d}F$ for some $a \ge 0$. This strong assumption was verified only in a few cases, as studied in Puccetti and Rüschendorf (2013) and Wang *et al.* (2013). Our asymptotic result does not require this assumption and, hence, gives a stronger result. In another recent paper Puccetti *et al.* (2013) studied this equivalence using the complete mixability, and obtained the asymptotic equivalence under different conditions, without estimates of the convergence rate. Their result requires a strictly positive and continuous density function of F bounded below on any finite intervals, which, interestingly, is not comparable to our condition of bounded (above) density. Note that this asymptotic equivalence can also be generalized to possible inhomogeneous portfolios with a finite number of choices of different marginal distributions (see Puccetti *et al.* (2013)).

Another interpretation of our result concerns the *superadditivity ratio* of VaR. It is well known that the risk measure VaR is often criticized for not being subadditive, and, hence, it is not coherent. It is then of interest to study the superadditive ratio $\delta_\alpha(n)$, defined as

$$\delta_\alpha(n) = \frac{\overline{\mathrm{VaR}}_\alpha(n, F)}{\mathrm{VaR}^+_\alpha(n, F)},$$

where $\mathrm{VaR}^+_\alpha(n, F) = n\mathrm{VaR}_\alpha(F)$ is called the VaR of comonotonic risks. For a discussion on $\delta_\alpha(n)$ in risk aggregation, we refer the reader to Embrechts *et al.* (2013), in which it was mentioned that numerical evidence suggests that $\delta_\alpha(n)$ converges to a limit quite fast, without theoretical proofs. Our result shows that this limit exists and it can be identified easily.

Corollary 3.4. *For F with a finite mean and a bounded density, and $F^{-1}(\alpha) > 0$,*

$$\delta_\alpha(n) = \frac{\overline{\mathrm{VaR}}_\alpha(n, F)}{\mathrm{VaR}^+_\alpha(n, F)} \to \frac{\mathrm{ES}_\alpha(F)}{\mathrm{VaR}_\alpha(F)} = \frac{1}{1-\alpha} \frac{\int_\alpha^1 F^{-1}(p)\,\mathrm{d}p}{F^{-1}(\alpha)}.$$

4. Dual representation of the complete mixability

In this section we give a dual representation of the recently developing concept of complete mixability, and provide *duality-based* proofs of properties of complete mixability proved in the literature using probability methods.

4.1. Preliminaries on complete mixability

We first give a summary of the existing results on completely mixable distributions which we will use in the remainder of this paper.

Definition 4.1. A distribution function F on $\mathbb{R}$ is called *n-completely mixable* (n-CM) if there exist n random variables $X_1, \dots, X_n$ identically distributed as F such that

$$X_1 + \cdots + X_n = n\mu \tag{4.1}$$

for some $\mu \in \mathbb{R}$. Any such μ is called a *center* of F and any vector $(X_1, \dots, X_n)$ satisfying (4.1) with $X_i \sim F$, $1 \le i \le n$, is called an *n-complete mix*.

It is obvious that if F is n-CM and has finite mean μ, then its center is unique and equal to μ. We denote by $\mathcal{M}_n(\mu)$ the set of all n-CM distributions with center μ, and by $\mathcal{M}_n = \bigcup_{\mu\in\mathbb{R}} \mathcal{M}_n(\mu)$ the set of all n-CM distributions on $\mathbb{R}$.

The following mean condition proposed in Wang and Wang (2011) is important to the CM distributions.

Definition 4.2. (*Mean condition.*) Let F be a distribution with finite mean μ, and let $[a, b]$ be the essential support of F, i.e. $a = \sup\{t \in \mathbb{R}\colon F(t) = 0\}$ and $b = \inf\{t \in \mathbb{R}\colon F(t) = 1\}$. We say that F satisfies the *mean condition* if

$$a + \frac{b-a}{n} \le \mu \le b - \frac{b-a}{n}. \tag{4.2}$$

In the above condition, a and b can be finite or infinite. It turns out that the mean condition is necessary for a CM distribution.

Proposition 4.1. (Wang and Wang (2011).) *Suppose that $F \in \mathcal{M}_n(\mu)$. Then F satisfies the mean condition (4.2).*

Some straightforward examples of CM distributions are given in Wang and Wang (2011). We summarize the existing theoretical results below.

Proposition 4.2. *The following statements hold.*

(a) *F is 1-CM if and only if F is the distribution of a constant.*

(b) *F is 2-CM if and only if F is symmetric, i.e. $X \sim F$ and $a - X \sim F$ for some constant $a \in \mathbb{R}$.*

(c) *Any linear transformation of an n-CM distribution is n-CM.*

(d) *If $F, G \in \mathcal{M}_n(\mu)$ then $\lambda F + (1-\lambda)G \in \mathcal{M}_n(\mu)$ for $\lambda \in [0, 1]$.*

(e) *If $F \in \mathcal{M}_n(\mu) \cup \mathcal{M}_k(\mu)$ for $n, k \in \mathbb{N}$ then $F \in \mathcal{M}_{n+k}(\mu)$.*

(f) *Any continuous distribution function F having a symmetric and unimodal density is n-CM for $n \ge 2$. (See Rüschendorf and Uckelmann (2002).)*

(g) *Suppose that F is a continuous distribution with a monotone density on its support. Then the mean condition (4.2) is sufficient. (See Wang and Wang (2011).)*

(h) *Suppose that F admits a concave density on its support. Then F is n-CM for $n \ge 3$. (See Puccetti et al. (2012).)*

For $n = 1$ or $n = 2$, $\mathcal{M}_n(\mu)$ is fully characterized. However, for $n \ge 3$, the full characterization on $\mathcal{M}_n(\mu)$ is still an open question and has been extremely challenging. In this paper we give a dual representation of complete mixability in the hope of giving another possible research direction to the study of complete mixability.

4.2. Dual representation of complete mixability

In this section we associate the duality to the complete mixability. By definition, we know that, for any distribution F, $F \in \mathcal{M}_n(\mu)$ is equivalent to $w_{n,F}(n\mu) = 0$. Moreover, for any distribution F with mean μ, $F \in \mathcal{M}_n(\mu)$ is equivalent to $m_{n,F}(n\mu) = 0$. This allows us to give two dual representations of complete mixability.

Using Lemma 2.1, we give a dual presentation of n-CM distributions.

Theorem 4.1. (Dual representation of complete mixability.) (a) *A probability distribution F is n-CM with center μ if and only if $\int f \,\mathrm{d}F \geq 1$ for all $f \in B_n(\mu)$.*

(b) *A probability distribution F with finite mean μ is n-CM if and only if $\int f \,\mathrm{d}F \geq 1$ for all $f \in A_n(\mu)$.*

Proof. (a) By the definition of n-CM distributions, $F \in \mathcal{M}_n(\mu)$ is equivalent to $w_{n,F}(n\mu) = 0$. By Lemma 2.1, this is again equivalent to $\inf\{\int f \,\mathrm{d}F \colon f \in B_n(\mu)\} = 1$. Since the function $f(x) = 1$ is always in $B_n(\mu)$, $\inf\{\int f \,\mathrm{d}F \colon f \in B_n(\mu)\} = 1$ is equivalent to $\inf\{\int f \,\mathrm{d}F \colon f \in B_n(\mu)\} \geq 1$.

(b) Suppose that $F \in \mathcal{M}_n(\mu)$. Since $A_n(\mu) \subset B_n(\mu)$, by (a) we have $\int f \,\mathrm{d}F \geq 1$ for all $f \in A_n(\mu)$. Now suppose that $\int f \,\mathrm{d}F \geq 1$ for all $f \in A_n(\mu)$. By Lemma 2.1 we have $m_{n,F}(n\mu) = 0$. Then there exist random variables $X_1, \ldots, X_n \sim F$ such that $\mathbb{P}(X_1 + \cdots + X_n \geq n\mu) = 1$ almost surely. Also, note that $\mathbb{E}[X] = \mu$; thus, $\mathbb{P}(X_1 + \cdots + X_n = n\mu) = 1$ and $F \in \mathcal{M}_n(\mu)$.

Remark 4.1. Although being very similar, Theorem 4.1(a) and (b) can be used in different situations. In general, when we consider the complete mixability of a distribution F with finite mean, the smaller set $A_n(\mu)$ is more convenient to use than the larger set $B_n(\mu)$. However, when the mean of F does not exist, (b) cannot be used. Also, note that, if we replace $[n\mu, \infty)$ in the definition of $A_n(\mu)$ by $(-\infty, n\mu]$, (b) still holds.

Remark 4.2. For a given function f, it is easy to check whether f is in $A_n(\mu)$ or $B_n(\mu)$. However, it is hard to characterize all the functions in $A_n(\mu)$ or $B_n(\mu)$. In general, when a distribution F is given, it is difficult to check whether $\int f \,\mathrm{d}F \geq 1$ for all f in $A_n(\mu)$ or $B_n(\mu)$.

Recall that, for any distribution F with mean μ, $F \in \mathcal{M}_n(\mu)$ is equivalent to $m_{n,F}(n\mu) = 0$. We can define the asymptotic mixability by the condition $m_{n,F}(n\mu) \to 0$ as $n \to \infty$.

Definition 4.3. A distribution F with mean μ is *asymptotically mixable* if $m_{n,F}(n\mu) \to 0$ as $n \to \infty$.

The asymptotic mixability of F states that, for any $\varepsilon > 0$, there exist $n \in \mathbb{N}$ random variables $X_1, \ldots, X_n$ from the distribution F such that $\mathbb{P}(X_1 + \cdots + X_n \geq n\mu) \geq 1 - \varepsilon$. By Corollary 3.2, it immediately follows that all distributions with a bounded density are asymptotically mixable. However, it is left open to answer whether all distributions are asymptotically mixable.

Corollary 4.1. *Any distribution with a bounded density is asymptotically mixable.*

4.3. Dual proofs of CM properties

In this section we give dual proofs of some theorems given in the literature of complete mixability. Some of the results are surprisingly simple to prove using the duality, but nontrivial to prove using probabilistic methods.

Theorem 4.2. (Completeness and convexity.) *Let $n \in \mathbb{N}$ and $\mu \in \mathbb{R}$.*

(a) *The (weak) limit of n-CM distributions with center μ is n-CM with center μ.*

(b) *A (possibly infinite) convex combination of n-CM distributions with center μ is n-CM with center μ.*

Proof. In the following suppose that $F_k \in M_n(\mu)$, $k = 1, 2, \ldots$. Then, for all $f \in B_n(\mu)$, $\int f \,\mathrm{d}F_k \geq 1$ for $k = 1, 2, \ldots$.

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(a) Suppose that $F_k \to F$. We have $\int f \,\mathrm{d}F = \lim_{k\to\infty} \int f \,\mathrm{d}F_k \ge 1$; thus, $F \in M_n(\mu)$.

(b) Suppose that $F = \sum_{k=1}^{\infty} a_k F_k$, where $a_k \ge 0$ and $\sum_{k=1}^{\infty} a_k = 1$. We have $\int f \,\mathrm{d}F = \int f \,\mathrm{d}(\sum_{k=1}^{\infty} a_k F_k) = \sum_{k=1}^{\infty} a_k \int f \,\mathrm{d}F_k \ge 1$; thus, $F \in M_n(\mu)$.

Remark 4.3. The above theorem summarizes the completeness theorems given in Puccetti *et al.* (2012), where a nontrivial probabilistic proof was given.

Proposition 4.3. *Let $n, k \in \mathbb{N}$ and $\mu \in \mathbb{R}$. If $F \in \mathcal{M}_n(\mu) \cup \mathcal{M}_k(\mu)$ then $F \in \mathcal{M}_{n+k}(\mu)$. In particular, $F \in \mathcal{M}_{dn}(\mu)$ for any $d \in \mathbb{N}$.*

Proof. By Proposition 2.3 we know that, for any $f \in B_{n+k}(\mu)$, we have $f \in B_n(\mu) \cup B_k(\mu)$. This implies that $\int f \,\mathrm{d}F \ge 1$ and, hence, $F \in \mathcal{M}_{n+k}(\mu)$.

Remark 4.4. The above proposition was also given as Proposition 2.1 of Wang and Wang (2011).

Very often, CM distributions on finite intervals are of interest. Since the complete mixability is affine invariant, we focus on distributions on [0, 1]. Necessary conditions for complete mixability are given in the following theorem.

Theorem 4.3. (Necessary conditions.) *Suppose that $F \in \mathcal{M}_n(\mu)$ is a probability distribution on* [0, 1]*. Then $F(n\mu/k) \ge (n-k+1)/n$ and $F((n\mu - n + k)/k) \le (k-1)/n$ for all $k = 1, \dots, n$. In particular,*

(a) $1/n \le \mu \le 1 - 1/n$, *given that* [0, 1] *is the essential support of F (see (4.2));*

(b) $1/n \le F(\mu) \le 1 - 1/n$.

Proof. Let $X \sim F$ be a random variable. Take $f = n\,\mathbf{1}_{(-\infty, n\mu/k]}/(n-k+1)$. When $x_1 + \cdots + x_n = n\mu$, since $x_1 + \cdots + x_n \ge x_1 + \cdots + x_k$, we have at most $k-1$ of $\{x_1, \dots, x_n\}$ greater than $n\mu/k$. Thus,

$$\sum_{i=1}^{n} \mathbf{1}_{(-\infty, n\mu/k]}(x_i) \ge n - k + 1;$$

hence, $f(x_1) + \cdots + f(x_n) \ge n$ and $f \in B_n(\mu)$. Then $\int f \,\mathrm{d}F \ge 1$ implies that $F(n\mu/k) \ge (n-k+1)/n$.

Similarly, take $f = n\,\mathbf{1}_{[(n\mu - n + k)/k, \infty)}/(n-k+1)$. When $x_1 + \cdots + x_n = n\mu$, since $x_1 + \cdots + x_n \le x_1 + \cdots + x_k + (n-k)$, we have at most $k-1$ of $\{x_1, \dots, x_n\}$ smaller than $(n\mu - n + k)/k$. Thus, $f(x_1) + \cdots + f(x_n) \ge n$, and $f \in B_n(\mu)$. Then $\int f \,\mathrm{d}F \ge 1$ implies that $1 - F((n\mu - n + k)/k) \ge (n-k+1)/n$; thus, $F((n\mu - n + k)/k) \le (k-1)/n$.

(a) Take $k = 1$. We have $F(n\mu) = 1$ and $F(n\mu - n + 1) = 0$. Then, if [0, 1] is the essential support of F, $1 \le n\mu \le n - 1$.

(b) Take $k = n$. We have $F(\mu) \ge 1/n$ and $F(\mu) \le (n-1)/n$.

Remark 4.5. The necessary conditions given in Theorem 4.3 can also be obtained using probabilistic methods. Theorem 4.3(a) is the mean condition (4.2) first given in Wang and Wang (2011). In the appendix of Puccetti *et al.* (2013), a probabilistic proof of these necessary conditions was given.

Theorem 4.4. (Unimodal and symmetric distributions.) *Any distribution with a unimodal and symmetric density is n-CM for $n \ge 2$.*

Proof. We first prove that a uniform distribution U on $[0, 1]$ is n-CM for $n \geq 2$ using the duality. For any $f \in A_n(\frac{1}{2})$, write

$$\int f \,\mathrm{d}U = \lim_{m\to\infty} \frac{1}{nm} \sum_{i=1}^{nm} f\left(\frac{i}{nm}\right) = \lim_{m\to\infty} \frac{1}{nm} \sum_{i=1}^{nm} f\left(\frac{i+1}{nm}\right).$$

It is easy to see that the numbers in the last summation (from 2 to $nm+1$) can be divided into m subgroups, such that there are n numbers with a sum at least $(1+nm)nm/2$ in each subgroup. Thus, since $f \in A_n(\frac{1}{2})$, we have $\sum_{i=1}^{nm} f((i+1)/nm) \geq nm$. Therefore, $\int f \,\mathrm{d}U \geq 1$ and U is n-CM for $n \geq 2$. Now, suppose that F is a distribution with a unimodal and symmetric density. It is obvious that F can be written as the limit of a convex combination of uniform distributions with the same mean as F, and, hence, by Theorem 4.2, F is n-CM for $n \geq 2$.

Remark 4.6. The above theorem summarizes the main result of Rüschendorf and Uckelmann (2002). We note that, for the other existing results, such as the main theorems in Wang and Wang (2011) and Puccetti *et al.* (2012) based on combinatorial techniques, a dual proof is not easy to find.

5. Conclusion

In this paper we studied the duality for the bounds on the distribution of aggregate risk with uncertainty of dependence, $m_{n,F}(s) = \inf\{\mathbb{P}(\psi(\boldsymbol{X}) < s)\colon X_i \sim F,\ i = 1, \dots, n\}$. It was proved for any continuous distribution F with a bounded density that

$$m_{n,F}(ns) \to F(a_0)$$

as $n \to \infty$, where $a_0 = \inf\{a \in \mathbb{R}\colon \mathbb{E}[X \mid X \geq a] \geq s,\ X \sim F\}$. We provided an upper bound on $m_{n,F}(ns)$ which turns out to converge to the real value of $m_{n,F}(ns)$ with a controlled convergence rate. An application of our result to risk management directly indicates that the worst-case value at risk is asymptotically equivalent to the worst-case expected shortfall with dependence uncertainty, and gives the asymptotic superadditivity ratio of value at risk. We also provided a dual representation of the complete mixability and proved existing theoretical results using the dual representation, which enriches the mathematical tools for the theory of complete mixability.

There are also many open questions in the related study. For the asymptotic bounds, it would be natural (and challenging) to generalize the bounds to inhomogeneous marginal distributions. Also, exact values (or more accurate bounds) of $m_{n,F}(ns)$ might be found through further study of the admissible sets $A_n(s)$. Although the rate of $n^{-1/3}$ is sufficient for the convergence in our asymptotic results, the rate might still be improved for more practical applications. For the dual representation of complete mixability, one research direction is to generate new classes of completely mixable distributions from the duality. Also, note that the question about the uniqueness of the center of complete mixability has been asked since complete mixability was first introduced, but has not yet been answered. The admissible sets $B_n(\mu)$ may help to study the uniqueness. That is, is there a distribution F with infinite mean such that $\int f \,\mathrm{d}F \geq 1$ for all $f \in B_n(\mu) \cup B_n(\nu)$, where $\mu \neq \nu$?

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