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DIVISION GRADED ALGEBRAS IN THE BRAUER-WALL GROUP

FRANCIS COGHLAN AND PETER HOFFMAN

ABSTRACT. We show that every element in the Brauer-Wall group of a field with characteristic different from 2 is represented uniquely by a division graded algebra, (*i.e. homogeneous* elements are invertible) but, of course, not necessarily by a graded (division algebra). This is a fairly direct consequence of Wall's structure theory for central simple $Z/2$-graded algebras.

Let F be a field in which $2 \neq 0$. Recall (from [**L**; pp. 70–75], for example) that the Brauer group, $\mathrm{Br}(F)$, of F can be taken to consist of congruence classes of central simple algebras over F under the operation of tensor product. Every such 'CSA' is isomorphic to the full matrix algebra, $D^{n\times n}$, for a unique n and a central division algebra D which is unique up to isomorphism. This means that every element of $\mathrm{Br}(F)$ is uniquely represented by a division algebra. We shall prove the analogue of this last statement for the Brauer-Wall group, $\mathrm{BW}(F)$. The motivation is an application of the Brauer-Wall group to the Schur index in representation theory [**H**; **T**].

DEFINITIONS. Below we review the definitions from pp. 76–77 of Lam [**L**]:

Graded algebra, or, more thoroughly, $\mathbb{Z}/2$-*graded algebra*, or, in stylish but uninformative jargon, *superalgebra* (abbreviated 'GA');

Graded centre, $\hat{Z}(A)$, of a graded algebra A;

CGA, *central graded algebra*;

SGA, *simple graded algebra*;

CSGA, *central simple graded algebra*.

We shall always regard a graded algebra as an ordered pair $A = (A_0, A_1)$ of finite dimensional F-vector spaces (plus a multiplication map); that is, we never bother with inhomogeneous elements. Thus $\hat{Z}_0(A)$ consists of those elements in A_0 which commute with everything in $A_0 \cup A_1$, whereas $\hat{Z}_1(A)$ consists of those elements in A_1 which commute with everything in A_0 but anti-commute with everything in A_1. A CGA is a GA with $\hat{Z}_1 = 0$ and $\hat{Z}_0 = F$. An SGA is one for which there is no pair of subspaces $(I_0, I_1) \subset (A_0, A_1)$ for which $I_0 \oplus I_1$ is a proper non-zero 2-sided ideal in the ungraded algebra $A_0 \oplus A_1$. A CSGA has both properties.

The group $\mathrm{BW}(F)$ consists of 'congruence' classes of CSGA's over F, with operation coming from the graded tensor product; see, for example, [**L**, pp. 95–97].

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A *division graded algebra* is a GA, (D_0, D_1), in which each non-zero element of $D_0 \cup D_1$ is invertible. Note that the ungraded algebra $D_0 \oplus D_1$ might not be a division algebra.

THEOREM 1. *Every element of* BW(F) *contains a division graded algebra; and only one, up to isomorphism.*

(Thus BW(F) may alternatively be regarded as the group whose elements are isomorphism classes of central division graded algebras over F, and whose multiplication has a somewhat awkward definition.)

DEFINITION. Given a division algebra D over F, a non-zero element d of D, and an F-automorphism θ of D which fixes d, define $D\langle\sqrt{d};\theta\rangle$ to be the GA which is (D, eD) as a graded F-module, and where multiplication is determined by that of D, plus requiring that $e^2 = d$ and $xe = e\theta(x)$ for all $x \in D$.

LEMMA 2. *a) For any such* (D, d, θ), *this defines a division graded algebra. Its graded centre is given by:*

$$\hat{Z}_0(D\langle\sqrt{d};\theta\rangle) = Z(D) \cap (+1 \text{ eigenspace of } \theta),$$

which has $Z(D)$ *as an extension of degree at most 2; and*

$$\hat{Z}_1(D\langle\sqrt{d};\theta\rangle) = 0.$$

b) Any division graded algebra either has the form $D\langle\sqrt{d};\theta\rangle$ *for some such* (D, d, θ), *or else has the form* $(D, 0)$.

c) If D *is a central division algebra over* F, *then* $D\langle\sqrt{d};\theta\rangle$ *is a central division graded algebra, hence a CSGA.*

d) Any central division graded algebra over F *is isomorphic to exactly one from the following four possibilities:*

(1) $(D, 0)$ *for a central division algebra* D *over* F;

(2) $D\langle\sqrt{f};\mathrm{id}\rangle$ *for a central division algebra* D *over* F *and a non-zero element* $f \in F$, *uniquely determined as an element of* $\dot{F}/\dot{F}^2$;

(3) $(C \otimes_F F[\sqrt{f}])\langle\sqrt{1};1 \otimes \gamma\rangle$, *where:*

C is a central division algebra over F;

f is a non-zero element in F which is not a square in C, uniquely determined as a (non-trivial) element of $\dot{F}/\dot{F}^2$; and

γ is the unique non-trivial F-automorphism of $F[\sqrt{f}]$;

(4) (D_0, D_1), *depending on* (D, f, s), *where:*

D is a central division algebra over F, uniquely determined up to isomorphism of (ungraded) algebras;

f is a non-zero element in F which is not a square in F, but is a square in D, where f is uniquely determined as a (non-trivial) element of $\dot{F}/\dot{F}^2$;

$s \in Z(D_0)$, and (D_0, D_1) is a grading of D with $s^2 = f$ and D_1 non-zero.

PROOF OF LEMMA 2. Most of the proof of *a)* is mechanical. It is clear that $\hat{Z}_1$ is always trivial for a division graded algebra, since $x^2 = -x^2$ implies that $x = 0$. To check the statement concerning the degree, note that θ maps $Z(D)$ into itself. Since θ^2 is inner automorphism using d, the restriction to $Z(D)$ of θ is an involution. If non-trivial (*i.e.*, if the degree is not 1), then multiplication by any fixed element in the (-1)-eigenspace of θ shows that the (± 1)-eigenspaces have the same dimension, so the degree is 2.

To prove *b)*, if the 1-grading is non-zero, for any non-zero $e \in D_1$, the map sending x to ex shows that $D_0 \cong D_1$. Pick any such e; set $d = e^2$; and define $\theta(\cdots) = e^{-1}(\cdots)e$.

Part *c)* follows immediately from *a)*.

The proof of *d)* consists of specializing Wall's structure theory for CSGA's in [W], by adding the assumption that the CSGA is also a division graded algebra. We shall use the formulation given in [**L**; IV, 3.6 and 3.8]. If the given central division graded algebra has odd type, then 3.6 ensures the existence of a non-zero element, z, of grading one in its *ungraded* centre, and such that $z^2 = (\text{say}) f \in F$. Furthermore, the graded isomorphism class of the given algebra depends exactly on the ungraded isomorphism class of its zero part and the class of f in $\dot{F}/\dot{F}^2$, as required, giving case (2). If the given central division graded algebra has even type, then either case (1) occurs, or the one-grading is non-zero. In the latter event, Lam's 3.8 applies. This yields a non-scalar z in the centre of the zero part, with $z^2 = (\text{say}) f \in F$. Now $f \notin \dot{F}^2$, since otherwise we wouldn't have a division graded algebra. It follows that (3) and (4) of 3.8 yield cases (3) and (4) respectively of our Lemma, as required.

REMARKS. We shall see in the next proof that the division graded algebra in *d)*(4) is actually independent of choice of s. The types in Lemma 2*d)* are partially distinguishable by their dimensions, which take the forms:

$$(1): (m^2, 0); \qquad (2): (m^2, m^2); \qquad (3) \text{ and } (4): (2m^2, 2m^2).$$

PROOF OF THEOREM 1. Here we shall write elements of the Brauer-Wall group using the bracket notation, [; ;], which parametrizes $BW(F)$ by $Br(F) \times \mathbb{Z}/2 \times \dot{F}/\dot{F}^2$ (as a *set*). See [**L**; p. 119, Defn. 3.8].

The classes in the Brauer-Wall group of the division graded algebras, A, in Lemma 2*d)* are as follows:

(1) $[D; 0; 1]$.

(2) $[D; 1; f]$. In this case, $A_0 \oplus A_1$ is not a central F-algebra, and is a division algebra if and only if $f \notin \dot{D}^2$.

(3) $[C; 0; f]$, with $f \notin \dot{C}^2$. In this case, $A_0 \oplus A_1$ is a central F-algebra, and is not a division algebra. It is $C^{2\times 2}$.

(4) $[D; 0; f]$, with $f \in \dot{D}^2 \setminus \dot{F}^2$. Here, $A_0 \oplus A_1$ is a central F-algebra, and is a division algebra.

In cases (1), (2) and (3), the division graded algebra representing the given Brauer-Wall element, as given in Lemma 2, is clearly unique up to isomorphism. In case (4), the set of isomorphism classes of division graded algebras representing $[D; 0; f]$ is evidently in 1-1 correspondence with the set of equivalence classes of elements $s \in D$ for which $s^2 = f$, in the following sense. Fix f in $\dot{F}$—this will really only depend on the class of f in $\dot{F}/\dot{F}^2$. Define two such elements s, t to be equivalent if and only if there is an automorphism $\phi: D \longrightarrow D$ of F-algebras which sends s to $\pm t$. (The next sentence is for motivation; it is not needed in the proof. By the Skolem-Noether Theorem, any automorphism ϕ as above is necessarily an *inner* automorphism). Thus, given two such elements s and t, we need only find a non-zero u such that $us = tu$. But $u = s + t$ clearly works when $s \neq -t$. For any c not commuting with s (which must exist because D is central), the element $u = c - s^{-1}cs$ will work when $s = -t$.

Existence in cases (1), (2) and (3) is obvious. As for (4), consider $[D; 0; f]$ where f is a non-zero element in F which is not a square in F, but is a square in D. Pick any $s \in D$ with $s^2 = f$. Construct a grading as in (4): $D_i := (-1)^i$-eigenspace of conjugation by s.

Since the cases (1) to (4) give a partition of BW(F) into four sets, this completes the proof.

REMARK. We have proved the theorem by making use of the theory which 'reduces' BW(F) to Br(F). Presumably there is a direct proof which proceeds in the graded case by *analogy* with the ungraded case.

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Mathematics Department
Manchester University
Manchester, England
M13 9PL

Pure Mathematics Department
Waterloo, Ontario
N2L 3G1

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