THE P^n-INTEGRAL

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(Received 3 July 1970; revised 20 April 1971)

(Communicated by B. Mond)

1. Introduction

In [5] James defined an nth order Perron integral, the P^n- ntegral, and developed its properties. His proofs are often indirect, using properties of the C_kP-integrals of Burkill, [3]. In this paper a simpler definition of the P^n-integral is given — the original and not completely equivalent definition, was probably chosen as James considered this integral as a special case of one defined in terms of certain symmetric derivatives, [5], when end points of the interval of definition had naturally to be avoided. We then give direct proofs of the basic results, give a characterization of P^n-primitives, and connect the integral with certain work of Denjoy, [4].

2. Peano Derivatives

Suppose F is a real-valued function defined on the bounded closed interval $[a,b]$ then if it is true that for $x_0 \in]a,b[$

$$F(x_0+h) - F(x_0) = \sum_{k=1}^{r} \alpha_k \frac{h^k}{k!} + 0(h^r), \quad \text{as } h \to 0 \tag{1}$$

where $\alpha_1, \cdots, \alpha_r$ depend on x_0 only, but not on h, or r then α_k, $1 \leqq k \leqq r$, is called the *Peano derivative of order k of F at* x_0, and we write $\alpha_k = F_{(k)}(x_0)$. If F possesses derivatives $F_{(k)}(x_0)$, $1 \leqq k \leqq r-1$, write

$$\frac{h^r}{r!}\gamma_r(F;x_0,h) = F(x_0+h) - F(x_0) - \sum_{k=1}^{r-1} \frac{h^k}{k!} F_{(k)}(x_0), \tag{2}$$

and define

$$\begin{aligned} \bar{F}_{(r)}(x_0) &= \limsup_{h\to 0} \gamma_r(F;x_0,h) \\ \underline{F}_{(r)}(x_0) &= \liminf_{h\to 0} \gamma_r(F;x_0,h) \end{aligned} \tag{3}$$

Further, by restricting h to be positive, or negative, in (1), or (3) we can define one-sided Peano derivatives, written $F_{(k),+}(x_0)$, $F_{(k),-}(x_0)$, $\bar{F}_{(k),+}(x_0)$ etc. If we say $F_{(k)}$, $1 \leqq k \leqq r$, exists in (a, b) we will mean that $F_{(k)}$ exists in $]a, b[$ and that the appropriate one sided derivates exist at those of the points a and b that are in (a, b).

3. Riemann Derivatives

Let $x_0, \cdots, x_r$ be $(r+1)$ distinct points from $[a, b]$ then the *rth divided difference of F at these* $(r+1)$ *points* is defined by

$$(4) \qquad V_r(F) = V_r(F; x_r) = V_r(F; \{x_k\}) = V_r(F; x_0, \cdots, x_r) = \sum_{k=0}^{r} \frac{F(x_k)}{w'(x_k)},$$

where

$$(5) \qquad w(x) = w_r(x) = w_r(x; x_k); \text{ etc.} = \prod_{k=0}^{r} (x - x_k).$$

Given the $(r+1)$ points P_k, $0 \leqq k \leqq r$, with coordinates $(x_k, F(x_k))$, $0 \leqq k \leqq r$, respectively, there is a unique polynomial of degree at most r passing through these points given by

$$(6) \qquad \pi_r(F; x; P_k) = \pi_r(x; P_k) = \pi_r(x; x_0, \cdots, x_r) \text{ etc.} = \sum_{k=0}^{r} F(x_k) \prod_{\substack{j=0 \\ j<k}}^{r} \frac{(x-x)_j}{(x_k - x_j)}.$$

Using the divided difference we now define another derivative. Suppose all of $x, x_1, \cdots, x_r$ are in $[a, b]$ and

$$(7) \qquad x_k = x + h_k, \ 0 \leqq k \leqq r, \text{ with } \quad 0 \leqq |h_0| < \cdots < |h_r|,$$

then the *rth Riemann derivative of F at x* is defined by

$$(8) \qquad D^r F(x) = \lim_{h_r \to 0} \cdots \lim_{h_0 \to 0} r! V_r(F; x_k)$$

if this iterated limit exists independently of the manner in which the h_k tend to zero, subject only to (7). In a similar manner we define the upper (lower) or one-sided derivatives by replacing all $r+1$ limits by upper (lower) or one-sided limits; these will be written $\bar{D}^r F(x)$, $\bar{D}^r_+ F(x)$ etc. If we say $D^r F$ exists in (a, b) we make the same gloss as for $F_{(r)}$. The following result can be found in [2].

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THEOREM 1. (a) *If* $x \in [a, b[$ *then* $D^r_+ F(x) = F_{(r),+}(x)$, *provided one side exists.*

(b) *If* $F_{(r)}$ *exists at all points of* $[a, b]$ *then* $F_{(r)}$ *possesses both the Darboux property and the mean-value property.*

The usual rth order derivative of F at x, $x \in (a, b)$, will be written $F^{(r)}(x)$.

4. *n*-Convexity

A real-valued function F defined on the closed bounded interval $[a, b]$ is said to be *n-convex on* $[a, b]$ iff for all choices of $n + 1$ distinct points, $x_0, \cdots, x_n$, in $[a, b]$, $V_n(F; x_k) \geqq 0$, [2, 5]. If $-F$ is n-convex then F is said to be *n-concave.* The only functions that are both n-convex and n-concave are polynomials of degree at most $n - 1$, [2, Lemma 1].

If $n = 1$ this is just the class of monotonic increasing functions and for $n = 2$ it is the class of convex functions; (the class $n = 0$ is just the class of non-negative functions, but we will usually only be interested in $n \geqq 1$).

Various properties of n-convex functions were obtained in [2]. We state them here for convenience.

THEOREM 2. *Let* $P_k = (x_k, y_k)$, $1 \leqq k \leqq n$, $n \geqq 2$, $a \leqq x_1 < \cdots < x_n \leqq b$, *be any n distinct points on the graph of the function F. Then F is n-convex iff for every such set of n points the graph lies alternately above and below the graph of the polynomial* π_{n-1} $F; x; P_{12})$, *lying below if* $x_{n-1} \leqq x \leqq x_n$. *If so then* $\pi_{n-1}(x; P_k) \leqq F(x)$, $x_n \leqq x \leqq b$; *and* $\pi_{n-1}(x; P_k) \leqq F(x)$ $(\geqq F(x))$ *if* $a \leqq x < x_1$, *n being even (odd).*

The definition, (6), of $\pi_r(x; P_k)$ can be extended to cover the case when not all of the P_k are distinct. Thus if only s of these points are distinct then besides giving the values at the s points, a total of $r + 1 - s$ derivatives must also be given — either $r + 1 - s$ derivatives all at one point, or $r + 1 - s$ first derivatives at $r + 1 - s$ distinct points, (when $r + 1 - s \leqq s$), etc. Of the many possible extensions to Theorem 2 we state

THEOREM 3. *Let* $P_k = (x_k, y_k)$, $1 \leqq k \leqq r$, $a_1 \leqq x_1 < \cdots < x_r \leqq b$, *be r distinct points on the graph of the function F. Suppose that* $F_{(s),+}(x_1)$ *exists,* $1 \leqq s \leqq n - r$. *Then Theorem 2 holds if* $\pi_{n-1}(x; P_k)$ *is taken to have*

$$\pi_{n-1}(x_s; P_k) = F(x_s),\ 1 \leqq s \leqq r,\ \pi^{(s)}_{n-1}(x_1; P_k) = F_{(s)+}(x_1),\ 1 \leqq s \leqq n - r,$$

and if P_1 *is considered as* $n - r + 1$ *points at and to the right of* P_1 *but to the left of* P_2.

THEOREM 4. (a) *If F is n-convex on* $[a, b]$ *and* $P_k = (x_k, y_k)$, $1 \leqq k \leqq n$ *are n distinct points on the graph of F,* $a \leqq x_1 < b$, *let* $x_k = x_1 + \varepsilon_k h$, $0 < \varepsilon_2 <$

$\cdots < \varepsilon_n$. *Then as* $h \to 0+$, $\pi_{n-1}(x; P_k)$ *converges uniformly to the right tangent polynomial at* x_1,

$$\tau_{n-1\,+}(F; x; x_1) = \tau_+(x) = F(x_1) + \sum_{k=1}^{n-2} \frac{(x-x_1)^k}{k!} F^{(k)}(x_1)$$
$$+ \frac{(x-x_1)^{n-1}}{(n-1)!} F_{(n-1)\,+}(x_1), \qquad x_1 \leqq x \leqq b.$$

Further on the right of x_1, $\tau_+ \leqq F$.

(b) *A similar result holds for the left tangent polynomial at* x_1, $\tau_-(x; x_1)$, $a \leqq x \leqq x_1$, $a < x_1 \leqq b$. *However in this case if n is even (odd) then on the left of* x_1 $\tau_- \leqq F(\geqq F)$.

(c) *At all but a countable set of points* x_1, *a similar result holds for the tangent polynomial at* x_1, $\tau(x_1; x)$, $a < x < b$, $a < x_1 < b$. *However if n is even the graph of* τ *lies below that of F, whereas if n is odd the graphs cross,* τ *being above on the left of* x_1, *and below on the right of* x_1.

THEOREM 5(a). *If F is an n-convex function on* $[a, b]$ *and*

$$a \leqq x_1 < \cdots < x_n \leqq b, \; a \leqq y_1 < \cdots < y_n \leqq b, \; x_k \leqq y_k, \; 1 \leqq k \leqq n,$$

then $V_{n-1}(F; x_k) \leqq V_{n-1}(F; y_k)$.

(b). *If F is n-convex in* $[a, b]$, $|F| \leqq K$, *then*

$$|F_{(k)}(x)| \leqq AK \sup \left\{\frac{1}{(b-x)^k}, \frac{1}{(x-a)^k}\right\}, \qquad 0 \leqq k \leqq n-1$$

where A is a constant independent of k, F and x, and where if $k = n-1$ *the derivative is to be interpreted as* $\sup(|F_{(n-1),+}(x)|, |F_{(n-1),-}(x)|)$.

(c). *If F is n-convex on* $[a, b]$, $a \leqq x \leqq y \leqq b$, $a \leqq x + h \leqq y$, *and* $x \leqq y + k \leqq b$ *then*

$$\gamma_{n-1}(F; x; h) \leqq F_{(n-1)\,-}(y) \text{ and } F_{(n-1),+}(x) \leqq \gamma_{n-1}(F; y; k),$$

where γ_{n-1} *is as defined in* (2).

THEOREM 6. *If F is n-convex on* $[a, b]$, $a < \alpha < \beta < b$, $E_\lambda = \{x : \alpha \leqq x \leqq \beta$ *and* $\bar{F}_{(n)}(x) \geqq \lambda\}$ *then*

$$km^*(E_\lambda) \leqq 2n\{F_{(n-1),-}(\beta) - F_{(n-1),+}(\alpha)\}. \tag{9}$$

THEOREM 7. *If F is n-convex then* (a)$F^{(r)}$ *is* $(n-r)$*-convex,* $1 \leqq r \leqq n-2$,
(b) $F^{(n-1)}$ *exists at all except a countable set of points,*
(c) $F^{(n)}$ *exist a.e.*

THEOREM 8. *If* $n \geqq 2$, *and*
(i) $F_{(1)}, \cdots, F_{(n-1)}$ *exist in* $[a, b]$,

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(ii) $\bar{F}_{(n),+}(x) \geqq 0,\ x \in [a,b] \sim E,\ |E| = 0,$

(iii) $\bar{F}_{(n),+}(x) > \infty,\ x \in [a,b] \sim C,$ *C countable then F is n-convex.*

If F is defined on $[a,b]$ as well as $F_{(k)}$, $1 \leqq k \leqq n-1$, let us write

$$\begin{aligned}\omega_n(F;a,b) &= \omega_n(a,b)\\ &= \max\Big\{ \sup_{a<x<b} |(x-a)\gamma_n(F;a;x-a)|,\\ &\qquad \sup_{a<x<b} |(b-x)\gamma_n(F;b;b-x)|\Big\};\end{aligned} \tag{10}$$

this quantity was introduced by Sargent, [10].

In [2] it was shown that F is the difference of two n-convex functions iff $\sum_{k=1}^{m} \omega_n(F;a_k,b_k) < K$ for all finite sets of non-overlapping intervals, $[a_k,b_k]$, $1 \leqq k \leqq m$. It was also shown in [2] that if F is n-convex then

$$\omega_n(F;a,b) \leqq n\{F_{(n-1)}(b) - F_{(n-1)}(a)\}. \tag{11}$$

5. The P^n-integral

Let f be a real-valued function on $[a,b]$ then a function M continuous on $[a,b]$ is called *a P^n-major function of f on* $[a,b]$, or just *a major function*, if there is no ambiguity, iff

(a) $M_{(k)}$ exists and is finite on $[a,b]$, $1 \leqq k \leqq n-1$,

(b) $\underline{M}_{(n)}(x) \geqq f(x),\ x \in [a,b] \sim E,\ |E| = 0,$

(c) $\underline{M}_{(n)}(x) > -\infty,\ x \in [a,b] \sim C,$ C countable,

(d) $M_{(k)}(a) = 0,\ 0 \leqq k \leqq n-1.$

If $-m$ is a major function of $-f$ then m is called *a minor function*, or more precisely, *a P^n-minor function of f on* $[a,b]$. It is clear from these definitions that f need only be finite a.e.

This definition differs from that in [5], in the use of the end-points in (a), in the existence of the sets E and C, and also in condition (d). In [5] the major function is normalized instead by requiring it to be zero on a given set of n distinct points $a_1, \cdots, a_n$; let us call these functions, *J-major functions over* $(a_k)_{1 \leqq k \leqq n}$.

Standard arguments, using Theorems 3 and 8, show that if M is any major function, m any minor function; then $M - m$ is a non-negative n-convex function, [compare with 5, Lemmas 5.1, 5.2].

For $a < c \leqq b$ define

$\bar{F}(b) = \bar{P}^n - \int_a^c f = \inf\{t : t = M(c),\ M$ is a major function of $f\}$, *the upper P^n-integral of f on* $[a,b]$; in a similar way we define *the lower P^n-integral*, $\underset{\sim}{P}^n - \int_a^c f$: if there is no ambiguity we will just write, $\int^c f$ or, $\int_a^c f$. If $\bar{F}(c) = \underset{\sim}{F}(c)$, we write the common value, $F(c)^c = P^n - \int_a^c f$ (or just $\int_a^c f$), and if further this value is finite we say f is *P^n-integrable on* $[a,c]$.

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If f is P^n-integrable in the sense of [5], let us say f is *$J-P^n$-integrable over* $(a_k\,;\,b)$.

THEOREM 9. (a) *f is P^n-integrable on $[a,b]$ iff for each $\varepsilon>0$ there exists a major function M and a minor function m, such that* $0 \leqq M(b)-m(b) \leqq \varepsilon$.

(b) *f is P^n-integrable on $[a,b]$ iff given $\varepsilon>0$ there exist continuous functions M, m on $[a,b]$ such that (i) $M_{(k)}$, $m_{(k)}$ exist and are finite in $[a,b]$,* $1 \leqq k \leqq n-1$ (ii) $-\infty \neq M_{(n)}(x) \geqq f(x) \geqq m_{(n)} \neq \infty$ (iii) $M_{(k)}(a) = m_{(k)}(a) = 0$, $0 \leqq k \leqq n-1$(iv) $0 \leqq M(b)-m(b) < \varepsilon$.

(c) *If f is P^n-intergrable on $[a,b]$, $f=g$ a.e. then g is P^n-integrable on $[a,b]$ and $\int_a^b f = \int_a^b g$.*

PROOF. (a) Immediate.

(b) The case $n=1$ is due to McGregor, [6].

Obviously we have to show that if f is P-integrable then there exist functions M, m satisfying the conditions (i)–(iv) of (b) with $n=1$. Since f is P-integrable there exists functions M, m as in (a), with $n=1$: if $F = P-\int_a^x f$ then F' exists and is finite almost everywhere (Theorem 20 below or [8, p. 202]), further $F-m$ and $M-F$ are monotonic increasing (Theorem 10(a) below) and so $m = F-(F-m)$, $M = F+(M-F)$ have finite derivatives almost everywhere. Let $E = \{x\colon$ either $M'\ (x) = \pm\infty$, $m\ (x) = \pm\infty$, $\underline{M}(x) = -\infty$ or $\bar{m}'\ (x) = \infty\}$ then E is of measure zero and can be covered by a set $\tilde{E}$ that is a G_δ is also of measure zero and hence by a result due to Zahorski, [12], there is a function w on $[a,b]$ such that (i) w is absolutely continuous, (ii) w' exists everywhere. (iii) if $x\in\tilde{E}$, $w'(x)=\infty$, (iv) if $x\notin\tilde{E}$, $0 \leqq w'(x) < \infty$, (v) $w(a)=0$, $w(b)<\varepsilon$. Now define $\tilde{m} = m-w$ $\tilde{M} = M+w$ then we see that they are the required functions since (i) $\tilde{M}$, $\tilde{m}$ are continuous on $[a,b]$, (ii) if $x\in\tilde{E}$, $\tilde{M}'(x) \geqq \underline{M}'$ $(x)+w'(x) = \infty$ and so $M'(x)$ exists with value ∞; if $x\notin\tilde{E}$ $\tilde{M}'(x)$ exists and is finite, (iii) similarly $\tilde{m}'$ exists everywhere in $[a,b]$, (iv) $\tilde{M}' \geqq f \geqq \tilde{m}'$,(v) $\tilde{m}(a) = \tilde{M}(a) = 0$, (vi) $0 \leqq M(b)-m(b) \leqq \varepsilon$. The general case follows simiarly using the extension of Zahorski's function introduced in [2, Theorem 16]

(c) Immediate.

THEOREM 10. (a) *For all major functions M, minor functions m, of f, $M-\tilde{F}$ and $F-m$ are non-negative n-convex functions.*

(b) *$\tilde{F}_{(k)}$ exists in $]a,b[$ $1 \leqq k \leqq n-2$; $\tilde{F}_{(n-1)}$ exists except on a countable set.*

(c) *If f is P^n-integrable then $F_{(n-1)}$ exists on $]a,b[$.*

(d) *If f is P^n-integrable $F(a) = F_{(k)}(a) = 0$, $1 \leqq k \leqq n-1$.*

PROOF. (a) Immediate.

(b) Immediate using (a), the definition of M and Theorem 7.

(c) By Theorem 1 and Theorem 5(a) and (b) if g is n-convex in $[a,b]$, $|g| < K$ then if $a<\alpha<x_1,\cdots,x_n \leqq \beta < b$, $|V_{n-1}(g;x_k)| < KA$, A depending on α, β but not on $x_1, \cdots, x_n$. So taking $g = F - M$, M a suitable major function of f we have

$$|V_{n-1}(F; x_k) - V_{n-1}(M; x_k)| < K\varepsilon.$$

Letting $x_k \to x$, $1 \leqq k \leqq n$, the existence of $F_{(n-1)}(x)$ follows from that of $M_{(n-1)}(x)$, and Theorem 1. Thus $F_{(n-1)}$ exists in $]a, b[$.

(d) Immediate since F lies between two functions M, and m, both of which are $0(x-a)^{n-1}$ near a.

Corollary 11. *If f is P^n-integrable then* (a) *for every major function M, and every minor function m, $M - F$ and $F - m$ are r-convex on $[a, b]$, $0 \leqq r \leqq n$,* (b) *$F_{(r)}(b)$ exists, $1 \leqq r \leqq n-1$.*

Proof. (a) The cases $r = 0$, n are just Theorem 10(a). By Theorem 5, and using the notation introduced there, since $M - F$ is n-convex, we have that $V_{n-1}(M - F; x_k) \geqq V_{n-1}(M - F; z_k)$. Letting $z_k \to a$, $1 \leqq k \leqq n$ we have by Theorem 10(d), that $V_{n-1}(M - F, x_k) \geqq 0$; that is, $M - F$ is $(n-1)$-convex. In a similar way we can show that $M - F$ is k-convex, $1 \leqq k \leqq n-2$, and that $F - m$ is k-convex, $1 \leqq k \leqq n-1$.

(b) Since, from (a), $M - F$ is $(k+1)$-convex, $1 \leqq k \leqq n-1$, and $V_k(M; x_j) = V_k(F; x_j) + V_k(M - F; x_j)$ it follows, by Theorem 5, that $\lim_{\substack{x_j \to b \\ 0 \leqq j \leqq k}} V_k(F; x_j)$ exists. Further $M - F$ is k-convex, so $V_k(M - F; x_j) \geqq 0$ and so $V_k(M : x_j) \geqq V_k(F; x_j)$; similarly $V_k(F; x_j) \geqq V_k(m; x_j)$ and so since both $M_{(k)}(b)$ and $m_{(k)}(b)$ exist the above limit is finite.

Theorem 12. (a) *If f is P^n-integrable on $[a, b]$ it is P^n-integrable on any sub-interval $[\alpha, \beta]$. Further, if F is the P^n-integral of f on $[a, b]$, then*

$$\int_\alpha^x f = \int_a^x f - \tau_{n-1}(F; x; \alpha), \qquad \alpha \leqq x \leqq \beta.$$

(b) *If f is P^n-integrable on $[a, b]$ then it is $J - P^n$-integrable over $(a_k : b)$ and*

$$J - P^n - \int_{(a_k)}^b f = P^n - \int_a^b f - \pi_{n-1}(F; b; a_k),$$

F being the P^n-integral of f.

Proof. (a) If $\varepsilon > 0$ and M a major function of f such that $0 \leqq M(b) - F(b) \leqq \varepsilon$, then since $M - F$ is k-convex we have by Theorem 5(b) that
$0 \leqq (M-F)_{(k)}(\alpha)\,(b-a)^k \leqq A\,\varepsilon$.

If we write M^* for $M - \tau^n(M; \alpha)$ and define F^* similarly then

$$0 \leqq M^*(\beta) - F^*(\beta) \leqq \underline{F}\varepsilon.$$

Since B does not depend on M this is sufficient to prove (a).

(b) Let M be a major function of the type occurring in Theorem 9 (b), then it is immediate that $M^* = M - \pi_{n-1}(M; a_k)$ is a J-major function. Defining m^* in a similar way, we have that

$$|M^*(b) - m^*(b)| \leqq M(b) - m(b) + |\pi_{(n-1)}(M - m; b; a_k)|$$

which by Theorem 9 (b), and Definition 5.3 of [5] is sufficient to complete the proof.

If $n > 1$ the converse of Theorem 12(b) is not true in general. Consider

$$f(x) = (1-x)^{-3/2}, \ -1 < x < 1;$$
$$= 0, \ x = \pm 1.$$

Then $F(x) = (\sqrt{3} - 2)x + 1 - (1 - x^2)^{1/2}$ is the $J - P^2$-integral of f over $(0, 1/2; x)$. However f is not P^2-integrable on $[-1, 1]$ since $F'(-1) = -\infty$.

Corollary 13. *If f is P^n-integrable on $[a, b]$, and F is its P^n-integral, $\varepsilon > 0$, then a major function M and a minor function m can be chosen so that if $R = M - F$, $r = F - m$* then

(12) $$0 \leqq \max\{R_{(k)}(x), r_{(k)}(x)\} \leqq \varepsilon, \qquad a \leqq x \leqq b, \ 0 \leqq k \leqq n-1.$$

(b) *If f is P^n-integrable on $[a, b]$ and on $[b, c]$ then f is P^n-integrable on $[a, c]$. Further if F^1 is the P^n-integral of f on $[a, b]$, F^2 the P^n-integral of f on $[b, c]$ then*

$$F(x) = F^1(x), \qquad a \leqq x \leqq b$$
$$= F^2(x) + \tau_{n-1,-}(F^1; x; b), \qquad b \leqq x \leqq c$$

is the P^n-integral of f on $[a, c]$.

Proof. (a) Since $R_{(k)}(x)$, $0 \leqq k \leqq n-1$, exists for $a \leqq x \leqq b$, it follows from Theorem 1(b) that it suffices to prove (12) for $a \leqq x < b$. If $[\alpha, \beta]$ is any subinterval of $[a, b]$, $a < \alpha < \beta < b$, then the first inequality obtained in the proof of Theorem 12(a) implies (12) holds in $[\alpha, \beta]$.

Let $\beta_0 = a < \beta < \beta_2 \cdots < b$, with $\lim_{j \to \infty} \beta_j = b$; and let ε_j, $j \geqq 0$ be a sequence of positive numbers to be specified later.

Let R^j be chosen to satisfy (12) in $[\beta_j, \beta_{j+1}]$, with $\varepsilon = \varepsilon_j$; since in fact R^j is defined on $[\beta_j, b]$ we can also require that $0 \leqq R^j \leqq \varepsilon_0$ on that interval.

Define the functions P^j and Q^j on $[\beta_j, \beta_{j+1}]$, $j \geqq 0$, inductively as follows.

$$P^0 = 0, \qquad Q^0 = P^0 + R^0$$
$$P^j(x) = \tau(Q^{j-1}; x; \beta_j), \qquad Q^j = P^j + R^j.$$

Then,

$$|Q^0_{(k)}| \leqq \varepsilon_0, \qquad 1 \leqq k \leqq n-1$$

$$|Q^j_{(k)}| \leqq \varepsilon_j + \sum_{i=k}^{n-1} Q^{j-1}_{(i)}(\beta_j) \frac{(\beta_{j+1}-\beta_j)^{i-k}}{(i-k)!}$$

Choose β_j, ε_j, $j \geqq 0$ so that

$$|Q^j_{(k)}| \leqq \varepsilon, \qquad 0 \leqq k \leqq n-1, \qquad j \geqq 0.$$

Now define

$$\begin{aligned} R(x) &= Q^j(x), \qquad \beta_j \leqq x < \beta_{j+1} \\ &= \lim_{y \to b} R(y), \qquad x = b. \end{aligned}$$

It can then be checked that $M = R + F$ is a major function of f with (12) satisfied.

A similar construction can be used to obtain a suitable minor function.

(b) Let M^1 be a major function of f on $[a, b]$ chosen so that (12) holds with $\varepsilon = \varepsilon_1 e^{c-b}$ and let M^2 be any major function of f on $[b, c]$. If then

$$\begin{aligned} M(x) &= M^1(x), \qquad a \leqq x \leqq b \\ &= M^2(x) + \tau_{n-1,-}(M^1; x; b), \qquad b \leqq x \leqq c, \end{aligned}$$

M is a major function of f on $[a, c]$ and

$$0 \leqq F(c) - M(c) \leqq F^2(c) - M^2(c) + \varepsilon_1 ;$$

this is sufficient to prove (b).

THEOREM 14. *If F is a real-valued function on $[a, b]$ such that* (a) $F_{(k)}$ *exists in* $[a, b]$, $1 \leqq k \leqq n-1$, (b) $F_{(n)}(x)$ *exists*, $x \in [a, b] \sim E$, $|E| = 0$, (c) $\bar{F}_{(n)}$, $\underline{F}_{(n)}$ *are finite everywhere on a countable set then if* $f(x) = F_{(n)}(x)$, $x \in [a, b] \sim E$, *and is zero elsewhere then f is P^n-integrable and*

$$\int_a^x f = F(x) - \tau_{n-1,+}(F; x; a).$$

PROOF. Immediate.

The converse of this is less immediate and is proved later, Theorem 20 below.

THEOREM 15. *If f is P^n-integrable on $[a, b]$ then f is P^{n+1}-integrable on $[a, b]$ and*

$$P^{n+1} - \int_a^b f = \int_a^b (P^n - \int_a^x f) dx.$$

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PROOF. This is given in [5, Theorem 7.2], although in the present case of unsymmetric derivatives the details are much simpler.

6. The P^n - and C_{n-1} -Integrals

As a result of the above modifications in the definition of the P^n-integral the relationship between this scale of integrals and the Cesàro-Perron scale of Burkill, [3], is much neater.

The C_0P–integral is the classical Perron integral. The C_nP-integral is defined by induction as follows.

(i) A function f is *C_n–continuous on* $[a,b]$ if it is $C_{n-1}P$-integrable and

$$\lim_{h\to 0} \frac{n}{h^n} C_{n-1}P - \int_x^{x+h} (x+h-t)^{n-1} f(t)dt = f(x),$$

for every x in $[a,b]$.

(ii) If f is $C_{n-1}P$-integrable on $[a,b]$ then the *upper C_n-derivative of f at x* is

$$C_n\bar{D}f(x) = \limsup_{h\to 0} \frac{n+1}{h}\left\{\frac{n}{h^n} C_{n-1}P - \int_x^{x+h} (x+h-t)^{n-1} f(t)dt - f(x)\right\}.$$

The *lower C_n-derivative of f at x* is similarly defined.

(iii) If f is defined on $[a,b]$ then M is called a *C_nP-major function of f on* $[a,b]$, iff

(a) M is C_n-continuous on $[a,b]$,

(b) $C_n\underline{D}M(x) \geqq f(x), \quad x \in [a,b] \sim E, \quad |E| = 0,$

(c) $C_n\underline{D}M(x) > -\infty, \quad x \in [a,b] \sim C, \quad C$ countable,

(d) $M(a) = 0$.

A *C_nP-minor function* is defined in a similar manner.

(iv) If for every $\varepsilon > 0$ there is a C_nP-major function M and a C_nP-minor function m such that $|M(b) - m(b)| < \varepsilon$ then f is said to be *C_nP-integrable in* $[a,b]$.

This definition is more general than that in [3] because of the existence of the exceptional sets E and C. However just as Theorem 9 (b) shows that the existence of these sets does not widen the scope of the P^n-integral it can also be shown that the above definition is equivalent to the usual one; see for instance the foot note on page 162 of [1].

THEOREM 16. *f is P^n-integrable on $[a,b]$ iff it is $C_{n-1}P$-integrable in $[a,b]$. If F is the P^n-integral of f then*

$$F_{(n-1)}(x) = C_{n-1}P - \int_a^x f \tag{13}$$

$$F(x) = P - \int_a^x C_1P - \int_a^x C_2P - \int_a^x \cdots C_{n-1}P - \int_a^x f.$$

PROOF. (a) If f is $C_{n-1}P$-integrable then the proof of Theorem 9.1 in [5] shows f is P^n-integrable. The proof now has fewer awkward details and can include the end points of $[a, b]$ in its argument.

(b) If f is P^n-integrable then as in [5, Theorem 11.1], if M is a P^n-major function then $M_{(n-1)}$ is a $C_{n-1}P$-major function. Further, by (12), we can choose M so that $0 \leqq F_{(n-1)}(x) - M_{(n-1)}(x) \leqq \varepsilon$ for all of x, $a \leqq x \leqq b$, which completes the proof.

It is seen from (13) that if F is a P^n-integral then $F_{(k)}$ is C_k-continuous, $0 \leqq k \leqq n-1$, [5, Lemma 11.1]. This is one place where C_k-concepts give information not obtainable directly; there seems to be no other continuity concept that describes the bounds set on the lack of ordinary continuity of Peano derivatives.

It follows from Theorem 16 and [9] that the P^n-integral can be given a descriptive definition. Following the spirit of this paper we will do this directly in the following section.

7. The D^n-Integral

Most of the concepts introduced in this section are based on ideas due to Sargent, [9, 10]; the notation has been changed slightly to agree better with the present work.

A function F is said to be AC^*_n *over (or on) a bounded set* E iff (a) $F_{(n-1)}$ exists in some interval containing E, and (b) for every $\varepsilon > 0$ there is an $\delta > 0$ such that, using notation of (10),

$$\sum_{k=1}^{m} \omega_n(a_k, b_k) < \varepsilon$$

for all finite sets of non-overlapping intervals, $[a_k, b_k]$, $1 \leqq k \leqq m$, with end points in E, and such that

$$\sum_{k=1}^{m} (b_k - a_k) < \delta.$$

A function F is ACG^*_n *over (or on) a bounded set* E iff (a) $F_{(n-1)}$ exists in some interval containing E and (b) $E = \cup_{k+N} E_k$ with f being AC^*_n on each E_k, $k \in N$; where N is the set of natural numbers.

If $n = 1$ these concepts reduce to the classical ones of AC^* and ACG^* respectively, [8]. The main properties of these classes of functions are collected in

LEMMA 17. (a) *If F is AC^*_n over a set E then* (i) *F is AC^*_n over $\bar{E}$,* (ii) *$F_{(n-1)}$ is AC over E,* (iii) *$F_{(n-1)}$ is approximately derivable a.e. on E, $F_{(n)} = AD\, F_{(n-1)}$ a.e., and $F_{(n)}$ is Lebesgue integrable on E,* (iv) *if E is a bounded closed set with contiguous intervals $[a_k, b_k]$, $k \in N$ then $\sum_{k \in N} \omega_n(a_k, b_k) < \infty$.*

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(b) *If F is such that $F_{(n-1)}$ exists in some interval containing a bounded closed set E and* (ii) *and* (iv) *of* (a) *hold then F is AC^*_n on E.*

(c) *F is ACG^*_n on $[a,b]$ iff* (i) *$F_{(n-1)}$ exists in $[a,b]$ and* (ii) *$[a,b] = \cup_{k+N}Q_k$, Q_k being closed and F being AC^*_n on Q_k, $k \in N$.*

(d) *If F, G are ACG^*_n over $[a,b]$ and $F_{(n)} = G_{(n)}$ a.e. then* (i) *$F-G$ is a polynomial of degree at most $(n-1)$,* (ii) *$\gamma_n(F;x;h) = \gamma_n(G;x;h)$, $a \leqq x \leqq b$, $a \leqq x+h \leqq b$.*

Proofs. The proofs of (a), (c), (d) are either immediate or are in [10]; the proof of (b) is an adaption of the proof of the similar result in [9].

A function f is said to be *D^n-integrable on* $[a,b]$ iff there is a function F such that (a) F is ACG^*_n on $[a,b]$, (b) $F_{(k)}(a) = 0$, $1 \leqq k \leqq n-1$, (c) $F_{(n)}(x) = f(x)$ a.e. Further we call F the D^n-integral of f, and write $F(x) = D^n - \int_a^x f$. It follows from Lemma 17 that if such an F exists it is unique and from Theorem 10 and [9, 10] that the P^n- and D^n-integrals are completely equivalent. This we now prove directly.

Theorem 18. *Suppose f is P^n-integrable on every $[\alpha,\beta]$, $a<\alpha<\beta<b$ and put $I(\alpha,\beta) = \int_\alpha^\beta f$. Suppose further that*

$$\text{(a)}\qquad \lim_{\alpha\to a} \frac{I(\alpha,\beta)}{(\alpha-a)^n{}_{-1}} = 0,$$

and (b) *there is a polynomial p of degree at most $n-1$ such that*

$$\lim_{\beta\to b} \frac{I(\alpha,\beta)-p(\beta)}{(b-\beta)^{n-1}} = 0,$$

then f is P^n-integrable on $[a,b]$ and

$$\int_a^b f = \lim_{\substack{\alpha\to a\\ \beta\to b}} I(\alpha,\beta).$$

Proof. Let us put

$$\begin{aligned} F(x) &= 0, \qquad x=a,\\ &= \lim_{\alpha\to a} I(\alpha,x), \qquad a<x<b,\\ &= \lim_{y\to b} F(y), \qquad x=b, \end{aligned}$$

Then $F_{(k)}(x)$ exists, $1 \leqq k \leqq n-1$, and $a \leqq x \leqq b$; further $F_{(k)}(a) = 0$, $1 \leqq k \leqq n-1$. We show that F is the P^n-integral of f on $[a,b]$.

Let $a < \cdots < x_{-1} < x_0 < x_1 < \cdots b$ with $a = \lim_{k\to-\infty} x_k$ $b = \lim_{k\to\infty} x_k$;

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write $I_k(x)$ for $I(x_{k-1}, x)$, $x_{k-1} \leqq x < x_k$; suppose $\varepsilon_k > 0$, $k \in Z$, (where Z is the set of integers), is a sequence of positive numbers to be specified later.

Let M_k be a major function of f on $[x_{k-1}, x_k]$ such that

$$0 \leqq M_k(x) - I_k(x) < \varepsilon_k \inf\{(x-a)^{n-1}, \quad (b-x)^{n-1}\}$$

and put $R_k = M_k - I_k$.

Now define

$$\begin{aligned} M(x) &= F(x) + \sum_{v=-\infty}^{k-1} R_v(x_v) + R_k(x), \qquad x_{k-1} \leqq x < x_k \\ &= 0, \qquad x = a, \\ &= F(b) + \sum_{v=-\infty}^{\infty} R_v(x_v), \qquad x = b. \end{aligned}$$

Then for $\alpha \leqq 0$ and $-\alpha$ large enough

$$0 \leqq M(x) - F(x) \leqq (x-a)^{n-1} \sum_{v=-\infty}^{k-1} \varepsilon_k, \qquad x_{k-1} \leqq x < x_k$$

and so by suitable choice of $\{\varepsilon_k\}$, $\alpha \leqq 0$, we see that $(M-F)_{(k)}(a) = 0$ and so that $M_{(k)}(a) = 0$, $1 \leqq k \leqq n-1$. Similarly if $\alpha \geqq 0$ and large enough

$$0 \leqq (M(b) - F(b)) - (M(x) - F(x)) \leqq (b-x)^{n-1} \sum_{v=k}^{\infty} \varepsilon_k,$$

$x_{k-1} \leqq x < x_k$; from which it is easy to deduce that $M_{(k)}(b)$ exists, $1 \leqq k \leqq n-1$, if $\{\varepsilon_k\}$, $\alpha \geqq 0$ are chosen suitably.

Finally we can still choose ε_k, $k \in N$ so that $0 \leqq M - F \leqq \varepsilon$, for any $\varepsilon > 0$.

This, together with a similar construction for a minor function completes the proof.

The conditions of Theorem 18 cannot be relaxed as is seen by the following example, [4]. Let

$$\begin{aligned} F(x) &= x^{n+\alpha} \sin x^{-p}, \qquad 0 < x \leqq 1, \\ &= 0, \qquad x = 0, \end{aligned}$$

$n \geqq 2$, an integer, $0 < \alpha < 1$, $p \geqq n + \alpha - 1$. Then $F_{(j)}(x)$ exists for all j $0 < x \leqq 1$, $F_{(j)}(0)$ exists $1 \leqq j \leqq n$. Thus if $f(x) = F_{(n+2)}(x)$, $\quad 0 < x \leqq 1$ Thus if

$$\begin{aligned} f(x) &= F_{(n+2)}(x), \qquad 0 < x \leqq 1 \\ &= 0, \qquad x = 0. \end{aligned}$$

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Then f is $P^{(n+2)}$-integrable on $[\varepsilon, 1]$ for all ε but is not $P^{(n+2)}$-integrable on $[0,1]$, since $F_{(n+1)}(0)$ does not exist.

LEMMA 19. *If E is a closed bounded set with end points a and b and contiguous intervals $[a_k, b_k]$ in $[a, b]$, $k = 1, 2, \cdots$ and if* (a) *f is Lebesgue integrable on E,*

(b) *f is P^n-integrable on each $[a_k, b_k]$, $k = 1, 2, \cdots$,*

(c) *$\sum_{k=1} \omega_n(F^k; a_k, b_k) < \infty$ then f is P^n-integrable on $[a, b]$, and*

$$P^n - \int_a^b f = \frac{1}{(n-1)!} L - \int_a^b 1_Q(t)(b-t)^{n-1} f(t)dt$$

$$+ \sum_k \tau_{n-1,-}(F^k; b; b_k),$$

(where $1\mu(t) = 1$, $t \in Q$, $= 0$, $t \notin Q$).
where F^k is the P^n-integral of f on $[a_k, b_k]$, $k = 1, 2, \cdots$.

PROOF. An adaption of a similar result of Sargent, [10].

THEOREM 20. *If f is P^n-integrable on $[a, b]$ and F is its P^n-integral then $\bar{F}_{(n)}$ exists and equals f a.e.*

PROOF. Let $\varepsilon > 0$ and M a major function chosen so that $0 \leqq R_{(k)} = (M - F)_{(k)} \leqq \varepsilon$, $0 \leqq k \leqq n-1$, (12).

Then R is n-convex and so by Theorem 6 $\bar{R}_{(n)} < \infty$ a.e. and hence $\underline{F}_{(n)} > -\infty$ a.e.

Now let $E = \{x; \bar{R}_{(n)}(x) \geqq \lambda\} \cap [\alpha, \beta]$, $a < \alpha < \beta < b$; then by Theorem 6,

$$_m{}^*E_\lambda \leqq \frac{2n\varepsilon}{\lambda}, \quad \text{hence} \quad m^*E\lambda = 0.$$

If $E_0 = E \cup C$, E, C being the sets associated with M by virtue of it being a major function and if $x \in [a, b] \sim (E_0 \cup E_k)$ then $\underline{F}_{(n)}(x) \geqq f(x) - k$, which implies that this last inequality holds almost everywhere. From this we easily deduce that $\underline{F}_{(n)}(x) \geqq f(x)$ almost everywhere.

Since $-f$ is also P^n-integrable we immediately see that $\bar{F}_{(n)}(x) \leqq f(x)$ and is finite, almost everywhere.

This completes the proof.

Before we state and prove the main result the concept of AC^*_n has to be extended as follows.

A function F is said to be *AC^*_n-below over (or on) a bounded set E* iff (a) $F_{(n-1)}$ exists in some interval containing E and (b) for every $\varepsilon > 0$ there is a $\delta > 0$ such that

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$$(14)\quad \sum_{k=1}^{m} \min\left\{ \inf_{a_k<x<b_k} [(x-a_k)\gamma_n(F;a_k;x-a_k)], \inf_{a_k<x<b_k} [(b_k-x)\gamma_n(F,b_k,b_k-x)] \right\} > -\varepsilon$$

for all finite sets of non-overlapping intervals $[a_k, b_k]$, $1 \leqq k \leqq m$, with end points in E and such that

$$\sum_{k=1}^{m} (b_k - a_k) < \delta.$$

In a similar way if (14) is replaced by

$$\sum_{k=1}^{m} \max \sup_{a_k<x<b_k} [(x-a_k)\gamma_n(F;a_k;x-a_k)], \sup_{a_k<x<b_k} [b_k-x]\gamma_n(F;b_k;b_k-x)] < \varepsilon$$

we say F is AC^*_n*-above over*, (*or on*), E.

The concepts of ACG^*_n-above and ACG^*_n-below are defined in the obvious way.

Clearly F is AC^*_n iff F is AC^*_n-above and AC^*_n-below. If $n = 1$ these concepts reduce to the classical ones of AC^*-above and AC^*-below, due to Ridder, [7].

LEMMA 21. *If* $F_{(k)}$, $1 \leqq k \leqq n-1$, *exists in some interval containing the bounded set* E *and if* $F_{(n)}(x) > -\infty$, $x \in E$ *then* F *is* ACG^*_n*-below on* E.

PROOF. Let m and j be integers, m positive

$$E_m(F) = E_m = \left\{ x; x \in E \text{ and } \gamma_n(F;x;h) > -m, \text{ for all } h \text{ such that, } 0 < |h| < \frac{1}{m} \right\},$$

$$E_m^j = E_m \cap \left[\frac{j}{m}, \frac{j+1}{m} \right];$$

then it is sufficient to show F to be AC^*_n-below over each E_m^j.

Let $[a_i, b_i]$, $i = 1, \cdots p$ be non-overlapping intervals with end points in E_m^j, (this set being assumed ,without loss of generality to have more than one point). Then

$$\gamma_n(F;a_i;x-a_i) > -m, \qquad a_i < x < b_i,$$

and so

$$\inf_{a_i<x<b_i} [(x-a_i)\gamma_n(F;a_i;x-a_i)] \geqq -m(b_i-a_i).$$

Thus if $\varepsilon > 0$,

$$\sum_{i=1}^{p} \inf_{a_i<x<b_i} [(x-a_i)\gamma_n(F;a_i;x-a_i)] \geqq -m \sum_{i=1}^{p} (b_i-a_i) > -\varepsilon,$$

provided $\Sigma_{i=1}^{m}(b_i - a_i) < \varepsilon/m$. In a similar way

$$\sum_{i=1}^{p} \inf_{a_i < x < b_i} [(bi - x)\gamma_n(F; bi; bi - x)] > -\varepsilon,$$

which completes the proof.

THEOREM 22. *If f is P^n-integrable on $[a,b]$ it is D^n-integrable on $[a,b]$ to the same value, and conversely.*

PROOF. (a) Let f be P^n-integrable, $\varepsilon > 0$ and M a major function such that

$$0 \leqq R_{(n-1)} = (M - F)_{(n-1)} \leqq \frac{\varepsilon}{2n}.$$

By Lemma 21, $[a,b] = \cup_{k\varepsilon N} E_k$, with M AC^*_n-below on each E_k, $k \in N$. Then there is a $\delta > 0$ such that if $[a_i, b_i]$, $i = 1, \cdots, p$ is any finite set of non-overlapping intervals with end points in E_k and

$\sum_{i=1}^{p} (b_i - a_i) < \delta$, then

$$\begin{aligned}(x - a_i)\gamma_n(F; a_i; (x - a_i) &= (x - a_i)\gamma_n(M; a_i; x - a_i) \\ &\quad - (x - a_i)\gamma_n(R; a_i; x - a_i) \\ &\geqq (x - a_i)\gamma_n(M; a_i; x - a_i) \\ &\quad - n\{R_{(n-1)}(b_i) - R_{(n-1)}(a_i)\} \quad \text{by (11).}\end{aligned}$$

Hence since $R_{(n-1)}$ is monotonic increasing

$$\begin{aligned}\sum_{i=1}^{p} \inf_{a_i < x < b_i} [(x - a_i)\gamma_n(F; a_i; x - a_i)] &\geqq -\frac{\varepsilon}{2} - n\{R_{(n-1)}(b) - R_{(n-1)}(a)\} \\ &\geqq -\varepsilon.\end{aligned}$$

In a similar way we see that

$$\sum_{i=1}^{p} \inf_{a_i < x < b_i} [(b_i - x)\gamma_n(F; b_i; b_i - x)] \geqq -\varepsilon$$

and so we have proved that F is ACG^*_n-below on $[a,b]$.

However since $-f$ is also P^n-integrable, F is also ACG^*_n-above on $[a,b]$ and hence ACG^*_n over $[a,b]$.

This and Theorem 20 shows that f is D^n-integrable and that

$$D^n - \int_a^x f = P^n - \int_a^x f, \qquad a \leqq x \leqq b.$$

(b) Suppose now f is D^n-integrable on $[a,b]$ and let $E = \{x; f$ is not P^n-integrable in any neighborhood of $x\}$. Clearly E is closed and let $[a_k, b_k]$ denote its contiguous intervals in $[a,b]$.

If $a_k < \alpha < \beta < b_k$ then f is P^n-integrable on $[\alpha, \beta]$ and if F is the D^n-integral of f on $[a, b]$ then since from the definition of the D^n-integral it is clear that $F - \tau_{n-1}(F; x)$ is the D^n-integral of f on $[\alpha, \beta]$ we have from (a) that

$$P^n - \int_\alpha^\beta f = F(\beta) - \tau_{n-1}(F; \beta; \alpha).$$

Since the right hand side of this equation satisfies the conditions of Theorem 18 on $[a_k, b_k]$ we have that f is P^n-integrable on $[a_k, b_k]$ and, of course,

$$P^n - \int_{a_k}^{b_k} f = F(b_k) - \tau_{n-1}(F; b_k; a_k).$$

Hence, by Corollary 13(b), E is a perfect set.

Suppose now that $E \neq \emptyset$. Since F is ACG^*_n over $[a, b]$ it follows from Lemma 17 that E contains a portion Q such that if c, d are the end points of $\bar{Q}$ and if $[c_k, d_k]$ are the contiguous intervals of $\bar{Q}$ in $[c, d]$ then (i) $F_{(n-1)}$ is AC on $\bar{Q}$ and (ii) $\sum_{k \varepsilon N} \omega_n(c_k, d_k) < \infty$. Thus by Theorem 20, and Lemmas 17 and 19 f is P^n-integrable on $[c, d]$.

This contradiction shows that $E \neq \emptyset$ and completes the proof of the theorem.

7. The P^n-Integral and the nth-Total of Denjoy

In [5] James suggested that the P^n-integral may be equivalent to the nth-order totalization of Denjoy, [4]. Since in the case $n = 1$ the P^n-integral is the classical Denjoy-Perron integral whereas the nth-order totalization is the Denjoy-Khintchine integral, [4, 8], this is not the case. Thus in this case the nth-order totalization is more general than the P^n-integral; this remains true for all n.

Suppose f is P^n-integrable with F its P^n-integral then

(a) $F_{(k)}$ exists in $[a, b]$, $1 \leqq k \leqq n - 1$, (Theorem 10 and Corollary 11);

(b) $F_{(n)} = AD\, F_{(n-1)} = f$ a.e. (Theorem 22 and Lemma 32);

(c) $F_{(n-1)}$ is ACG on $[a, b]$, (Theorem 22 and Lemma 17).

This implies that f is nth-order totalizable and that F is an nth-order total of f.

Denjoy's process is clearly strictly more general for all n. Take F to be a Denjoy-Khintchine integral that is not a $C_{n-1}P$-integral, [11], and let $\tilde{F}$ be the integral of order $(n - 1)$ of F. Then $\tilde{F}$ is an nth-order total of $f = ADF$ but f is not P^n-integrable, by Theorem 10.

A Perron type integral that is equivalent to the nth-order totalization and its related generalization of the Cesàro-Perron integral scale will be considered in a later paper.

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https://doi.org/10.1017/S1446788700010065 Published online by Cambridge University Press