ON CERTAIN TRIPLE INTEGRAL EQUATIONS WITH TRIGONOMETRIC KERNELS

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1. In this note we formally solve the following triple integral equations,

$$\int_0^\infty g(\lambda)\sin\lambda x\,d\lambda = f_1(x) \quad (0 < x < \alpha), \tag{1}$$

$$\int_0^\infty \lambda^{-1} g(\lambda)\tanh\lambda h\sin\lambda x\,d\lambda = f_2(x) \quad (\alpha < x < \beta), \tag{2}$$

$$\int_0^\infty g(\lambda)\sin\lambda x\,d\lambda = f_3(x) \quad (\beta < x < \infty), \tag{3}$$

where $f_1(x)$, $f_2(x)$ and $f_3(x)$ are integrable for $0 < x < \alpha$, $\alpha < x < \beta$ and $\beta < x < \infty$, respectively, and the function $g(\lambda)$ is assumed to satisfy sufficient conditions for the Fourier sine transform to exist. A special case of this system arose in a problem concerned with transistors.

2. Solution of equations. We follow the normal procedure for triple integral equations (see **[3]**, for example), and write

$$\int_0^\infty g(\lambda)\sin\lambda x\,d\lambda = p(x) \quad (\alpha < x < \beta), \tag{4}$$

so that $p(x)$ is integrable over $[\alpha, \beta]$. By using the inversion theorem for the Fourier sine transform we obtain

$$g(\lambda) = \frac{2}{\pi}\int_0^\alpha f_1(x)\sin\lambda x\,dx + \frac{2}{\pi}\int_\alpha^\beta p(x)\sin\lambda x\,dx + \frac{2}{\pi}\int_\beta^\infty f_3(x)\sin\lambda x\,dx. \tag{5}$$

Substitute (5) into (2) and interchange the order of integration of the resulting double integrals to obtain

$$\int_0^\alpha f_1(y)H(x,y)dy + \int_\alpha^\beta p(y)H(x,y)dy + \int_\beta^\infty f_3(y)H(x,y)dy = \frac{\pi}{2}f_2(x) \quad (\alpha < x < \beta), \tag{6}$$

where

$$H(x,y) = \int_0^\infty \lambda^{-1}\tanh\lambda h\sin\lambda y\sin\lambda x\,d\lambda. \tag{7}$$

The interchanges in the order of integrations can be justified by applying the results of sections 4.3, 4.431(I) and 4.44(II) of [**4**]. However, by [**1**, p. 516],

$$H(x,y)=\tfrac{1}{2}\int_0^\infty \lambda^{-1}\tanh\lambda h\,\{\cos(x-y)\lambda-\cos(x+y)\lambda\}d\lambda$$
$$=\tfrac{1}{2}\log\left|\coth\left\{\frac{\pi}{4h}(x-y)\right\}\Big/\coth\left\{\frac{\pi}{4h}(x+y)\right\}\right|. \qquad (8)$$

We may rewrite the right-hand side of (8) as

$$\tfrac{1}{2}\log\left|\frac{\sinh\gamma x+\sinh\gamma y}{\sinh\gamma x-\sinh\gamma y}\right|=\tfrac{1}{2}\psi(x,y),\quad \text{say},$$

where $\gamma=\pi/2h$, and hence we can rewrite (6) as

$$\int_0^\alpha f_1(y)\psi(x,y)dy+\int_\alpha^\beta p(y)\psi(x,y)dy+\int_\beta^\infty f_3(y)\psi(x,y)dy=\pi f_2(x)\quad(\alpha<x<\beta). \qquad (9)$$

Let

$$\pi^2L(x)=\pi f_2(x)-\int_0^\alpha f_1(y)\psi(x,y)dy-\int_\beta^\infty f_3(y)\psi(x,y)dy. \qquad (10)$$

Then we can rewrite (9) as

$$\int_\alpha^\beta p(y)\log\left|\frac{\sinh\gamma x+\sinh\gamma y}{\sinh\gamma x-\sinh\gamma y}\right|dy=\pi^2L(x)\quad(\alpha<x<\beta). \qquad (11)$$

Now, since $\sinh\gamma x$ is a positive monotonic increasing function in (α,β), (11) can be solved by a result due to Parihar [**2**]. The solution is

$$p(y)=\frac{s'(y)}{m(y)}\left\{\int_\alpha^\beta\frac{m(x)L'(x)}{s(y)-s(x)}dx+\tfrac{1}{2}B(s(\beta)^{\frac{1}{2}})/F\left(\frac{\pi}{2},\left(1-\frac{s(\alpha)}{s(\beta)}\right)^{\frac{1}{2}}\right)\right\}\quad(\alpha<y<\beta) \qquad (12)$$

where

$$s(y)=\sinh^2\gamma y,\ m(y)=\sinh\gamma y\,\{(\sinh^2\gamma y-\sinh^2\gamma\alpha)(\sinh^2\gamma\beta-\sinh^2\gamma y)\}^{\frac{1}{2}}$$

and

$$B=\frac{\pi\sinh\gamma\beta}{F\left[\frac{\pi}{2},\frac{\sinh\gamma\alpha}{\sinh\gamma\beta}\right]}\int_\alpha^\beta\frac{s'(x)L(x)dx}{m(x)}-2\int_\alpha^\beta\frac{s'(y)dy}{m(y)}\int_\alpha^\beta\frac{m(x)L'(x)dx}{s(y)-s(x)}, \qquad (13)$$

where the first integral in (12) and the last integral in (13) are to be understood in the sense of their principal values. Once $p(y)$ has been obtained we use (5) to obtain $g(\lambda)$.

In the problem about transistors the functions $f_1(x)$, $f_2(x)$ and $f_3(x)$ have the values 0, -1 and 0 respectively. For this case the analysis is greatly simplified, and we find that the particular form of $p(y)$ is given by the expression:

$$p(y)=\frac{-\gamma\cosh\gamma y\sinh\gamma\beta}{\{(\sinh^2\gamma y-\sinh^2\gamma\alpha)(\sinh^2\gamma\beta-\sinh^2\gamma y)\}^{\frac{1}{2}}F(k)}\quad(\alpha<y<\beta), \qquad (14)$$

where $\gamma=\pi/2h$, $k=\sinh\gamma\alpha/\sinh\gamma\beta$ and $F(k)$ is the complete elliptic integral of the first kind.

Hence

$$g(\lambda)=\frac{-\gamma\sinh\gamma\beta}{F(k)}\int_\alpha^\beta\frac{\cosh\gamma y\sin\lambda y\,dy}{\{(\sinh^2\gamma y-\sinh^2\gamma\alpha)(\sinh^2\gamma\beta-\sinh^2\gamma y)\}^{\frac{1}{2}}}. \tag{15}$$

3. A further result. If in the relevant intervals, $f_1(x)$, $f_2(x)$ and $f_3(x)$ are non-constant differentiable functions, we can obtain the solution of (1)–(3) with cos λx instead of sin λx by differentiating with respect to x. However if $f_1(x)$, $f_2(x)$ and $f_3(x)$ are constants we cannot solve the problem in this manner and we have to obtain the solution by a method similar to that of section 2. The solution for the particular case $f_1(x)=0$ for $0<x<\alpha$, $f_2(x)=-1$, for $\alpha<x<\beta$, and $f_3(x)=0$ for $\beta<x<\infty$, and with $h=\pi$, is given by:

$$p(x)=\frac{-\cosh\frac{\beta}{2}}{16\cosh\frac{x}{2}\left\{\left(\cosh^2\frac{x}{2}-\cosh^2\frac{\alpha}{2}\right)\left(\cosh^2\frac{\beta}{2}-\cosh^2\frac{x}{2}\right)\right\}^{\frac{1}{2}}}\frac{1}{F(k)},$$

where $k=\cosh(\alpha/2)/\cosh(\beta/2)$; hence $g(\lambda)$ is given by

$$g(\lambda)=\frac{8}{\pi\lambda}\int_\alpha^\beta p(x)\sinh x\cos x\lambda\,dx.$$

REFERENCES

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