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Diophantine Approximation for Certain Algebraic Formal Power Series in Positive Characteristic

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Abstract. In this paper, we study rational approximations for certain algebraic power series over a finite field. We obtain results for irrational elements of strictly positive degree satisfying an equation of the type

$$\alpha = \frac{A\alpha^q + B}{C\alpha^q},$$

where $(A, B, C) \in (\mathbb{F}_q[X])^2 \times \mathbb{F}_q^\star[X]$. In particular, under some conditions on the polynomials A, B and C, we will give well approximated elements satisfying this equation.

1 Introduction

Let p be a given prime number and $\mathbb{F}_q$ the finite field of characteristic p having q elements. We denote by $\mathbb{F}_q[X]$ the ring of polynomials with coefficients in $\mathbb{F}_q$ and by $\mathbb{F}_q(X)$ the field of fractions of $\mathbb{F}_q[X]$. Let $\mathbb{F}_q((X^{-1}))$ be the field of formal power series

$$\mathbb{F}_q((X^{-1})) = \Big\{ \alpha = \sum_{i \geq n_0} \alpha_i X^{-i} : n_0 \in \mathbb{Z} \quad \text{and} \quad \alpha_i \in \mathbb{F}_q \Big\}.$$

Let $\alpha = \sum \alpha_i X^{-i}$ be any formal power series; we define its polynomial part, denoted $[\alpha]$, by $[\alpha] := \sum_{i \leq 0} \alpha_i X^{-i}$. If $\alpha \neq 0$, then the degree of α is $\deg(\alpha) = \sup\{-i : \alpha_i \neq 0\}$ and $\deg(0) = -\infty$. Thus, we define the non-archimedean absolute value over $\mathbb{F}_q((X^{-1}))$ by $|\alpha| = q^{\deg(\alpha)}$ and $|0| = 0$.

As in the classical context of real numbers, we have a continued fraction algorithm in $\mathbb{F}_q((X^{-1}))$. If $\alpha \in \mathbb{F}_q((X^{-1}))$, we can write

$$\alpha = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots}}} = [a_0, a_1, a_2, \dots]$$

where $a_i \in \mathbb{F}_q[X]$. The a_i are called the partial quotients, and we have $\deg a_i > 0$ for $i > 0$. This continued fraction is finite if and only if $\alpha \in \mathbb{F}_q(X)$. We define two

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sequences of polynomial (P_n) and (Q_n) by $P_0 = a_0$, $Q_0 = 1$, $P_1 = a_0a_1 + 1$, $Q_1 = a_1$, and for any $n \geq 2$,

$$P_n = a_nP_{n-1} + P_{n-2}, \qquad Q_n = a_nQ_{n-1} + Q_{n-2}.$$

We notice that

$$P_nQ_{n-1} - P_{n-1}Q_n = (-1)^{n-1} \quad \text{and} \quad \frac{P_n}{Q_n} = [a_0, a_1, a_2, \dots, a_n];$$

$\frac{P_n}{Q_n}$ is called the n-th convergent of α. We have the important equality

$$\left|\alpha - \frac{P_n}{Q_n}\right| = |a_{n+1}|^{-1}|Q_n|^{-2}.$$

We study the approximation of the elements of $\mathbb{F}_q((X^{-1}))$ by the elements of $\mathbb{F}_q(X)$. In particular, we consider this approximation for the elements of $\mathbb{F}_q((X^{-1}))$ that are algebraic over $\mathbb{F}_q(X)$. In order to measure the quality of rational approximation, we introduce the following notation and definition. Let α be an irrational element of $\mathbb{F}_q((X^{-1}))$.

For all real numbers μ, we define

$$B(\alpha, \mu) = \liminf_{|Q| \to \infty} |Q|^{\mu}|Q\alpha - P|,$$

where P and Q run over polynomials in $\mathbb{F}_q[X]$ with $Q \neq 0$. Now the approximation exponent of α is defined by

$$\nu(\alpha) = \sup\{\mu \in \mathbb{R} : B(\alpha, \mu) < \infty\}.$$

We recall that if $\frac{P_n}{Q_n}$ is a convergent to α, we have

$$|Q_n\alpha - P_n| = |Q_n|^{-\deg Q_{n+1}/\deg Q_n}.$$

Since the best rational approximation to α are its convergents, in the above notation we have

$$\nu(\alpha) = \limsup\Big(\frac{\deg Q_{k+1}}{\deg Q_k}\Big) = 1 + \limsup\Big(\frac{\deg a_{k+1}}{\sum_{1\leq i\leq k}\deg a_i}\Big).$$

It is clear that the approximation exponent can be determined when the continued fraction of the element is explicitly known. Since $|Q_n\alpha - P_n| \leq 1/|Q_n|$, for all irrational $\alpha \in \mathbb{F}_q((X^{-1}))$ we have $\nu(\alpha) \geq 1$. Furthermore Mahler's version of Liouville's theorem says that if $\alpha \in \mathbb{F}_q((X^{-1}))$ is algebraic over $\mathbb{F}_q(X)$ of degree $n > 1$, then $B(\alpha, n-1) > 0$. Consequently, for $\alpha \in \mathbb{F}_q((X^{-1}))$ algebraic over $\mathbb{F}_q(X)$ of degree $n > 1$ we have $\nu(\alpha) \in [1, n-1]$.

We now use the following vocabulary. If $\alpha \in \mathbb{F}_q((X^{-1}))$, we say that:

- α is *badly approximable* if $\nu(\alpha) = 1$ and $B(\alpha, 1) > 0$. This is equivalent to saying that the partial quotients in the continued fraction for α are bounded.
- α is *normally approximable* if $\nu(\alpha) = 1$ and $B(\alpha, 1) = 0$.
- α is *well approximable* if $\nu(\alpha) > 1$.

In 1975, L. Baum and M. Sweet [1] gave the first example of algebraic formal series of degree 3 with bounded partial quotients; it remains the most famous example of algebraic formal series with bounded quotients, along with, of course, the quadratic formal series. Some years later, W. Mills and D. Robbins [8] gave the explicit continued fraction expansion of this series; they have also given more example of other algebraic formal series by explaining their continued fraction expansion. For this they highlighted an algorithm to give the explicit continued fraction expansion of a special class of formal series, the class of hyperquadratics that we will call $\mathcal{H}$. It consists of the irrational elements in $\mathbb{F}_q((X^{-1}))$, satisfying an algebraic equation of the form

$$\alpha = \frac{A\alpha^q + B}{C\alpha^q + D}$$

where $A, B, C, D \in \mathbb{F}_q[X]$ and $q \in p^{\mathbb{N}\setminus\{0\}}$. Notice that the cubic of Baum and Sweet belongs to this class. There are other examples with $\nu(\alpha) = 1$, but no criterion is known. The class $\mathcal{H}$ contains elements which are very well approximated by rationals. A famous example in $\mathbb{F}_p((X^{-1}))$, which was given by K. Mahler in 1949 [7], satisfies the algebraic equation $\alpha = X^{-1} + \alpha^p$.

Later, M. Mkaouar [9] described an algorithm to compute the partial quotients of continued fraction expansion for algebraic formal power series. This enabled him to give the explicit continued fraction expansion of element of class $\mathcal{H}$ if $\deg A > \sup(\deg B, \deg C, \deg D)$. He proved the following theorem.

Theorem 1.1 *Let $P(Y) = A_m Y^m + A_{m-1}Y^{m-1} + \cdots + A_0$ with $A_i \in \mathbb{F}_q[X]$ and $|A_{m-1}| > |A_i|$ for all $i \neq m-1$. Then P has a unique $\alpha \in \mathbb{F}_q((X^{-1}))$ satisfying $|\alpha| > 1$. Moreover, $[\alpha] = -[A_{m-1}/A_m]$ and the formal power series $\beta = 1/(\alpha - [\alpha])$ has the same property as α.*

It gradually became apparent that the elements of class $\mathcal{H}$ deserve special consideration. A. Lasjaunias [6] explored the property of the sequence of partial quotients being bounded or unbounded for the elements of the class $\mathcal{H}$. He showed that a great number of irrational elements of this class will have an unbounded sequence of partial quotients. This is certainly the case if the critical bound $1/(q-1)(\deg \Delta)$ (where $\Delta = AB - AC$) is less than q, and it is necessarily so if $\deg \Delta$ is fixed and q is large enough. In general, if in the continued fraction expansion of an element α of class $\mathcal{H}$ there exists a partial quotient with degree more than $1/(q-1)(\deg \Delta)$, then α admits unbounded partial quotients. In other words, the bound $1/(q-1)(\deg \Delta)$ is a critical value. We can also see this result in [10]. We express this result in the following lemma.

Lemma 1.2 *Let α be an irrational element in $\mathbb{F}_q((X^{-1}))$ satisfying*

$$\alpha = \frac{A\alpha^q + B}{C\alpha^q}, \quad \text{where } (A, B, C) \in \left(\mathbb{F}_q[X]\right)^2 \times \mathbb{F}_q^\star[X]. \tag{1.1}$$

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If α has a partial quotient, other than the first, with degree $> \deg(BC)/(q-1)$, then α has unbounded partial quotients.

The rational approximation of elements of class $\mathcal{H}$ has been studied also by J. Voloch [11] and more deeply by B. de Mathan [2]. They showed that if the partial quotients in the continued fraction expansion of such elements α are unbounded, then $\nu(\alpha) > 1$. In other words, there are no normally approximable elements of class $\mathcal{H}$. By the work of de Mathan [2], we know, moreover, that for elements of class $\mathcal{H}$, the approximation exponent $\nu(\alpha)$ is a rational number and $B(\alpha, \nu(\alpha)) \neq 0, \infty$. Many elements of class $\mathcal{H}$ are well approximable, but the question of determining which elements of class $\mathcal{H}$ are badly approximable remains open.

Now we will show how it is possible in some cases to compute the approximation exponent for an algebraic element, without knowing the whole continued fraction. This is possible if this approximation exponent is large enough, that is to say not close to 1. Lasjaunias [5] has given applications to algebraic elements that are of class $\mathcal{H}$ and also to others that are not. The basic idea in the following result is due to Voloch [11]. It has been improved by de Mathan [3].

Theorem 1.3 *Let $\alpha \in \mathbb{F}_q((X^{-1}))$. Assume that there is a sequence $(P_n, Q_n)_{n\geq 0}$, with $P_n, Q_n \in \mathbb{F}_q[X]$, satisfying the following conditions:*

(i) *There are two real constants $\lambda > 0$ and $\mu > 1$, such that*

$$|Q_n| = \lambda |Q_{n-1}|^{\mu} \quad \textit{and} \quad |Q_n| > |Q_{n-1}| \textit{ for all } n \geq 1.$$

(ii) *There are two real constants $\rho > 0$ and $\gamma > 1 + \sqrt{\mu}$, such that*

$$\left| \alpha - \frac{P_n}{Q_n} \right| = \rho |Q_n|^{-\gamma} \textit{ for all } n \geq 0.$$

Then we have $\nu(\alpha) = \gamma - 1$. Further, if $\gcd(P_n, Q_n) = 1$ for $n \geq 0$, we have $B(\alpha, \nu(\alpha)) = \rho$, and if the sequence $(\gcd(P_n, Q_n))_{n\geq 0}$ is bounded, then $B(\alpha, \nu(\alpha)) \neq 0, \infty$.

Note that equation (1.1) belongs to class $\mathcal{H}$ with $D = 0$. In this paper we are interested in rational approximation to the unique irrational solution of the equation (1.1) of strictly positive degree. In fact, in each result obtained, if we have $\deg(A) > \max(\deg(C), \deg(B))$ as assumption, then by the following lemma we get results for the unique solution α of equation (1.1) with $|\alpha| > 1$.

Lemma 1.4 ([4]) *Let $P(Y) = A_d Y^d + A_{d-1} Y^{d-1} + \cdots + A_0$, where $A_i \in \mathbb{F}_q[X]$, then P has a unique root in $\mathbb{F}_q((X^{-1}))$ with absolute value > 1 if and only if $|A_{d-1}| > |A_i|$ for all $i \neq d-1$.*

First, we will give some results concerning the behavior of the partial quotients of the formal power series α satisfying (1.1). Secondly, we will study the approximation exponent of such α.

2 Results

In the following two theorems, we give, under some assumption on the polynomials A, B, and C, elements satisfying equation (1.1) that are well approximated by rationals, without giving the exact value of their approximation exponents.

Theorem 2.1 *Let $\alpha \notin \mathbb{F}_q(X)$ be a solution of equation (1.1). If*

$$(q-1)\deg A > q\deg C + \deg B,$$

then α admits unbounded partial quotients.

Proof Set $|A| = |X|^n$, $|C| = |X|^i$, and $|B| = |X|^m$, where n, m, and i are positive integers, then $|\alpha| = |X|^{n-i}$. Furthermore ,

$$|\alpha - \frac{A}{C}| = \frac{|B|}{|C||\alpha|^q} = \frac{|X|^m}{|X|^i|X|^{q(n-i)}} = \frac{1}{|X|^{2i}|X|^{q(n-i)-i}|X|^{-m}}.$$

We have $qn - qi - i - m > (i+m)/(q-1)$ equivalent to $(q-1)n > qi + m$. Then by Lemma 1.2 α admits unbounded partial quotients. ∎

Theorem 2.2 *Let $\alpha \notin \mathbb{F}_q(X)$ be a solution of (1.1) such that $\deg A > \deg C$ and $B = -1$. Let $P = [A/C]$. Suppose that there exists $D \in \mathbb{F}_q[X]$ such that $P^qA - DC = 1$. Then α admits unbounded partial quotients.*

Proof We have

$$|\alpha - \frac{D}{P^q}| = |\frac{A}{C} - \frac{D}{P^q} - \frac{1}{(C)\alpha^q}| = |\frac{1}{(C)P^q} - \frac{1}{(C)\alpha^q}| = \frac{1}{|CP^q|}|1 - \frac{\alpha}{P}|^q.$$

Consider that $\alpha = P + a_1X^{-1} + a_2X^{-2} + \cdots + a_qX^{-q} + a_{q+1}X^{-q-1} + \cdots$. Assume that $a_1 \neq 0$. Then

$$\frac{\alpha}{P} = 1 + \frac{a_1X^{-1} + \cdots}{P}.$$

Thus,

$$|1 - \frac{\alpha}{P}|^q = \frac{1}{|XP|^q} = \frac{1}{|X|^q|P|^q}.$$

Consequently,

$$|\alpha - \frac{D}{P^q}| = \frac{1}{|C||P^q|}\frac{1}{|X|^q|P|^q}.$$

Then there exists a partial quotient of α of degree $\deg C + q > \frac{\deg C}{q-1}$. So we conclude by Lemma 1.2 that α admits unbounded partial quotients. ∎

Example 2.3 *Let $\alpha \notin \mathbb{F}_q(X)$ be a solution of the equation*

$$(X^q + 1)\alpha^{q+1} - (X^{q+1} + X - 1)\alpha^q - 1 = 0.$$

Then α admits unbounded partial quotients.

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For this example, it suffices to note that $[\alpha] = X = P$ and

$$X^q(X^{q+1} + X - 1) - (X^{q+1} - 1)(X^q + 1) = 1.$$

Continued fractions are naturally useful in this setting, since the best rational approximations are obtained by truncating the continued fraction expansion. Unfortunately, it is generally difficult to obtain the continued fraction expansion of a given algebraic power series. Indeed, the effect of basic operations such as addition or multiplication on the continued fraction expansion is not clear at all. In the following theorem we show that using the algorithm given by Theorem 1.1 we can, under some assumptions on polynomials A, C, and B, describe explicitly the continued fraction expansion of the root α of equation (1.1). Then by computing the value of the approximation exponent of α, we find that it is maximal, and so we get a family of formal series very well approximated.

Theorem 2.4 *Let $\alpha \notin \mathbb{F}_q(X)$ be a solution of equation (1.1). Suppose that B and C divides A. Then the continued fraction expansion of α is $[a_0, \dots, a_n, \dots]$, where*

$$a_n = \Big(\frac{A}{C}\Big)^{\frac{q^{n+1}+(-1)^n}{q+1}} \Big(\frac{A}{B}\Big)^{\frac{q^n+(-1)^{n+1}}{q+1}}$$

and $\nu(\alpha) = q$.

Proof Let $\alpha_0 = \alpha$ and $\alpha_{s+1} = \frac{1}{\alpha_s - [\alpha_s]}$, then α_s verifies the equation

$$A_s\alpha_s^{q+1} + B_s\alpha_s^q + C_s\alpha_s + D_s = 0,$$

where

$$\begin{aligned}
A_{s+1} &= A_s a_s^{q+1} + B_s a_s^q + C_s a_s + D_s\\
B_{s+1} &= A_s a_s^q + C_s\\
C_{s+1} &= A_s a_s + B_s\\
D_{s+1} &= A_s.
\end{aligned}$$

Then

$$a_0 = -\Big[\frac{-A}{C}\Big] = \frac{A}{C} \quad \text{and} \quad a_{s+1} = -\Big[\frac{B_{s+1}}{A_{s+1}}\Big]$$

for all $s \geq 1$. Then we show using a simple recursion on s that

$$a_s = (\frac{A}{C})^{\frac{q^{s+1}+(-1)^s}{q+1}} (\frac{A}{B})^{\frac{q^s+(-1)^{s+1}}{q+1}}.$$

Consequently the approximation exponent for α is:

$$\begin{aligned}
\nu(\alpha) &= 1 + \limsup\Big(\frac{\deg a_{n+1}}{\sum_{1\leq s\leq n}\deg a_s}\Big)\\
&= 1 + \limsup\Big(\frac{(q^{n+2} + (-1)^{n+1})\deg\frac{A}{C} + (q^{n+1} + (-1)^n)\deg\frac{A}{B}}{(\sum_{1\leq s\leq n} q^{s+1} + (-1)^s)\deg\frac{A}{C} + (\sum_{1\leq s\leq n} q^s + (-1)^{s+1})\deg\frac{A}{B}}\Big)\\
&= q.
\end{aligned}$$

∎

In [2], de Mathan gave a method for determining the approximation exponents for formal power series belonging to the class $\mathcal{H}$ that is based on the construction of chains of convergents. At the end of his paper he gave application in $\mathbb{F}_2((X^{-1}))$ to the irrational root of the equation $\alpha^3 = R$ where $R \in \mathbb{F}_2(X)$ with degree a multiple of 3. Lasjaunias [5] continued this work by studying the approximation exponent of the irrational root of the equation $\alpha^n = R$ in $\mathbb{F}_q((X^{-1}))$, with $R \in \mathbb{F}_q(X)$, where $\deg(R)$ is a multiple of n and $\gcd(n, p) = 1$. We see that the results of de Mathan are very interesting and allow us to determine the approximation exponent of several other power series. The following theorem is a direct application of the results of de Mathan and gives us a good minoration of the approximation exponent of the solution of equation (1.1) in terms of $\deg A$, $\deg B$, and $\deg C$, and then gives us a family of very well approximated power series.

Theorem 2.5 *Let $\alpha \notin \mathbb{F}_q(X)$ be a solution of equation (1.1). Suppose that*

$$\deg A > q \deg C + \deg B.$$

Then

$$\nu(\alpha) \geq q - \frac{q \deg C + \deg B}{\deg A}.$$

Proof With the same notations of [2], by [2, Lemma 3] we set $P_0 = A$, $Q_0 = C$, and we have

$$\left| \alpha - \frac{A}{C} \right| = \left| \frac{A\alpha^q + B}{C\alpha^q} - \frac{A}{C} \right| = \left| \frac{B}{C\alpha^q} \right| = \left| \frac{BC^{q-1}}{A^q} \right|.$$

Then (A, C) verifies [2, Lemma 2(i), (ii), and (iii)] if $\deg A > q \deg C + \deg B$. Thus we construct from $P_0/Q_0 = A/C$ a sequence of rational approximations P_n/Q_n. We obtain

$$\begin{cases} r_n = \dfrac{((q+1)\deg A - q \deg C - \deg B)q^{n+1}}{q-1} - \dfrac{(2m - c)}{q-1}, \\ r_0 = q \deg A - (q-1) \deg C - \deg B, \\ m = q \deg A - (q-1) \deg C, \\ c = \deg B + \deg C, \end{cases}$$

where $r_n = -\deg(\alpha - P_n/Q_n)$, $m = \deg(C\alpha^q)$ and $c = \deg(BC)$.

Since we have

$$\deg(Q_n) = q^n \deg C + \frac{m(q^n - 1)}{q-1} = \frac{(\deg A)q^{n+1}}{q-1} - \frac{m}{q-1},$$

then

$$\deg Q_n^* = \frac{(q \deg A - q \deg C - \deg B)q^{n+1}}{q-1} - \frac{(m-c)}{q-1},$$

where Q_n^* is the denominator of the next convergent. So

$$\nu(\alpha) \geq \frac{q \deg A - q \deg C - \deg B}{\deg A} \geq q - \frac{q \deg C + \deg B}{\deg A}.$$

Since $\frac{q \deg C + \deg B}{\deg A} < 1$, $\nu(\alpha) > q - 1$, hence $\nu(\alpha) \in]q-1, q]$. ∎

Theorem 1.3 allows us to find the approximation exponent of well approximated formal series. Despite its importance, there are few applications of this result. In the following theorem and under suitable assumptions on $\deg A$, $\deg C$, and the polynomial B, we give interesting results for the approximation exponent of the solution of equation (1.1) by giving a precise value of the exponent. So this gives us a new family of well approximated formal series.

Theorem 2.6 *Let $\alpha \notin \mathbb{F}_q(X)$ be a solution of equation (1.1) such that $\gcd(A, C) = 1$ and $B = 1$.*

(i) *If $q \geq 3$ and $\deg A = q \deg C$, then $\nu(\alpha) = q - 1$ and*

$$B(\alpha, q-1) = \frac{1}{|C|^{\frac{q^3 - 3q^2 + 3q - 1}{q-1}}}.$$

(ii) *If $q \geq 4$ and $\deg A = (q-1)\deg C$, then $\nu(\alpha) = \frac{q^2 - 2q}{q-1}$ and*

$$B\Big(\alpha, \frac{q^2 - 2q}{q-1}\Big) = \frac{1}{|C|^{\frac{q^3 - 4q^2 + 4q}{q-1}}}.$$

Proof (i) We consider the following sequence: $P_0 = A$ $Q_0 = C$ and for $n \geq 1$,

$$P_n = AP_{n-1}^q + Q_{n-1}^q \qquad Q_n = CP_{n-1}^q.$$

Then for all $n \geq 0$,

$$\Big|\alpha - \frac{P_n}{Q_n}\Big| = \Big|\frac{1}{C\alpha^q} - \frac{Q_{n-1}^q}{CP_{n-1}^q}\Big| = \frac{1}{|C|^{q^2-q+1}}\Big|\frac{\alpha}{C^{q-1}} - \frac{P_{n-1}}{C^{q-1}Q_{n-1}}\Big|^q$$
$$= \frac{1}{|C|^{2q^2-2q+1}}\Big|\alpha - \frac{P_{n-1}}{Q_{n-1}}\Big|^q.$$

We show by recursion that for all $n \geq 0$,

$$\Big|\alpha - \frac{P_n}{Q_n}\Big| = \frac{1}{|C|^{\frac{(2q^2-2q+1)(q^n-1)}{q-1}}}\Big|\alpha - \frac{P_0}{Q_0}\Big|^{q^n},$$

since

$$\Big|\alpha - \frac{P_0}{Q_0}\Big| = |\alpha - \frac{A}{C}| = \frac{1}{|C|^{q^2-q+1}}.$$

Then

$$\Big|\alpha - \frac{P_0}{Q_0}\Big|^{q^n} = \frac{1}{|C|^{q^n(q^2-q+1)}}.$$

So

$$\Big|\alpha - \frac{P_n}{Q_n}\Big| = \frac{1}{|C|^{\frac{q^{n+3} - 2q^2 + 2q - 1}{q-1}}}.$$

Secondly, we have for all $n \geq 1$ $Q_n = CP_{n-1}^q$ and since $|P_{n-1}| = |C|^{q-1}|Q_{n-1}|$, $|Q_n| = |C|^{q^2-q+1}|Q_{n-1}|^q$. Again by recursion we show that

$$|Q_n| = |C|^{\frac{(q^2-q+1)(q^n-1)}{q-1}}|Q_0|^{q^n} = |C|^{\frac{q^{n+2}-q^2+q-1}{q-1}}.$$

So we obtain for all $n \geq 0$,

$$\left|\alpha - \frac{P_n}{Q_n}\right| = \frac{1}{|C|^{\frac{q^3-3q^2+3q-1}{q-1}}|Q_n|^q} = \frac{1}{|C|^{(q-1)^2}|Q_n|^q}.$$

Then by Theorem 1.3 we conclude that $\nu(\alpha) = q - 1$. Further, since $\gcd(A, C) = 1$, $\gcd(P_n, Q_n) = 1$ for all $n \geq 0$, and so $B(\alpha, q-1) = \frac{1}{|C|^{\frac{q^3-3q^2+3q-1}{q-1}}}$.

(ii) With the same method as the proof of (i), we consider the following sequence: $P_0 = A$ $Q_0 = C$ and for $n \geq 1$,

$$P_n = AP_{n-1}^q + Q_{n-1}^q \qquad Q_n = CP_{n-1}^q.$$

We obtain for all $n \geq 0$,

$$\left|\alpha - \frac{P_n}{Q_n}\right| = \frac{1}{|C|^{\frac{q^{n+3}-q^{n+2}-q^{n+1}-2q^2+4q-1}{q-1}}}.$$

We show also that $|Q_n| = |C|^{q^{n+1}-q+1}$. So we obtain for all $n \geq 0$,

$$\left|\alpha - \frac{P_n}{Q_n}\right| = \frac{1}{|C|^{\frac{q^3-4q^2+4q}{q-1}}|Q_n|^{\frac{q^2-q-1}{q-1}}}.$$

Then by Theorem 1.3 we conclude that $\nu(\alpha) = (q^2 - 2q)/(q - 1)$. Further, since $\gcd(A, C) = 1$ then $\gcd(P_n, Q_n) = 1$ for all $n \geq 0$, and so

$$B\left(\alpha, \frac{q^2-2q}{q-1}\right) = \frac{1}{|C|^{\frac{q^3-4q^2+4q}{q-1}}}. \qquad \blacksquare$$

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