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NON-UNIQUENESS OF THE SOLUTION TO A GENERALIZED DIRICHLET PROBLEM

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It is generally known [1] that the singular partial differential equation

$$\frac{\partial^2 v}{\partial r^2}+\frac{2\nu}{r}\frac{\partial v}{\partial r}+\frac{\partial^2 v}{\partial z^2}=0, \qquad \nu<-\tfrac{1}{2} \tag{1}$$

may not have a unique solution because of the existence of nontrivial representations of zero. This situation arises even more remarkably (e.g. ν need not be $<-\frac{1}{2}$) when the boundary conditions are distributional in nature, i.e. $v(r, z)$ converges in some generalized sense to certain Schwartz distributions at the boundaries. In this note we give an example of a Dirichlet problem with distributional conditions whose solution is not unique.

The following problem was solved in [2]:

Find a function $v(r, z)$ on the domain $0<r<\infty$, $0<z<\infty$, that satisfies Laplace's equation

$$\frac{\partial^2 v}{\partial r^2}+\frac{1}{r}\frac{\partial v}{\partial r}+\frac{\partial^2 v}{\partial z^2}=0 \tag{2}$$

and the boundary conditions:

(a) as $z\to 0^+$, $v(r, z)$ converges in some generalized sense to the distribution $f(r)$ whose support is a compact subset of $0<r<\infty$.

(b) as $z\to\infty$, $v(r, z)$ converges to zero uniformly on $0<r<\infty$.

(c) as $r\to\infty$, $v(r, z)$ converges to zero for every $z>0$.

(d) as $r\to 0^+$, $v(r, z)$ remains finite.

To show non-uniqueness of the solution to the above problem, we replace condition (a) by

$$(\text{a}') \text{ as } z\to 0^+,\ v(r, z)\to 0 \text{ uniformly on } 0<r<\infty,$$

and find a nontrivial solution $v_h(r, z)$ to the resulting problem. Thus by the principle of superposition, some multiple of $v_h(r, z)$ added to the solution in [2] will yield another solution.

As in [2], we set $u(r, z)=(r)^{1/2}v(r, z)$ in (2) and apply the zero-order Hankel transformation with respect to r. The Hankel transform of $u(r, z)$ so obtained can now be inverted by an appeal to the Lipschitz-Hankel integral [3, p. 9]. Thus,

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it is easily shown that a solution to the problem with condition (a′) is given by

$$(3)\qquad v_h(r,z)=\frac{n!}{(z^2+r^2)^{(n+1)/2}}P_n\left[\frac{z}{(z^2+r^2)^{1/2}}\right],\qquad n=\text{odd integer}$$

where $P_n(x)$ is the Legendre polynomial of x of degree n.

It might be mentioned in passing that equation (3) represents the potential on the (r, z) plane due to a multipole located at the origin. The special case of $n=1$ gives the potential due to a dipole:

$$(4)\qquad v_h(r,z)=\frac{z}{(z^2+r^2)^{3/2}}\qquad \text{(see [4, p. 302]).}$$

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