

ATOMIC DECOMPOSITION OF WEIGHTED TRIEBEL–LIZORKIN SPACES ON SPACES OF HOMOGENEOUS TYPE

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(Received 14 June 2009; accepted 14 September 2010)

Communicated by A. J. Pryde

Abstract

We obtain an atomic decomposition for weighted Triebel–Lizorkin spaces on spaces of homogeneous type, using the area function, the discrete Calderón reproducing formula and discrete sequence spaces.

2010 *Mathematics subject classification*: primary 42B35; secondary 42B20, 46E.

Keywords and phrases: weighted Triebel–Lizorkin spaces, atomic decomposition, Calderón reproducing formula, spaces of homogeneous type.

1. Introduction and statement of main result

Spaces of homogeneous type were introduced by Coifman and Weiss [3] in the 1970s in order to extend the Calderón–Zygmund theory of singular integrals to a more general setting. These spaces have no translations or dilations, no analogues of the Fourier transform or convolution, and no group structure. Examples of spaces of homogeneous type include Euclidean space, the n -torus, smooth compact Riemannian manifolds, the boundaries of Lipschitz domains, and d -sets in \mathbb{R}^n .

Recently, based on the works of Christ [1] and David *et al.* [4], Deng and Han [5] developed a version of harmonic analysis on spaces of homogeneous type, using discrete Littlewood–Paley–Stein analysis. They first defined test function spaces and the so-called distribution spaces on spaces of homogeneous type, and then proved a new ‘ $T1$ -theorem’, namely, the boundedness on a test function space of a certain class of Calderón–Zygmund operators whose kernels satisfy an additional second-order smoothness condition. Next, they established several discrete Calderón reproducing formulas on spaces of homogeneous type. As an application of these results, they

The author is supported by the Young Faculty Career Start Program (Grant No. 34000-3126172) and by NNSF of China (Grant No. 11001275).

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studied function spaces on spaces of homogeneous type, including L^p , where $1 < p < \infty$, the generalized Sobolev spaces $\dot{L}^{p,s}$, the Hardy spaces H^p , the bounded mean oscillation space and the Besov spaces. Theorems of $T1$ type on these spaces are also proved. Han [10, 11] studied Triebel–Lizorkin spaces on spaces of homogeneous type, using discrete Littlewood–Paley–Stein analysis. See also [16, 18, 22].

We recall the definition of Muckenhoupt weights. A weight w belongs to A_q , where $1 < q < \infty$, if there is a constant C_q such that

$$\sup_{B \subset X} \left(\frac{1}{\mu(B)} \int_B w(x) d\mu(x) \right) \left(\frac{1}{\mu(B)} \int_B w(x)^{-1/(q-1)} d\mu(x) \right)^{q-1} \leq C_q. \quad (1.1)$$

The class A_1 is defined by letting q tend to 1, that is,

$$\sup_{B \subset X} \left(\frac{1}{\mu(B)} \int_B w(x) d\mu(x) \right) \|w^{-1}\|_{L^\infty(B)} \leq C_1,$$

where C_1 depends only on w . These classes, in the Euclidean setting, were introduced by Muckenhoupt [20] and developed by Coifman and Fefferman [2]; see also García-Cuerva and Rubio de Francia [9]. For more information on Muckenhoupt weights, we refer the reader to Stein [21].

In this paper, we consider weights that belong to the class A_∞ , which is the union of the classes A_q when $1 \leq q < \infty$; we define the critical index q_w of w by

$$q_w = \inf\{q > 1 : w \in A_q\}.$$

García-Cuerva and Martell [8] found the wavelet characterization of the weighted Hardy space $H_w^p(\mathbb{R})$. Deng *et al.* [6] gave an atomic characterization of the weighted Triebel–Lizorkin spaces $\dot{F}_{p,w}^{\alpha,q}(\mathbb{R}^n)$, using an idea of [8], namely, of combining wavelet analysis and the theory of vector-valued Calderón–Zygmund operators. See [6, Theorem 1].

The aim of this paper is to study weighted Triebel–Lizorkin spaces on spaces of homogeneous type and present an atomic decomposition of functions in these spaces. The main tools used are the area function and the discrete Calderón reproducing formula. We also study the duality between these weighted Triebel–Lizorkin spaces by considering weighted sequence spaces, and the lifting and projection operators on the weighted Triebel–Lizorkin spaces. Sequence spaces originated in the work of Frazier and Jawerth [7], and were developed by Deng and Han [5] in the single-parameter case and by Han, Lu and the author [13, 14] in the multi-parameter cases.

We now begin by recalling the definitions necessary for introducing weighted Triebel–Lizorkin spaces on spaces of homogeneous type. A quasi-metric d on a set X is a function $d : X \times X \rightarrow [0, \infty)$ satisfying the conditions:

- (i) $d(x, y) = 0$ if and only if $x = y$;
- (ii) $d(x, y) = d(y, x)$ for all $x, y \in X$;
- (iii) there exists a constant $A \in [1, \infty)$ such that $d(x, y) \leq A[d(x, z) + d(z, y)]$ for all $x, y, z \in X$.

Any quasi-metric defines a topology, whose balls $B(x, r) = \{y \in X : d(y, x) < r\}$ form a base. But the balls need not be open if $A > 1$.

We now state the definition of a space of homogeneous type.

DEFINITION 1.1 [3]. A space of homogeneous type (X, d, μ) is a set X with a quasi-metric d and a nonnegative Borel regular measure μ on X such that $0 < \mu(B(x, r)) < \infty$ (for all $x \in X$ and all $r > 0$) and there exists a positive constant A' such that

$$\mu(B(x, 2r)) \leq A' \mu(B(x, r))$$

for all $x \in X$ and $r > 0$. Here μ is assumed to be defined on a σ -algebra which contains all Borel sets and all balls $B(x, r)$.

We suppose that $\mu(X) = \infty$ and $\mu(\{x\}) = 0$ for all $x \in X$. It was shown by Macías and Segovia [19, Theorems 2 and 3] that, in this case, $\rho'(x, y)$, the infimum of the measures $\mu(B)$ where B runs over the balls containing x and y , is a quasi-metric on X yielding the same topology as d . Further, there is a quasi-metric ρ equivalent to ρ' in the sense that there is a constant C such that $C^{-1}\rho'(x, y) \leq \rho(x, y) \leq C\rho'(x, y)$ for all $x, y \in X$, and there are constants C and $0 < \theta < 1$ such that

$$C^{-1}r \leq \mu(B(x, r)) \leq Cr$$

for all $x \in X$ and $r > 0$, and

$$|\rho(x, y) - \rho(x', y)| \leq C\rho(x, x')^\theta [\rho(x, y) + \rho(x', y)]^{1-\theta}$$

for all $x, x', y \in X$. For more on spaces of homogeneous type, see [5, Section 1.2].

We now recall the notion of an approximation to the identity on X .

DEFINITION 1.2 [5]. A sequence $\{S_k\}_{k \in \mathbb{Z}}$ of operators is said to be an approximation to the identity of order $\epsilon \in (0, \theta]$ if there exists a constant $C > 0$ such that for all $k \in \mathbb{Z}$ and all $x, x', y, y' \in X$, the kernel $S_k(\cdot, \cdot)$ of the operator S_k is a function from $X \times X$ to \mathbb{C} satisfying

$$|S_k(x, y)| \leq C \frac{2^{-k\epsilon}}{(2^{-k} + \rho(x, y))^{1+\epsilon}}; \quad (1.2)$$

$$|S_k(x, y) - S_k(x', y)| \leq C \left(\frac{\rho(x, x')}{2^{-k} + \rho(x, y)} \right)^\epsilon \frac{2^{-k\epsilon}}{(2^{-k} + \rho(x, y))^{1+\epsilon}} \quad (1.3)$$

when $\rho(x, x') \leq (2A)^{-1}(2^{-k} + \rho(x, y))$;

$$|S_k(x, y) - S_k(x, y')| \leq C \left(\frac{\rho(y, y')}{2^{-k} + \rho(x, y)} \right)^\epsilon \frac{2^{-k\epsilon}}{(2^{-k} + \rho(x, y))^{1+\epsilon}} \quad (1.4)$$

when $\rho(y, y') \leq (2A)^{-1}(2^{-k} + \rho(x, y))$;

$$\begin{aligned} & |S_k(x, y) - S_k(x, y') - S_k(x', y) + S_k(x', y')| \\ & \leq C \left(\frac{\rho(x, x')}{2^{-k} + \rho(x, y)} \right)^\epsilon \left(\frac{\rho(y, y')}{2^{-k} + \rho(x, y)} \right)^\epsilon \frac{2^{-k\epsilon}}{(2^{-k} + \rho(x, y))^{1+\epsilon}} \end{aligned} \quad (1.5)$$

when $\rho(x, x'), \rho(y, y') \leq (2A)^{-1}(2^{-k} + \rho(x, y))$;

$$\int_X S_k(x, y) d\mu(y) = \int_X S_k(x, y) d\mu(x) = 1. \tag{1.6}$$

REMARK 1.3. An approximation to the identity was constructed by Coifman as follows. Take a nonnegative smooth function h equal to 1 on $[1, 9]$ and supported in $[0, 10]$. Let H_k be the operator with kernel $2^k h(2^k \rho(\cdot, \cdot))$. The doubling condition on μ and the construction of h imply that there is a constant $C \geq 1$ such that $C^{-1} \leq H_k(1) \leq C$ for all $k \in \mathbb{Z}$. Let M_k and W_k be the operators of pointwise multiplication by $(H_k(1))^{-1}$ and $w_k = (H_k(H_k(1))^{-1})^{-1}$ respectively, and finally let $\tilde{S}_k = M_k H_k W_k H_k M_k$. It is easy to check that this \tilde{S}_k satisfies conditions (i)–(iv) of Definition 1.2. Examples of approximations to the identity on Euclidean space include the Gaussian kernel and the Poisson kernel.

DEFINITION 1.4 [5]. Fix $\gamma > 0$ and $0 < \beta \leq \epsilon$. We say that a function f defined on X belongs to $\mathcal{M}(x_0, r, \beta, \gamma)$ if it satisfies the following conditions:

- (i) $|f(x)| \leq C \frac{r^\gamma}{(r + \rho(x, x_0))^{1+\gamma}}$;
- (ii) $|f(x) - f(y)| \leq C \left(\frac{\rho(x, y)}{r + \rho(x, x_0)} \right)^\beta \frac{r^\gamma}{(r + \rho(x, x_0))^{1+\gamma}}$

for all $x, y \in X$ with $\rho(x, y) \leq (2A)^{-1}(r + \rho(x, x_0))$.

For a fixed $x_0 \in X$, we write $\mathcal{M}(\beta, \gamma) = \mathcal{M}(x_0, 1, \beta, \gamma)$, and define

$$\|f\|_{\mathcal{M}(\beta, \gamma)} = \inf\{C > 0 : \text{(i) and (ii) hold}\}.$$

Let $\mathcal{M}_0(\beta, \gamma)$ be the set of all $f \in \mathcal{M}(\beta, \gamma)$ such that $\int_X f(x) d\mu(x) = 0$. Its dual space, denoted by $(\mathcal{M}_0(\beta, \gamma))'$, consists of all linear functionals \mathcal{L} from $\mathcal{M}_0(\beta, \gamma)$ to \mathbb{C} for which there exists a constant C such that $|\mathcal{L}(f)| \leq C \|f\|_{\mathcal{M}_0(\beta, \gamma)}$ for all $f \in \mathcal{M}_0(\beta, \gamma)$.

Let $\{S_k\}_k$ be an approximation to the identity of order ϵ , and set $D_k = S_k - S_{k-1}$. Then for every $f \in (\mathcal{M}_0(\beta, \gamma))'$, where $0 < \beta, \gamma < \epsilon$, we define the Littlewood–Paley \mathcal{G} -function and the S -function by

$$\mathcal{G}_{\alpha, q}(f)(x) = \left\{ \sum_k (2^{k\alpha} |D_k(f)(x)|)^q \right\}^{1/q}; \tag{1.7}$$

$$S_{\alpha, q}(f)(x) = \left\{ \sum_k \int_{\rho(x, y) \leq C_3 2^{-k}} 2^k (2^{k\alpha} |D_k(f)(y)|)^q d\mu(y) \right\}^{1/q}. \tag{1.8}$$

We now introduce the weighted Triebel–Lizorkin spaces $\dot{F}_{p, w}^{\alpha, q}(X)$.

DEFINITION 1.5. Suppose that $-\epsilon < \alpha < \epsilon$ while $\min(1 + \alpha + \epsilon, 1 + \epsilon)^{-1} < p < \infty$ and $1 < q < \infty$. Given $w \in A_\infty$ with $q_w \leq q$, the weighted Triebel–Lizorkin space $\dot{F}_{p,w}^{\alpha,q}(X)$, where $0 < \beta, \gamma < \epsilon$, is the collection of all $f \in (\mathcal{M}_0(\beta, \gamma))'$ that satisfy

$$\|f\|_{\dot{F}_{p,w}^{\alpha,q}(X)} = \|\mathcal{G}_{\alpha,q}(f)\|_{L_w^p(X)} < \infty, \tag{1.9}$$

where the $L_w^p(X)$ are weighted L^p spaces.

In particular, when $w = 1$, we denote $\dot{F}_{p,w}^{\alpha,q}(X)$ by $\dot{F}_p^{\alpha,q}(X)$.

DEFINITION 1.6. Suppose that $-\epsilon < \alpha < \epsilon$ and $\min(1 + \alpha + \epsilon, 1 + \epsilon)^{-1} < p \leq 1 < q < \infty$. A function $a \in (\mathcal{M}_0(\beta, \gamma))'$, where $0 < \beta, \gamma < \epsilon$, is said to be a (p, q, α) -atom for $\dot{F}_{p,w}^{\alpha,q}(X)$ if

$$\text{supp } a \subset B = B(x_0, r); \tag{1.10}$$

$$\|a\|_{\dot{F}_{q,w}^{\alpha,q}(X)} \leq w(B)^{1/q-1/p}; \tag{1.11}$$

$$\int_X a(x) \, d\mu(x) = 0. \tag{1.12}$$

Here is the main result of this paper.

THEOREM 1.7. Suppose that $-\epsilon < \alpha < \epsilon$ and $\min(1 + \alpha + \epsilon, 1 + \epsilon)^{-1} < p \leq 1 < q < \infty$. Given $w \in A_\infty$ for which $q_w \leq q$, a function f defined on X belongs to $\dot{F}_{p,w}^{\alpha,q}(X)$ if and only if f has a decomposition

$$f(x) = \sum_{k=0}^{\infty} \lambda_k a_k(x),$$

where the series converges in the sense of $(\mathcal{M}_0(\beta, \gamma))'$ where $0 < \beta, \gamma < \epsilon$, the a_k are (p, q, α) -atoms, and $\sum_k |\lambda_k|^p < \infty$. Furthermore,

$$\|f\|_{\dot{F}_{p,w}^{\alpha,q}(X)} \approx \inf \left(\sum_{k=0}^{\infty} |\lambda_k|^p \right)^{1/p},$$

where the infimum is taken over all such decompositions.

This paper is organized as follows. In Section 2 we recall the continuous and discrete Calderón reproducing formulas on spaces of homogeneous type, and obtain the Plancherel–Pólya inequalities for $\dot{F}_{p,w}^{\alpha,q}(X)$ and an equivalence of $\dot{F}_{p,w}^{\alpha,q}(X)$ in terms of the S -function. In Section 3, we use the sequence spaces of Frazier and Jawerth to obtain the duality of $\dot{F}_{p,w}^{\alpha,q}(X)$ (see Theorem 3.5 below). The proof of our main result, Theorem 1.7, is given in Section 4 by combining the results of Section 3, estimates of the area function, and the continuous Calderón reproducing formula.

Throughout, C will denote (possibly different) constants that are independent of the essential variables. Further, $A \approx B$ means that the ratio A/B is bounded and bounded away from zero by constants that do not depend on the relevant variables in A and B . And q' denotes the conjugate index of $q \in (1, \infty)$, that is, $1/q + 1/q' = 1$.

2. Some basic results

We first recall a result of Christ [1], which gives an analogue of the Euclidean dyadic cubes.

LEMMA 2.1 [1]. *Let (X, ρ, μ) be a space of homogeneous type. Then there exist a collection $\{Q_\alpha^k \subset X : k \in \mathbb{Z}, \alpha \in I_k\}$ of open subsets, where I_k is some index set, and constants $C_1, C_2 > 0$, such that:*

- (i) $\mu(X \setminus \bigcup_\alpha Q_\alpha^k) = 0$ for each fixed k and $Q_\alpha^k \cap Q_\beta^k = \emptyset$ if $\alpha \neq \beta$;
- (ii) for all α, β, k, l where $l \geq k$, either $Q_\beta^l \subset Q_\alpha^k$ or $Q_\beta^l \cap Q_\alpha^k = \emptyset$;
- (iii) for each (k, α) and each $l < k$, there is a unique β such that $Q_\alpha^k \subset Q_\beta^l$;
- (iv) $\text{diam}(Q_\alpha^k) \leq C_1 2^{-k}$;
- (v) each Q_α^k contains a ball $B(y_\alpha^k, C_2 2^{-k})$, where $y_\alpha^k \in X$.

We think of Q_α^k as a dyadic cube with diameter roughly 2^{-k} and center y_α^k . We define CQ_α^k to be the dyadic cube with the same center as Q_α^k and diameter $C \text{diam}(Q_\alpha^k)$. When $k \in \mathbb{Z}$ and $\tau \in I_k$, we denote by $Q_\tau^{k,v}$, where $v = 1, 2, \dots, N(k, \tau)$, the set of all cubes $Q_\tau^{k+J} \subset Q_\tau^k$, where J is a fixed large positive integer, and by $y_\tau^{k,v}$ a point in $Q_\tau^{k,v}$.

We now state the continuous and the discrete Calderón reproducing formulas on spaces of homogeneous type, as developed in [5].

PROPOSITION 2.2 (Continuous Calderón reproducing formula). *Suppose that $\{S_k\}_k$ is an approximation to the identity of order ϵ . Set $D_k = S_k - S_{k-1}$ for all $k \in \mathbb{Z}$. Then there exists a family of operators $\{\bar{D}_k\}_k$ such that*

$$f = \sum_k D_k \bar{D}_k(f), \tag{2.1}$$

for all $f \in (\mathcal{M}_0(\beta, \gamma))'$, the dual of $\mathcal{M}_0(\beta, \gamma)$, and the series converges in the sense that

$$\lim_{M \rightarrow \infty} \left\langle \sum_{|k| \leq M} D_k \bar{D}_k(f), g \right\rangle = 0$$

for all $g \in \mathcal{M}_0(\beta', \gamma')$, where $0 < \beta < \beta'$ and $0 < \gamma < \gamma'$. Moreover, the kernel of \bar{D}_k satisfies (1.2) and (1.4) with ϵ replaced by ϵ' , where $0 < \epsilon' < \epsilon$, and

$$\int_X \bar{D}_k(x, y) d\mu(x) = \int_X \bar{D}_k(x, y) d\mu(y) = 0.$$

PROPOSITION 2.3 (Discrete Calderón reproducing formula). *Suppose that $\{S_k\}_k$ is an approximation to the identity of order ϵ , and set $D_k = S_k - S_{k-1}$ for all $k \in \mathbb{Z}$. Then there exists a family of operators $\{\tilde{D}_k\}_k$ such that, for all $f \in \mathcal{M}_0(\beta, \gamma)$,*

$$f(x) = \sum_{k \in \mathbb{Z}} \sum_{\tau \in I_k} \sum_{v=1}^{N(k, \tau)} \mu(Q_\tau^{k,v}) \tilde{D}_k(x, y_\tau^{k,v}) D_k(f)(y_\tau^{k,v}), \tag{2.2}$$

where the series converges in the norm of $\mathcal{M}_0(\beta', \gamma')$, where $0 < \beta' < \beta$ and $0 < \gamma' < \gamma$, and also in the space $\mathcal{M}_0(\beta'', \gamma'')$ where $0 < \beta < \beta''$ and $0 < \gamma < \gamma''$. Moreover, the kernel of \tilde{D}_k satisfies (1.2) and (1.3) with ϵ replaced by ϵ' , where $0 < \epsilon' < \epsilon$, and

$$\int_X \tilde{D}_k(x, y) d\mu(x) = \int_X \tilde{D}_k(x, y) d\mu(y) = 0.$$

In order to verify that the definition of $\dot{F}_{p,w}^{\alpha,q}(X)$ is independent of the choices of approximations to the identity, we need the following Plancherel–Pólya inequalities.

LEMMA 2.4. *Suppose that $\{S_k\}_{k \in \mathbb{Z}}$ and $\{P_k\}_{k \in \mathbb{Z}}$ are approximations to the identity of order ϵ , $D_k = S_k - S_{k-1}$ and $E_k = P_k - P_{k-1}$. Then when $f \in (\mathcal{M}_0(\beta, \gamma))'$ where $0 < \beta, \gamma < \epsilon$, $-\epsilon < \alpha < \epsilon$ and $\min(1 + \alpha + \epsilon, 1 + \epsilon)^{-1} < p, q < \infty$,*

$$\begin{aligned} & \left\| \left\{ \sum_{k \in \mathbb{Z}} \sum_{\tau \in I_k} \sum_{v=1}^{N(k,\tau)} \left[\mu(Q_\tau^{k,v})^{-\alpha} \sup_{x \in Q_\tau^{k,v}} |D_k(f)(x)| \chi_{Q_\tau^{k,v}}(\cdot) \right]^q \right\}^{1/q} \right\|_{L_w^p} \\ & \approx \left\| \left\{ \sum_{k \in \mathbb{Z}} \sum_{\tau \in I_k} \sum_{v=1}^{N(k,\tau)} \left[\mu(Q_\tau^{k,v})^{-\alpha} \inf_{x \in Q_\tau^{k,v}} |E_k(f)(x)| \chi_{Q_\tau^{k,v}}(\cdot) \right]^q \right\}^{1/q} \right\|_{L_w^p}. \end{aligned} \tag{2.3}$$

The proof of Lemma 2.4 is similar to that of [5, Theorem 4.6] with only minor modifications. We omit the details here.

Using the definition of $Q_\tau^{k,v}$ and Lemma 2.4, we see that when $f \in \dot{F}_{p,w}^{\alpha,q}(X)$,

$$\begin{aligned} & \left\| \left\{ \sum_{k \in \mathbb{Z}} \sum_{\tau \in I_k} \sum_{v=1}^{N(k,\tau)} \left[\mu(Q_\tau^{k,v})^{-\alpha} \inf_{x \in Q_\tau^{k,v}} |D_k(f)(x)| \chi_{Q_\tau^{k,v}}(\cdot) \right]^q \right\}^{1/q} \right\|_{L_w^p} \\ & \approx \left\| \left\{ \sum_{k \in \mathbb{Z}} \sum_{\tau \in I_k} \sum_{v=1}^{N(k,\tau)} \left[\mu(Q_\tau^{k,v})^{-\alpha} \sup_{x \in Q_\tau^{k,v}} |E_k(f)(x)| \chi_{Q_\tau^{k,v}}(\cdot) \right]^q \right\}^{1/q} \right\|_{L_w^p} \\ & \approx \left\| \left\{ \sum_k (2^{k\alpha} |D_k(f)(\cdot)|)^q \right\}^{1/q} \right\|_{L_w^p} \\ & = \|\mathcal{G}_{\alpha,q}(f)\|_{\dot{F}_{p,w}^{\alpha,q}(X)} \\ & = \|f\|_{\dot{F}_{p,w}^{\alpha,q}(X)}. \end{aligned}$$

This shows that the definition of $\dot{F}_{p,w}^{\alpha,q}(X)$ is independent of the choice of the approximation to the identity.

To obtain an atomic decomposition for $\dot{F}_{p,w}^{\alpha,q}(X)$, we need an equivalent characterization of $\dot{F}_{p,w}^{\alpha,q}(X)$ in terms of area function. More precisely, we have the following result.

LEMMA 2.5. *For all $f \in \dot{F}_{p,w}^{\alpha,q}(X)$,*

$$\|f\|_{\dot{F}_{p,w}^{\alpha,q}(X)} = \|\mathcal{G}_{\alpha,q}(f)\|_{L_w^p} \approx \|S_{\alpha,q}(f)\|_{L_w^p}.$$

PROOF. From (1.8) and Lemma 2.4, we obtain

$$\begin{aligned}
 & \|S_{\alpha,q}(f)\|_{L_w^p} \\
 &= \left\| \left\{ \sum_k \int_{\rho(x,y) \leq C_3 2^{-k}} 2^k (2^{k\alpha} |D_k(f)(y)|)^q d\mu(y) \right\}^{1/q} \right\|_{L_w^p} \\
 &= \left\| \left\{ \sum_k \sum_{\tau \in I_k} \sum_{v=1}^{N(k,\tau)} \int_{\rho(x,y) \leq C_3 2^{-k}} 2^k (2^{k\alpha} |D_k(f)(y)|)^q d\mu(y) \chi_{Q_\tau^{k,v}}(x) \right\}^{1/q} \right\|_{L_w^p} \\
 &\leq \left\| \left\{ \sum_k \sum_{\tau \in I_k} \sum_{v=1}^{N(k,\tau)} \int_{\rho(x,y) \leq C_3 2^{-k}} 2^k \right. \right. \\
 &\quad \left. \left. \times \left(2^{k\alpha} \sup_{z \in CQ_\tau^{k,v}} |D_k(f)(z)| \right)^q d\mu(y) \chi_{Q_\tau^{k,v}}(x) \right\}^{1/q} \right\|_{L_w^p} \\
 &\leq C \left\| \left\{ \sum_k \sum_{\tau \in I_k} \sum_{v=1}^{N(k,\tau)} \left(2^{k\alpha} \sup_{z \in CQ_\tau^{k,v}} |D_k(f)(z)| \right)^q \chi_{Q_\tau^{k,v}}(x) \right\}^{1/q} \right\|_{L_w^p} \\
 &\leq C \|G_{\alpha,q}(f)\|_{L_w^p}.
 \end{aligned}$$

On the other hand,

$$\begin{aligned}
 & \|S_{\alpha,q}(f)\|_{L_w^p} \\
 &\geq \left\| \left\{ \sum_k \sum_{\tau \in I_k} \sum_{v=1}^{N(k,\tau)} \int_{\rho(x,y) \leq C_3 2^{-k}} 2^k \right. \right. \\
 &\quad \left. \left. \times \left(2^{k\alpha} \inf_{z \in CQ_\tau^{k,v}} |D_k(f)(z)| \right)^q d\mu(y) \chi_{Q_\tau^{k,v}}(x) \right\}^{1/q} \right\|_{L_w^p} \\
 &\geq C \left\| \left\{ \sum_k \sum_{\tau \in I_k} \sum_{v=1}^{N(k,\tau)} \left(2^{k\alpha} \inf_{z \in CQ_\tau^{k,v}} |D_k(f)(z)| \right)^q \chi_{Q_\tau^{k,v}}(x) \right\}^{1/q} \right\|_{L_w^p} \\
 &\geq C \|G_{\alpha,q}(f)\|_{L_w^p},
 \end{aligned}$$

which proves Lemma 2.5. □

LEMMA 2.6. *If $w \in A_q$, where $1 < q < \infty$, then*

$$\mu(Q) \leq w(Q)^{1/q} (w^b(Q))^{(q-1)/q} \leq C_q \mu(Q), \tag{2.4}$$

where $w(Q) = \int_Q w(x) d\mu(x)$, $w^b = w^{-1/(q-1)}$ and $w^b(Q) = \int_Q w(x)^{-1/(q-1)} d\mu(x)$.

PROOF. From (1.1),

$$\begin{aligned} \frac{1}{\mu(Q)} \int_Q d\mu(x) &= 1/\mu(Q) \int_Q w(x)^{1/q} w(x)^{-1/q} d\mu(x) \\ &\leq \left(\frac{1}{\mu(Q)} \int_Q w(x) d\mu(x) \right)^{1/q} \\ &\quad \times \left(\frac{1}{\mu(Q)} \int_Q w(x)^{-1/(q-1)} d\mu(x) \right)^{(q-1)/q} \\ &\leq C_q. \end{aligned}$$

This proves (2.4). □

3. Duality of weighted Triebel–Lizorkin spaces

In recent years Han and Lu have developed an approach to the Hardy spaces H^p and Carleson measure spaces CMO^p , where $p \leq 1$ and near 1, and obtained the duality, using discrete Littlewood–Paley–Stein analysis. More precisely, they defined a type of sequence spaces s^p and c^p , and proved that H^p can be lifted to s^p and s^p can be projected to H^p . Moreover, the composition of the lifting and projection operators is equal to the identity operator on H^p . Similar results hold for CMO^p and c^p . Then, they showed the duality between s^p and c^p . Finally, by working on the level of sequence spaces, they obtained the duality between H^p and CMO^p . Their methods can be applied to the multi-parameter product case [13], the multi-parameter case with implicit flag structures [14] and also the multi-parameter case with Zygmund dilations [15].

In this section, we will show that the dual space of $\dot{F}_{q,w}^{\alpha,q}(X)$ is $\dot{F}_{q',w'}^{-\alpha,q'}(X)$ when $-\epsilon < \alpha < \epsilon$, $1 < q < \infty$ and $w \in A_\infty$ where $q_w \leq q$ and $w' = w^{-1/(q-1)} = w^b \in A_{q'}$. Following [5], we introduce sequence spaces.

DEFINITION 3.1. Let $\tilde{\chi}_Q(x) = \mu(Q)^{-1/2} \chi(x)$. When $-\epsilon < \alpha < \epsilon$ and $1 < p, q < \infty$, the sequence space $\dot{f}_{p,w}^{\alpha,q}$ is defined to be the set of all complex-valued sequences

$$s = \{s_{Q_\tau^{k,v}}\}_{k \in \mathbb{Z}; \tau \in I_k; v=1, \dots, N(k,\tau)}$$

such that

$$\|s\|_{\dot{f}_{p,w}^{\alpha,q}} = \left\| \left\{ \sum_k \sum_{\tau \in I_k} \sum_{v=1}^{N(k,\tau)} (\mu(Q_\tau^{k,v})^{-\alpha} |s_{Q_\tau^{k,v}}| \tilde{\chi}_{Q_\tau^{k,v}}(\cdot))^q \right\}^{1/q} \right\|_{L_w^p} < \infty.$$

We now prove the following theorem.

THEOREM 3.2. When $-\epsilon < \alpha < \epsilon$ and $1 < q < \infty$,

$$(\dot{f}_{q,w}^{\alpha,q})' = \dot{f}_{q',w'}^{-\alpha,q'}. \tag{3.1}$$

PROOF. Given a sequence $t \in \dot{f}_{q',w'}^{-\alpha,q'}$, set

$$L(s) = \sum_k \sum_{\tau \in I_k} \sum_{v=1}^{N(k,\tau)} s_{Q_\tau^{k,v}} \bar{t}_{Q_\tau^{k,v}},$$

for all sequences $s \in \dot{f}_{q,w}^{\alpha,q}$. By Hölder’s inequality,

$$\begin{aligned} |L(s)| &\leq \int \sum_k \sum_{\tau \in I_k} \sum_{v=1}^{N(k,\tau)} \mu(Q_\tau^{k,v})^{-\alpha} |s_{Q_\tau^{k,v}}| \tilde{\chi}_{Q_\tau^{k,v}}(x) w^{1/q}(x) \\ &\quad \times \mu(Q_\tau^{k,v})^\alpha |t_{Q_\tau^{k,v}}| \tilde{\chi}_{Q_\tau^{k,v}}(x) w^{-1/q}(x) d\mu(x) \\ &\leq \|s\|_{\dot{f}_{q,w}^{\alpha,q}} \|t\|_{\dot{f}_{q',w'}^{-\alpha,q'}}, \end{aligned}$$

and hence $\dot{f}_{q',w'}^{-\alpha,q'} \subseteq (\dot{f}_{q,w}^{\alpha,q})'$.

Conversely, for every $L \in (\dot{f}_{q,w}^{\alpha,q})'$, we see that

$$L(s) = \sum_k \sum_{\tau \in I_k} \sum_{v=1}^{N(k,\tau)} s_{Q_\tau^{k,v}} \bar{t}_{Q_\tau^{k,v}}$$

for some sequence $t = \{t_{Q_\tau^{k,v}}\}_{k \in \mathbb{Z}; \tau \in I_k; v=1, \dots, N(k,\tau)}$. Define

$$L_w^q(l^q) = \left\{ f = \{f_k\} : \|f\|_{L_w^q(l^q)} \equiv \left\| \left(\sum_{k \in \mathbb{Z}} |f_k|^q \right)^{1/q} \right\|_{L_w^q} \right\}.$$

Then $(L_w^q(l^q))' = L_w^{q'}(l^{q'})$ when $1 < q < \infty$, with the obvious pairing, namely,

$$f \mapsto \int_X \sum_k f_k(x) \bar{g}_k(x) d\mu(x)$$

when $g = \{g_k\} \in L_w^{q'}(l^{q'})$, and the map In from $\dot{f}_{q,w}^{\alpha,q}$ to $L_w^q(l^q)$ defined by $\text{In}(s) = \{f_k(s)\}$, where

$$f_k(s) = \sum_{\tau \in I_k} \sum_{v=1}^{N(k,\tau)} \mu(Q_\tau^{k,v})^{-\alpha} s_{Q_\tau^{k,v}} \tilde{\chi}_{Q_\tau^{k,v}},$$

is a linear isometry onto a subspace of $L_w^q(l^q)$. By the Hahn–Banach theorem, there exists $\tilde{L} \in L_w^{q'}(l^{q'})$ with $\|\tilde{L}\| = \|L\|$ such that $\tilde{L} \circ \text{In} = L$. In other words, there exists $g = \{g_k\} \in L_w^{q'}(l^{q'})$ with $\|g\|_{L_w^{q'}(l^{q'})} \leq \|L\|$ such that

$$\tilde{L} \circ \text{In}(s) = L(s) = \int_X \sum_k f_k(x) \bar{g}_k(x) d\mu(x)$$

for all $s \in \dot{f}_{q,w}^{\alpha,q}$. Substituting the expression for f_k into this formula, we obtain

$$L(s) = \sum_k \sum_{\tau \in I_k} \sum_{v=1}^{N(k,\tau)} s_{Q_\tau^{k,v}} \bar{t}_{Q_\tau^{k,v}},$$

where

$$t_{Q_\tau^{k,v}} = \int_{Q_\tau^{k,v}} g_k(y) d\mu(y) / \mu(Q_\tau^{k,v})^{\alpha+1/2}.$$

Denote by M the Hardy–Littlewood maximal operator. Using the vector-valued maximal inequality, we see that

$$\|t\|_{\dot{f}_{q',w'}^{-\alpha,q'}} \leq \|M(g)\|_{L_{w'}^{q'}(I_{q'})} \leq C \|g\|_{L_{w'}^{q'}(I_{q'})} \leq C \|L\|.$$

This completes the proof of Theorem 3.2. □

We recall the lifting and projection operators defined in [5].

DEFINITION 3.3. Let $y_\tau^{k,v}$ be the center of $Q_\tau^{k,v}$ as in Lemma 2.1.

(i) For a function $f \in (\mathcal{M}_0(\beta, \gamma))'$, we define

$$S_D(f) = \{\mu(Q_\tau^{k,v})^{1/2} D_k(f)(y_\tau^{k,v})\}_{k \in \mathbb{Z}; \tau \in I_k; v=1, \dots, N(k,\tau)}. \tag{3.2}$$

(ii) For a sequence $s \in \dot{f}_{q,w}^{\alpha,q}$, we define

$$T_{\tilde{D}}(s)(x) = \sum_k \sum_{\tau \in I_k} \sum_{v=1}^{N(k,\tau)} s_{Q_\tau^{k,v}} \mu(Q_\tau^{k,v})^{1/2} \tilde{D}_k(x, y_\tau^{k,v}). \tag{3.3}$$

To obtain the duality of weighted Triebel–Lizorkin spaces, we need to work at the level of sequence spaces.

PROPOSITION 3.4. Let S_D and $T_{\tilde{D}}$ be the operators in Definition 3.3. Then for all $f \in \dot{F}_{q,w}^{\alpha,q}(X)$,

$$\|S_D(f)\|_{\dot{f}_{q,w}^{\alpha,q}} \leq C \|f\|_{\dot{F}_{q,w}^{\alpha,q}(X)}. \tag{3.4}$$

Conversely, for all sequences $s \in \dot{f}_{q,w}^{\alpha,q}$,

$$\|T_{\tilde{D}}(s)\|_{\dot{F}_{q,w}^{\alpha,q}(X)} \leq C \|s\|_{\dot{f}_{q,w}^{\alpha,q}}. \tag{3.5}$$

Moreover, $T_{\tilde{D}} \circ S_D$ is equal to the identity on $\dot{F}_{q,w}^{\alpha,q}(X)$.

PROOF. Estimate (3.4) follows directly from Lemma 2.4 and Definition 3.3.

We now prove (3.5). To simplify our notation, we work with dyadic cubes of the form $\{Q_\tau^k : k \in \mathbb{Z}, \tau \in I_{k+J}\}$. Set

$$m_{Q_\tau^k}(x) = \mu(Q_\tau^k)^{1/2} \tilde{D}_k(x, y_\tau^k).$$

From the size and cancellation conditions on D_j and \tilde{D}_k ,

$$D_j(m_{Q_\tau^k})(x) \leq C\mu(Q_\tau^k)^{1/2}2^{-(j-k)\epsilon'} \frac{1}{(1 + 2^k \rho(x, y_\tau^k))^{1+\epsilon}}. \tag{3.6}$$

Write

$$\begin{aligned} \|T_{\tilde{D}}(s)\|_{\dot{F}_{q,w}^{\alpha,q}(X)} &= \left\| \left\{ \sum_j (2^{j\alpha} |D_j(T_{\tilde{D}}(s))|)^q \right\}^{1/q} \right\|_{L_w^q} \\ &\leq \left\| \left\{ \sum_j \left(2^{j\alpha} \sum_{k>j} \sum_{\tau \in I_{k+j}} |s_{Q_\tau^k}| |D_j(m_{Q_\tau^k})(\cdot)| \right)^q \right\}^{1/q} \right\|_{L_w^q} \\ &\quad + \left\| \left\{ \sum_j \left(2^{j\alpha} \sum_{k \leq j} \sum_{\tau \in I_{k+j}} |s_{Q_\tau^k}| |D_j(m_{Q_\tau^k})(\cdot)| \right)^q \right\}^{1/q} \right\|_{L_w^q} \\ &=: T_1 + T_2, \end{aligned}$$

say. We first estimate the term T_1 . Denote

$$\begin{aligned} a &= \{a_k\}_k = \left\{ M \left(\mu(Q_\tau^k)^{-\alpha} \sum_{\tau \in I_{k+j}} |s_{Q_\tau^k}| \tilde{\chi}_{Q_\tau^k} \right) \right\}_k; \\ b &= \{b_k\}_k = \{2^{k(\epsilon'+\alpha)} \chi_{\{k \in \mathbb{Z}: k < 0\}}(k)\}_k. \end{aligned}$$

Then Young’s inequality and the vector-valued maximal inequality, together with the formula

$$(a * b)_j(x) = \sum_k a_k(x) b_{j-k},$$

give

$$\begin{aligned} T_1 &\leq C \|a * b(\cdot)\|_{l^q} \|L_w^q\| \leq C \|a(\cdot)\|_{l^q} \|b\|_{l^1} \|L_w^q\| \leq C \|a(\cdot)\|_{l^q} \|L_w^q\| \\ &\leq C \left\| \left\{ \sum_k \left(M \left(\mu(Q_\tau^k)^{-\alpha} \sum_{\tau \in I_{k+j}} |s_{Q_\tau^k}| \tilde{\chi}_{Q_\tau^k} \right) (\cdot) \right)^q \right\}^{1/q} \right\|_{L_w^q} \\ &\leq C \left\| \left\{ \sum_k \left(\mu(Q_\tau^k)^{-\alpha} \sum_{\tau \in I_{k+j}} |s_{Q_\tau^k}| \tilde{\chi}_{Q_\tau^k}(\cdot) \right)^q \right\}^{1/q} \right\|_{L_w^q} \\ &\leq C \|s\|_{\dot{F}_{q,w}^{\alpha,q}}, \end{aligned}$$

where the first inequality is because $\mu(Q_\tau^k)^{-\alpha} \approx 2^{k\alpha}$ and

$$\sum_{k>j} \sum_{\tau \in I_{k+j}} |s_{Q_\tau^k}| |D_j(m_{Q_\tau^k})(x)| \leq C \sum_{k>j} 2^{(j-k)\epsilon'} M \left(\sum_{\tau \in I_{k+j}} |s_{Q_\tau^k}| \tilde{\chi}_{Q_\tau^k} \right) (x).$$

A similar argument shows that $T_2 \leq C \|s\|_{\dot{F}_{q,w}^{\alpha,q}}$. This completes the proof of (3.5). \square

Finally, we prove a duality result of weighted Triebel–Lizorkin spaces.

THEOREM 3.5. *Suppose that $0 < \beta, \gamma < \epsilon$, while $-\epsilon < \alpha < \epsilon$ and $1 < q < \infty$. Then*

$$(\dot{F}_{q,w}^{\alpha,q}(X))' = \dot{F}_{q',w'}^{-\alpha,q'}(X).$$

More precisely, if $g \in \dot{F}_{q',w'}^{-\alpha,q'}(X)$, then the map L_g , defined by $L_g(f) = \langle f, g \rangle$, initially for $f \in \mathcal{M}_0(\beta, \gamma)$, extends to a continuous linear functional on $\dot{F}_{q,w}^{\alpha,q}(X)$ and

$$\|L_g\| \approx \|g\|_{\dot{F}_{q',w'}^{-\alpha,q'}(X)}.$$

Conversely, every $L \in (\dot{F}_{q,w}^{\alpha,q}(X))'$ is equal to L_g for some $g \in \dot{F}_{q',w'}^{-\alpha,q'}(X)$.

PROOF. If $g \in \dot{F}_{q',w'}^{-\alpha,q'}(X)$ and $f \in \mathcal{M}_0(\beta, \gamma)$, then from Proposition 3.4,

$$\langle f, g \rangle = \langle T_{\tilde{D}} \circ S_D(f), g \rangle \equiv \langle S_D(f), S_{\tilde{D}}(g) \rangle,$$

where

$$S_{\tilde{D}}(g) = \{\mu(Q_\tau^{k,v})^{1/2} \tilde{D}_k(g)(y_\tau^{k,v})\}_{k \in \mathbb{Z}; \tau \in I_k; v=1, \dots, N(k, \tau)}.$$

By Lemma 2.4 and Definition 3.3,

$$\|S_{\tilde{D}}(g)\|_{\dot{F}_{q',w'}^{-\alpha,q'}} \leq C \|g\|_{\dot{F}_{q',w'}^{-\alpha,q'}(X)},$$

and then

$$|\langle f, g \rangle| \leq \|S_D(f)\|_{\dot{F}_{q,w}^{\alpha,q}} \|S_{\tilde{D}}(g)\|_{\dot{F}_{q',w'}^{-\alpha,q'}} \leq \|f\|_{\dot{F}_{q,w}^{\alpha,q}(X)} \|g\|_{\dot{F}_{q',w'}^{-\alpha,q'}(X)}.$$

This proves that

$$\|L_g\| \leq C \|g\|_{\dot{F}_{q',w'}^{-\alpha,q'}(X)}.$$

Conversely, suppose that $L \in (\dot{F}_{q,w}^{\alpha,q}(X))'$. Then

$$L_1 \equiv L \circ T_{\tilde{D}} \in (\dot{f}_{q,w}^{\alpha,q})'.$$

By Theorem 3.2, there exists $t \in \dot{f}_{q',w'}^{-\alpha,q'}$ such that $L_1(s) = \langle t, s \rangle$ for all $s \in \dot{f}_{q,w}^{\alpha,q}$, and

$$\|t\|_{\dot{f}_{q',w'}^{-\alpha,q'}} \approx \|L_1\| \leq C \|L\|$$

since $T_{\tilde{D}}$ is bounded. Now $L_1 \circ S_D = L \circ T_{\tilde{D}} \circ S_D = L$ by Proposition 3.4. Hence,

$$L(f) = L \circ T_{\tilde{D}}(S_D(f)) = \langle S_D(f), t \rangle = \langle f, T_D(t) \rangle$$

for all $f \in \dot{F}_{q,w}^{\alpha,q}(X)$, where

$$T_D(t) = \sum_k \sum_{\tau \in I_k} \sum_{v=1}^{N(k, \tau)} t_{Q_\tau^{k,v}} \mu(Q_\tau^{k,v})^{1/2} D_k(x, y_\tau^{k,v}).$$

By Lemma 2.4 and Definition 3.3, we obtain

$$\|T_D(t)\|_{\dot{f}_{q',w'}^{-\alpha,q'}} \leq C \|t\|_{\dot{f}_{q',w'}^{-\alpha,q'}} \leq C \|L\|.$$

Let $g = T_D(t) \in \dot{f}_{q',w'}^{-\alpha,q'}$. This completes the proof of the theorem. □

4. Proof of Theorem 1.7

To prove the ‘if’ part of Theorem 1.7, we need the following lemma.

LEMMA 4.1. *Let $S_{\alpha,q}$ be the area function defined in (1.8). Then there exists a positive constant C such that for all (p, q, α) -atoms a ,*

$$\|S_{\alpha,q}(a)\|_{L_w^p}^p \leq C. \tag{4.1}$$

Let us take this lemma for granted for the moment.

PROOF OF THEOREM 1.7. Let $f(x) = \sum_k \lambda_k a_k(x)$, where each a_k is a (p, q, α) -atom, as in Definition 1.6. From Lemma 4.1,

$$\|f\|_{\dot{F}_{p,w}^{\alpha,q}(X)}^p = \|S_{\alpha,q}(f)\|_{L_w^p}^p \leq \sum_k |\lambda_k|^p \|S_{\alpha,q}(a_k)\|_{L_w^p}^p \leq C \sum_k |\lambda_k|^p.$$

Conversely, we use the construction of an approximation to the identity by Coifman, as in Remark 1.3, to obtain $\{S_k\}_k$ on X for which there exists a constant C_4 such that, for all $k \in \mathbb{Z}$ and $x, x', y \in X$:

- (i) $S_k(x, y) = 0$ if $\rho(x, y) > C_4 2^{-k}$ and $\|S_k\|_{\infty} \leq C_4 2^k$;
- (ii) $|S_k(x, y) - S_k(x', y)| \leq C_4 2^{k(1+\epsilon)} \rho(x, x')^\epsilon$;
- (iii) $\int_X S_k(x, y) d\mu(y) = 1$;
- (iv) $S_k(x, y) = S_k(y, x)$;

We can check that such $\{S_k\}_k$ satisfies all the conditions in Definition 1.2. Moreover, we can see that for each fixed y , when $S_k(x, y)$ is considered as a function of x , it is supported on $\{x \in X : \rho(x, y) \leq C_4 2^{-k}\}$. Set $D_k = S_k - S_{k-1}$. Then we can see that similar results hold for D_k but with (iii) replaced by:

(iii)' $\int_X D_k(x, y) d\mu(y) = 0.$

Substituting this $\{D_k\}_k$ into (1.8), Definition 1.2 and Proposition 2.2, for a function $f \in \dot{F}_{p,w}^{\alpha,q}(X)$, we let

$$\begin{aligned} \Omega_k &= \{x \in \mathcal{X} : S_{\alpha,q}(f)(x) > 2^k\}, \\ B_k &= \{Q : w(Q \cap \Omega_k) > w(Q)/2, w(Q \cap \Omega_{k+1}) \leq w(Q)/2\}, \\ \tilde{\Omega}_k &= \{x \in \mathcal{X} : M(\chi_\Omega)(x) > 1/2\}. \end{aligned}$$

From Lemma 2.1, we know that for each dyadic cube Q in X , there is a unique $k \in \mathbb{Z}$ such that $Q = Q_\alpha^k$ for some $\alpha \in I_k$. We denote such k by k_Q . Also, for each dyadic

cube Q , there is a unique $k \in \mathbb{Z}$ such that $Q \in B_k$. For each dyadic cube $Q \in B_k$, there is a unique maximal dyadic cube $Q' \in B_k$ such that $Q \subseteq Q'$. Denote the collection of all maximal dyadic cubes in B_k by $Q_i^k, i \in J_k$, an index set which depends on k (it may be finite). We then have for all dyadic cubes Q ,

$$\bigcup_Q Q = \bigcup_k \bigcup_{i \in J_k} \bigcup_{Q \subseteq Q_i^k, Q \in B_k} Q.$$

Applying the Calderón reproducing formula (Proposition 2.2), we see that

$$\begin{aligned} f(x) &= \sum_k D_k \bar{D}_k(f)(x) \\ &= \sum_k \sum_{i \in J_k} \sum_{Q \subseteq Q_i^k, Q \in B_k} \int_Q D_{k_Q}(x, y) \bar{D}_{k_Q}(f)(y) d\mu(y). \end{aligned}$$

To obtain an atomic decomposition, we first claim that

$$\sum_{Q \in B_k} \int_Q 2^{k_Q} (2^{k_Q \alpha} |\bar{D}_{k_Q}(f)(y)|)^q w(Q) d\mu(y) \leq C 2^{kq} w(\Omega_k), \tag{4.2}$$

where C is a constant independent of k and i .

Indeed, by (1.8) and the definition of B_k ,

$$\begin{aligned} &\int_{\tilde{\Omega}_k \setminus \Omega_{k+1}} S_{\alpha,q}(f)(x)^q w(x) d\mu(x) \\ &\geq C \int_{\tilde{\Omega}_k \setminus \Omega_{k+1}} \sum_j \int_{\rho(x,y) \leq C_3 2^{-j}} (2^{j\alpha} |\bar{D}_j(f)(y)|)^q d\mu(y) w(x) d\mu(x) \\ &\geq C \sum_j \int_{\mathcal{X}} 2^j (2^{j\alpha} |\bar{D}_j(f)(y)|)^q \\ &\quad \times w(\{x \in \tilde{\Omega}_k \setminus \Omega_{k+1} : \rho(x, y) \leq C_3 2^{-j}\}) d\mu(y) \\ &\geq C \sum_{Q \in B_k} \int_Q 2^{k_Q} (2^{k_Q \alpha} |\bar{D}_{k_Q}(f)(y)|)^q \\ &\quad \times w(\{x \in \tilde{\Omega}_k \setminus \Omega_{k+1} : \rho(x, y) \leq C_3 2^{-k_Q}\}) d\mu(y) \\ &\geq C \sum_{Q \in B_k} \int_Q 2^{k_Q} (2^{k_Q \alpha} |\bar{D}_{k_Q}(f)(y)|)^q w(Q) d\mu(y). \end{aligned} \tag{4.3}$$

On the other hand, it follows from the definition of Ω_k that

$$\int_{\tilde{\Omega}_k \setminus \Omega_{k+1}} S_{\alpha,q}(f)(x)^q w(x) d\mu(x) \leq 2^{(k+1)q} w(\tilde{\Omega}_k \setminus \Omega_{k+1}) \leq C 2^{kq} w(\Omega_k). \tag{4.4}$$

Estimate (4.3), together with (4.4), yields (4.2).

Define

$$\lambda_{k,i} = w(Q_i^k)^{1/p-1/q} \left\{ \sum_{Q \subset Q_i^k, Q \in B_k} \int_Q 2^{kQ} (2^{kQ\alpha} |\bar{D}_{kQ}(f)(y)|)^q w(Q) d\mu(y) \right\}^{1/q}, \tag{4.5}$$

$$a_{k,i}(x) = \begin{cases} \frac{\tilde{C}}{\lambda_{k,i}} \sum_{Q \subset Q_i^k, Q \in B_k} \int_Q D_{kQ}(x, y) \bar{D}_{kQ}(f)(y) d\mu(y) & \text{if } \lambda_{k,i} \neq 0, \\ 0 & \text{if } \lambda_{k,i} = 0. \end{cases} \tag{4.6}$$

Here \tilde{C} is a fixed constant to be chosen later, which is independent of k and i .

Let us verify that each $a_{k,i}$ is a (p, q, α) -atom. By (iii)', we can see that each $a_{k,i}$ satisfies (1.12). And by the construction of D_k ,

$$\text{supp } a_{k,i} \subset \bigcup_{y \in Q \in Q_i^k \cap B_k} \{x \in X : \rho(x, y) \leq C_3 2^{-kQ}\} \subset \bar{C} Q_i^k,$$

where \bar{C} is a constant independent of x, y and Q_i^k . This gives (1.10).

To show the size condition of each $a_{k,i}$, we claim that

$$\sup_{\|h\|_{\dot{F}_{q',w'}^{-\alpha,q'}(X)} \leq 1} |\langle a_{k,i}, h \rangle| \leq w(Q_i^k)^{1/q-1/p}. \tag{4.7}$$

Let us assume (4.7) first; then (1.11) follows directly from Theorem 3.5. Indeed, for all h satisfying $\|h\|_{\dot{F}_{q',w'}^{-\alpha,q'}(X)} \leq 1$, by combining (1.8), (4.5), Lemma 2.6 and Hölder's inequality, we see that

$$\begin{aligned} |\langle a_{k,i}, h \rangle| &= \frac{\tilde{C}}{\lambda_{k,i}} \left| \left\langle \sum_{Q \subset Q_i^k, Q \in B_k} \int_Q D_{kQ}(x, y) \bar{D}_{kQ}(f)(y) d\mu(y), h(x) \right\rangle \right| \\ &\leq \frac{\tilde{C}}{\lambda_{k,i}} \sum_{Q \subset Q_i^k, Q \in B_k} \int_Q |D_{kQ}(h)(y)| |\bar{D}_{kQ}(f)(y)| d\mu(y) \\ &\leq \frac{\tilde{C}}{\lambda_{k,i}} \left\{ \sum_{Q \subset Q_i^k, Q \in B_k} \int_Q 2^{kQ} [2^{kQ\alpha} |\bar{D}_{kQ}(f)(y)|]^q w(Q) d\mu(y) \right\}^{1/q} \\ &\quad \times \left\{ \sum_{Q \subset Q_i^k, Q \in B_k} \int_Q 2^{kQ} [2^{-kQ\alpha} |D_{kQ}(h)(y)|]^{q'} w^b(Q) d\mu(y) \right\}^{1/q'} \\ &\leq \tilde{C} w(Q_i^k)^{1/q-1/p} \|S_{\alpha,q}(h)\|_{L_{w'}^{q'}} \\ &\leq \tilde{C} C w(Q_i^k)^{1/q-1/p} \|h\|_{\dot{F}_{q',w'}^{-\alpha,q'}(X)} \\ &\leq w(Q_i^k)^{1/q-1/p}, \end{aligned} \tag{4.8}$$

where C is a constant independent of k and i , and we choose \tilde{C} such that $\tilde{C}C \leq 1$. This gives (4.7).

Finally, from (4.2), we obtain that

$$\begin{aligned} & \sum_k \sum_{i \in I_k} |\lambda_{k,i}|^p \\ & \leq C \sum_k \sum_{i \in I_k} w(Q_i^k)^{1-p/q} \\ & \quad \times \left\{ \sum_{Q \subset Q_i^k, Q \in B_k} \int_Q 2^{kQ} (2^{kQ\alpha} |\bar{D}_{kQ}(f)(y)|)^q w(Q) d\mu(y) \right\}^{p/q} \\ & \leq C \sum_k \left(\sum_{i \in I_k} w(Q_i^k) \right)^{1-p/q} \\ & \quad \times \left\{ \sum_{i \in I_k} \sum_{Q \subset Q_i^k, Q \in B_k} \int_Q 2^{kQ} (2^{kQ\alpha} |\bar{D}_{kQ}(f)(y)|)^q w(Q) d\mu(y) \right\}^{p/q} \\ & \leq C \sum_k 2^{kp} w(\Omega_k) \\ & \leq C \|f\|_{\dot{F}_{p,w}^{\alpha,q}(X)}^p. \end{aligned}$$

This concludes the proof of Theorem 1.7, modulo the proof of Lemma 4.1, which we will give now. □

PROOF OF LEMMA 4.1. Suppose that $\text{supp } a \subseteq B = B(z_0, r)$. Let

$$\begin{aligned} \|S_{\alpha,q}(a)\|_{L_w^p}^p &= \int_{\mathcal{X}} |S_{\alpha,q}(a)|^p w(x) d\mu(x) \\ &= \left(\int_{2B} + \int_{(2B)^c} \right) |S_{\alpha,q}(a)|^p w(x) d\mu(x) =: T_1 + T_2, \end{aligned}$$

say. For the term T_1 , by Hölder’s inequality and (1.11),

$$\begin{aligned} T_1 &\leq \left(\int_{2B} |S_{\alpha,q}(a)|^q w(x) d\mu(x) \right)^{p/q} \left(\int_{2B} w(x) d\mu(x) \right)^{1-p/q} \\ &\leq C \|a\|_{\dot{F}_{q,w}^{\alpha,q}(X)}^p w(B)^{1-p/q} \\ &\leq C w(B)^{p/q-1} w(B)^{1-p/q} \\ &\leq C, \end{aligned}$$

where C depends only on p and \tilde{C} in (4.8).

Now let us turn to the term T_2 . By Definition 1.8, one can write

$$T_2 = \int_{(2B)^c} \left\{ \sum_j \int_{\rho(x,y) \leq C_3 2^{-j}} 2^j (2^{j\alpha} |D_j(a)(y)|)^q d\mu(y) \right\}^{p/q} w(x) d\mu(x). \tag{4.9}$$

By (1.10) and the construction of D_j in the proof of Theorem 1.7, we deduce that for each j , the function $y \mapsto D_j(a)(y)$ is supported in $\{y : \rho(y, z) \leq C_6 2^{-j}\}$, where $z \in B$ and C_6 is a constant independent of j and B . In the integral in (4.9), moreover, y should be in $\{y : \rho(x, y) \leq C_3 2^{-j}\}$. Hence there exists a constant C_7 , which depends only on C_3 and C_6 , such that $D_j(a)(y) = 0$ when $2^{-j} \leq C_7|x - z_0|$. So

$$T_2 = \int_{(2B)^c} \left\{ \sum_{2^j < C_7/\rho(x-z_0)} \int_{\rho(x,y) \leq C_3 2^{-j}} 2^j (2^{j\alpha} |D_j(a)(y)|)^q d\mu(y) \right\}^{p/q} w(x) d\mu(x). \tag{4.10}$$

To estimate the right-hand side of (4.10), we first consider $D_j(a)$. By (1.10) and (1.12),

$$D_j(a)(y) = \int_X D_j(y, z)a(z) d\mu(z) = \int_X [D_j(y, z) - \psi_j(y)]\eta(z)a(z) d\mu(z),$$

where $\eta(z) = \chi_B(z)$ and

$$\psi_j(y) = \frac{\int D_j(y, z)\eta(z) d\mu(z)}{\int \eta(z) d\mu(z)}.$$

Set $h(z) = [D_j(y, z) - \psi_j(y)]\eta(z)$. Then, for all $z, z' \in B$,

$$|h(z)| \leq C2^{j(1+\epsilon)} \mu(B)^\epsilon; \tag{4.11}$$

$$|h(z) - h(z')| \leq C2^{j(1+\epsilon)} \rho(z, z')^\epsilon; \tag{4.12}$$

$$\int_X h(z) d\mu(z) = 0. \tag{4.13}$$

Now $h(z) = \sum_k \tilde{D}_k D_k(h)(z)$, from Lemma 2.2. Together with Hölder’s inequality, this shows that

$$\begin{aligned} |D_j(a)(y)| &= \left| \int_X h(z)a(z) d\mu(z) \right| \\ &\leq C \|a\|_{\dot{F}_{q,w}^{\alpha,q}(X)} \left\{ \int_X \sum_k (2^{-k\alpha} |D_k(h)(z')|)^{q'} w^b(z') d\mu(z') \right\}^{1/q'}. \end{aligned} \tag{4.14}$$

To estimate the right-hand side of (4.14), note that

$$\begin{aligned} &\left\{ \int_X \sum_k (2^{-k\alpha} |D_k(h)(z')|)^{q'} w^b(z') d\mu(z') \right\}^{1/q'} \\ &\leq \left\{ \int_{4B} \sum_{2^{-k} \leq \mu(B)} (2^{-k\alpha} |D_k(h)(z')|)^{q'} w^b(z') d\mu(z') \right\}^{1/q'} \\ &\quad + \left\{ \int_{4B} \sum_{2^{-k} > \mu(B)} (2^{-k\alpha} |D_k(h)(z')|)^{q'} w^b(z') d\mu(z') \right\}^{1/q'} \\ &\quad + \left\{ \int_{(4B)^c} \sum_{2^{-k} \leq \mu(B)} (2^{-k\alpha} |D_k(h)(z')|)^{q'} w^b(z') d\mu(z') \right\}^{1/q'} \end{aligned}$$

$$\begin{aligned}
 & + \left\{ \int_{(4B)^c} \sum_{2^{-k} > \mu(B)} (2^{-k\alpha} |D_k(h)(z')|)^{q'} w^b(z') d\mu(z') \right\}^{1/q'} \\
 & =: V_1 + V_2 + V_3 + V_4,
 \end{aligned}$$

say. For the term V_1 , by (4.12),

$$|D_k(h)(z')| = \left| \int_X D_k(z', x)[h(x) - h(z')] d\mu(x) \right| \leq C2^{-k\epsilon} 2^{j(1+\epsilon)},$$

which gives $V_1 \leq C(w^b(B))^{1/q'} \mu(B)^{\alpha+\epsilon} 2^{j(1+\epsilon)}$.

Now consider term V_2 . By (4.11),

$$|D_k(h)(z')| \leq \int_X |D_k(z', x)||h(x)| d\mu(x) \leq C2^k 2^{j(1+\epsilon)} \mu(B)^{1+\epsilon},$$

which proves that $V_2 \leq C(w^b(B))^{1/q'} \mu(B)^{\alpha+\epsilon} 2^{j(1+\epsilon)}$.

For the term V_3 , note that $\text{supp } h \subseteq B$ and $2^{-k} \leq C\mu(B)$, so $V_3 = 0$.

Finally, let us estimate the term V_4 . By (4.13),

$$\begin{aligned}
 |D_k(h)(z')| & \leq \int_X |D_k(z', x) - D_k(z', z_0)||h(x)| d\mu(x) \\
 & \leq C \int_B \left(\frac{\rho(x, z_0)}{2^{-k} + \rho(z', z_0)} \right)^\epsilon \frac{2^{-k\epsilon}}{(2^{-k} + \rho(z', z_0))^{1+\epsilon}} 2^{j(1+\epsilon)} \mu(B)^\epsilon d\mu(x) \\
 & \leq C2^{j(1+\epsilon)} \mu(B)^{1+2\epsilon} \frac{2^{-k\epsilon}}{(2^{-k} + \rho(z', z_0))^{1+2\epsilon}}.
 \end{aligned}$$

This shows that V_4 is at most

$$\left\{ \sum_{2^{-k} > \mu(B)} \frac{2^{j(1+\epsilon)q'}}{2^{k\alpha q'}} \mu(B)^{(1+2\epsilon)q'} \int_{(4B)^c} \left[\frac{2^{-k\epsilon}}{(2^{-k} + \rho(z', z_0))^{1+2\epsilon}} \right]^{q'} w^b(z') d\mu(z') \right\}^{1/q'}.$$

Noting that $\mu(B) \leq \rho(z', z_0) \leq c2^{-k}$ and $\rho(z', z_0) \geq c2^{-k}$, we see that

$$V_4 \leq C(w^b(B))^{1/q'} \mu(B)^{\alpha+\epsilon} 2^{j(1+\epsilon)}.$$

Combining (4.14) and the estimates of V_1, V_2, V_3 and V_4 , we obtain

$$|D_j(a)(y)| \leq Cw(B)^{1/q-1/p} (w^b(B))^{1/q'} \mu(B)^{\alpha+\epsilon} 2^{j(1+\epsilon)}. \tag{4.15}$$

Substituting (4.15) back into (4.10), and using Lemma 2.6, we see that

$$\begin{aligned}
 T_2 & \leq w(B)^{p/q-1} (w^b(B))^{p/q'} \mu(B)^{(\alpha+\epsilon)p} \\
 & \quad \times \int_{(2B)^c} \left\{ \sum_{2^j < C_7/\rho(x, z_0)} 2^{j(1+\epsilon+\alpha)q} \right\}^{p/q} w(x) d\mu(x)
 \end{aligned}$$

$$\begin{aligned} &\leq C\mu(B)^{(1+\alpha+\epsilon)p}w(B)^{-1}\int_{(2B)^c}\frac{w(x)}{\rho(x,z_0)^{(1+\epsilon+\alpha)p}}d\mu(x) \\ &\leq C. \end{aligned}$$

Our estimates of T_1 and T_2 yield (4.1), and the proof of Lemma 4.1 is complete. \square

Acknowledgements

The author thanks Professors Y. X. Han and L. X. Yan for helpful discussions and suggestions on this paper. The author would like to thank the referee for a careful reading of the manuscript, giving numerous valuable suggestions to improve its mathematical accuracy.

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