HEREDITARY SEMI-PRIMARY RINGS AND TRI-ANGULAR MATRIX RINGS

MANABU HARADA

To the memory of Professor TADASI NAKAYAMA

It is well known that the semi-simple rings with minimum conditions coincide with the rings of global homological dimension zero and that the hereditary rings coincide with the rings of global dimension one. Eilenberg, Jans, Nagao and Nakayama gave some properties of hereditary rings in [4] and [11], which relate to global dimension of factor rings[0]. As an example of non-commutative hereditary ring we know a tri-angular matrix ring over a semi-simple ring.

Let Λ be a ring with radical N. If N is nilpotent and Λ/N satisfies the minimum conditions, then we call Λ a semi-primary ring.

The purpose of this paper is to give a visible form of hereditary semi-primary rings which is similar to the fact that a simple ring with minimum condition is isomorphic to a matrix ring over a division ring.

In §2 we shall define a generalized tri-angular matrix ring over semi-simple rings and give properties of such a ring, which is a generalization of [4], Theorem 8 and [11], Proposition 7.

In §3 we shall show that every hereditary semi-primary ring is isomorphic to a generalized tri-angular matrix ring over semi-simple rings and we show, conversely, every generalized tri-angular matrix ring is a homomorphic image of an hereditary semi-primary ring by modifying slightly the method in [11], §2.

As an application of results in §§1–3, we show in §4 that if Λ is an hereditary semi-primary ring, then so is $e\Lambda e$ for any idempotent e and Λ con-

Received Dec. 5, 1964.

[0] Added in Proof.
Chase generalized those results to a generalized triangular matrix ring in [2] and Nakano also studied such a ring in [4]. Some results in this paper will overlap with them.

https://doi.org/10.1017/S0027763000026313 Published online by Cambridge University Press

tains a minimal faithful left ideal which is contained in any faithful Λ-projective module as a direct summand.

We consider, in §5, an hereditary ring with minimum conditions, which is semi-primary. However, we note that we have two ways in such a ring to obtain a relation between hereditary rings and generalized tri-angular matrix rings; namely by using the nilpotency of the radical and the length of composition series of indecomposable left ideals. In general there are no relations between them, however we show that two ways coincide under some assumptions.

We always assume that a ring Λ has the unit element and any Λ-module is unitary. Furthermore, any ring is semi-primary except in §1.

1. basic rings

Let Λ be a ring with unit element 1 and N is the radical of Λ. In this section we always assume that every idempotent element in Λ/N is lifted from those elements in Λ, e.g. Λ satisfies the minimum conditions or N is nil and so on. Furthermore, we assume that Λ/N is a semi-simple ring with minimum conditions. In such a situation we have

$$\Lambda = \sum_{i=1}^{n}\sum_{j=1}^{n(i)} \Lambda e_{i,j}$$

where the $\Lambda e_{i,j}$'s are indecomposable left ideals in Λ, $e_{i,j}^2 = e_{i,j}$ and $\Lambda e_{i,j} \approx \Lambda e_{i,k}$, $\Lambda e_{i,j} \not\approx \Lambda e_{i',k}$ if $i \neq i'$. We put $e_i = e_{i,1}$ and $e = \sum_i e_i$. Then $\tau_\Lambda(\Lambda e) = \Lambda$, (see the definition of trace ideal in [1]). Furthermore, it is clear that $\mathrm{Hom}_\Lambda^l(\Lambda e, \Lambda e) = e\Lambda e$. Since $\tau_\Lambda(\Lambda e) = \Lambda$, we obtain from [1], Theorem A. 2.

LEMMA 1. *Λe is a finitely generated $e\Lambda e$-projective module and* $\Lambda = \mathrm{Hom}_{e\Lambda e}^r(\Lambda e, \Lambda e)$.

PROPOSITION 1. *Let e be as above. Λ is a semi-primary ring such that* $N^t = (0)$ *if and only if so is $e\Lambda e$ and* $N'^t = (0)$, *where N' is the radical of $e\Lambda e$.*

Proof. Since $N' = eNe$, $e\Lambda e$ satisfies the above condition if so does Λ. We assume that $\Gamma = e\Lambda e$ is a semi-primary ring with $N'^t = (0)$. From an exact sequence $0 \to \Lambda eN' \to \Lambda e \to \Lambda e/\Lambda eN' \to 0$ we obtain the exact sequence $0 \to \mathrm{Hom}_\Gamma(\Lambda e, \Lambda eN')$ $\to \mathrm{Hom}_\Gamma(\Lambda e, \Lambda e) \to \mathrm{Hom}_\Gamma(\Lambda e, \Lambda e/\Lambda eN') = \mathrm{Hom}_{\Gamma/N'}(\Lambda e/\Lambda eN', \Lambda e/\Lambda eN') \to 0$, since Λe is Γ-projective. $\Lambda e/\Lambda eN'$ is a finitely generated Γ/N'-module by Lemma 1 and hence, $\mathrm{Hom}_{\Gamma/N'}(\Lambda e/\Lambda eN', \Lambda e/\Lambda eN')$ is a semi-simple ring with minimum conditions. Therefore, $\mathrm{Hom}_\Gamma(\Lambda e, \Lambda eN')$ contains the radical N of $\Lambda = \mathrm{Hom}_\Gamma(\Lambda e, \Lambda e)$. However $(\mathrm{Hom}_\Gamma(\Lambda e, \Lambda eN'))^t \subseteq \mathrm{Hom}_\Gamma(\Lambda e, \Lambda eN'^t) = (0)$. Hence Λ is a semi-primary ring such that $N^t = (0)$.

PROPOSITION 2. *Let e be as above. Then*

$$\text{l.gl.dim } \Lambda = \text{l.gl.dim } e\Lambda e,$$
$$\text{r.gl.dim } \Lambda = \text{r.gl.dim } e\Lambda e.$$

Proof. Since Λe is right $e\Lambda e$-projective, we obtain $\text{l.gl.dim } \Lambda \geq \text{l.gl.dim } e\Lambda e$ by [5], Theorem 7. On the other hand $\text{l.gl.dim } e\Lambda e \geq \text{l.gl.dim } \Lambda$ by [7], Lemma 1.2. Replacing Λe by $e\Lambda$, we have the second half.

COROLLARY 1. *Λ is left hereditary if and only if so is $e\Lambda e$. Furthermore, $e\Lambda e/N'$ is a directsum of division rings.*

Remark 1. In Proposition 2 we only need that $\tau_\Lambda(\Lambda e) = \Lambda$.

Remark 2. If we consider an hereditary semi-primary ring, we may restrict ourselves to the case where the factor ring with respect to its radical is a direct sum of division rings from the above results.

2. Generalized tri-angular matrix rings

From now on we always consider a semi-primary ring Λ and denote its radical by $N(\Lambda)$. In the next section we shall study an hereditary semi-primary ring and show that it is isomorphic to a generalized tri-angular matrix ring over semi-simple rings (see the below). Thus, we study, in this section, some properties of such a ring.

Let $R_1, R_2, \ldots, R_n$ be rings and $M_{i,j}$ a left R_i- and right R_j-module for $i > j$ and $M_{i,i} = R_i$. We consider a family of bilinear $R_i - R_j$ homomorphisms.

$$(1) \qquad \begin{aligned} \varphi^l_{i,j} &: M_{i,l} \otimes_{R_l} M_{l,k} \to M_{i,k} \\ \varphi^t_{i,t} &: M_{i,t} \otimes_{R_t} R_t \approx M_{i,t} \\ \varphi^i_{i,t} &: R_i \otimes_{R_i} M_{i,t} \approx M_{i,t} \end{aligned}$$

and a family of diagrams

$$(2) \qquad \begin{array}{ccc} M_{i,j} \otimes_{R_j} M_{j,l} \otimes_{R_l} M_{l,k} & \xrightarrow{I_{i,j} \otimes \varphi^l_{j,k}} & M_{i,j} \otimes_{R_j} M_{j,k} \\ \downarrow \varphi^j_{i,l} \otimes I_{l,k} & & \downarrow \varphi^j_{i,k} \\ M_{i,l} \otimes_{R_l} M_{l,k} & \xrightarrow{\varphi^l_{i,k}} & M_{i,k} \end{array}$$

where I means the identity mapping.

Next, we consider the following sets

$$T_n(R_i\ ;\ M_{i,j}) = \left\{ \begin{pmatrix} r_{1,1} & & & 0 \\ m_{2,1} & r_{2,2} & & \\ \cdots & & \ddots & \\ \cdots & & & \\ m_{n,1} & \cdots & m_{n,n-1} & r_{n,n} \end{pmatrix} r_{i,i} \in R_i,\ m_{i,j} \in M_{i,j} \right\}.$$

We can make it a ring as follows:

$$(3) \qquad \begin{aligned} (m_{i,j}) \pm (m'_{i,j}) &= (m_{i,j} \pm m'_{i,j}) \\ (m_{i,j}) \cdot (m'_{i,j}) &= (\textstyle\sum \varphi^l_{i,j}(m_{i,l} \otimes m'_{l,j})). \end{aligned}$$

It is clear that this product is associative if and only if the diagrams of (2) are commutative.

In this case, we call it a *generalized tri-angular matrix ring* over R_i, and we denote it briefly by *g.t.a. matrix ring* over R_i. $M_{i,k}M_{k,j}$ means the image of $M_{i,k} \otimes M_{k,j}$ by $\varphi^k_{i,j}$.

We are only interested, in this paper, in a case where all R_i are semi-primary. Then a g.t.a. matrix ring over R_i is also semi-primary.

LEMMA 2. *Let Λ be a g.t.a. matrix ring $T_n(R_i\ ;\ M_{i,j})$. If Λ is hereditary, then every R_i-submodule in $M_{i,j}$ is R_i-projective, and hence, all R_i are hereditary.*

Proof. Let $M'_{i,j}$ be an R_i-submodule in $M_{i,j}$, or a left ideal in R.

Let $$L_0 = \left\{ i \overset{\quad j}{\begin{pmatrix} 0 & & 0 \\ \vdots & m_{i,j} & \vdots \\ 0 & & 0 \end{pmatrix}},\ m_{i,j} \in M'_{i,j} \right\},$$

$L = \Lambda L_0$ and $$\mathfrak{A} = i \begin{pmatrix} & 0 & & \\ M_{i,1} \cdots M_{i,i-1} & & 0 & \\ M_{i+1,1} & \cdots & R_{i+1} & 0 \\ & \cdots\cdots & & \\ M_{n,1} \cdots & & & R_n \end{pmatrix}.$$

Then $\mathfrak{A}$ is a two-sided ideal in Λ. Since L is Λ-projective, $L/\mathfrak{A}L$ is $\Lambda/\mathfrak{A}$-projective. From types of $L/\mathfrak{A}L$ and $\Lambda/\mathfrak{A}$, we know that $M'_{i,j}$ is R_i-projective. Especially we obtain that every left ideal in R_i is R_i-projective. Hence, R_i is hereditary.

By replacing A and L by

$$i\begin{pmatrix} 0 & \overset{k}{0} \\ 0 & M_{l,t} \end{pmatrix} \text{ and } \begin{bmatrix} & \overset{k}{N_k} & \\ 0 & M_{k+1,k} & 0 \\ & \vdots & \\ & M_{n,k} & \end{bmatrix}, \text{ where } N_k = N(R_k).$$

We obtain

LEMMA 3. *Let Λ be as above. If Λ is hereditary, then* $\sum_{t=1}^{i-k+1} M_{i,k+t}M_{k+t,k}$ *is a direct summand of $M_{i,k}$ as an R_i-module for all $>k$, where $M_{i,i}=N(R_i)$.*

LEMMA 4. *Let $\Lambda = T_n(R_i\ ;\ M_{i,j})$ and $e_j = T_n(0, \ldots, 0, \overset{j}{1_j}, 0, \ldots; 0)$, $e=\sum_{j=i}^{n} e_j$. Let A be a left Λ-module such that $eA = A$. Then* $\mathrm{l.dim}_{\Lambda}A = \mathrm{l.dim}_{e\Lambda e}A$. *and* $\mathrm{gl.dim}\ \Lambda \geq \mathrm{gl.dim}\ e\Lambda e$, *where 1_i is the unit element of R_i.*

Proof. It is clear that $\Lambda e = e\Lambda e$. Hence $\Lambda e \otimes_{e\Lambda e} A = e\Lambda e \otimes_{e\Lambda e} A = A$ as a left Λ-module. Since $\mathrm{r.dim}_{e\Lambda e}\Lambda e = \mathrm{r.dim}_{e\Lambda e}\, e\Lambda e = 0$, we have $\mathrm{l.dim}_{e\Lambda e}A = \mathrm{l.dim}_{\Lambda}\, \Lambda e \otimes A = \mathrm{l.dim}_{\Lambda}A$ and $\mathrm{gl.dim}\Lambda \geq \mathrm{gl.dim}\ e\Lambda e$ by [5], Proposition 15 and Theorem 7.

THEOREM 1. *Let R_i be semi-primary rings with radical N_i. A g.t.a. matrix ring $T_n(R_i\ ;\ M_{i,j})$ over R_i is hereditary if and only if the following conditions are satisfied.*

a) *All R_i are hereditary,*

b) *$\varphi^{j}_{i,k}$ is monomorphic,*

c) *$M_{i,j}/\sum_{t=j}^{i-1} M_{i,t}M_{t,j} (\approx \overline{M}_{i,j})$ is R_i-projective; $\overline{M}_{i,j} \subseteq M_{i,j}$,*

d) *$M_{i,j} = \overline{M}_{i,j} \oplus M_{i,j+1}\overline{M}_{j+1,j} \oplus \cdots \oplus M_{i,i-1}\overline{M}_{i-1,j} \oplus M_{i,j}N_j$ as a left R_i-module, where $M_{i,i}=N_i$.*

Proof. Let $\Lambda = T_n(R_i\ ;\ M_{i,j})$. It is clear that $N' = N(\Lambda) = T_n(N_1, \ldots, N_n\ ;$ $M_{i,j})$ and

$$N' = \sum \oplus \begin{bmatrix} & \overset{i}{0} & \\ 0 & N_i & 0 \\ & M_{i+1,i} & \\ & \vdots & \\ & M_{n,i} & \end{bmatrix}. \text{ We put } L_i = \begin{bmatrix} N_i \\ M_{i+1,i} \\ \vdots \\ M_{n,i} \end{bmatrix}$$

and $\Lambda_i = T_{n-i+1}(R_i, \ldots, R_n\ ;\ M_{i,j})$. Then we may regard that L_i is a left Λ_i-module. Let e_i be as above. Then $e_iL_i = L_i$ and $\Lambda_i = e_i\Lambda e_i$. Hence $\mathrm{l.dim}_{\Lambda}L_i = \mathrm{l.dim}_{\Lambda_i}L_i$ by Lemma 4. Therefore, we know that Λ is hereditary if and only if L_i is Λ_i-projective for all i. For the sake of simpleness to explain, we consider a case of $i=1$. We denote L_1, Λ_1 by L, Λ. Then

$$\bar{L}=L/NL=\begin{pmatrix} M_{1,1}^{*} \\ M_{2,1}^{*} \\ \cdot \\ \cdot \\ \cdot \\ M_{n,1}^{*} \end{pmatrix}$$

where $M_{1,1}^{*}=N_1/N_1^2$, $M_{i,1}^{*}=M_{i,1}/(M_{i,1}N_1+\cdots+N_iM_{i,1})$.

Let $\{f_k^{(j)}\}$ be a set of non-isomorphic primitive idempotent elements in R_j. We put $\tilde{f}_k^{(j)}=T_n(0,\ldots,0,\overset{j}{f_k^{(j)}},\ldots,0\,;\,0)$. Since $M_{i,1}^{*}$ is an $\bar{R}_i=R_i/N_i$-module, $M_{i,1}^{*}=\sum_t\oplus L_t^{(i)}$; $L_t^{(i)}=\bar{R}_iy_t^{(i)}\approx\bar{R}_if_{l(t)}^{(i)}$, $y_t^{(i)}\in M_{i,1}$. From the idea of minimal projective resolution (cf. [3]), we know that L is Λ-projective if and only if

(4) $$L=\sum_i\sum_t\oplus\Lambda\tilde{y}_t^{(i)},\ \text{where}\ \tilde{y}_t^{(i)}=\begin{pmatrix} 0 \\ \cdot \\ \cdot \\ 0 \\ y_t^{(i)} \\ 0 \\ \cdot \\ \cdot \end{pmatrix} i \ \text{and}$$

(5) $$\Lambda\tilde{y}_t^{(i)}=\begin{pmatrix} 0 \\ 0 \\ R_iy_t^{(i)} \\ M_{i+1,i}y_t^{(i)} \\ M_{n,i}y_t^{(i)} \end{pmatrix}\approx\Lambda\tilde{f}_{l(t)}^{(i)}=\begin{pmatrix} 0 \\ 0 \\ R_if_{l(t)}^{(i)} \\ M_{i+1,i}f_{l(t)}^{(i)} \\ M_{n,i}f_{l(t)}^{(i)} \end{pmatrix}$$

(natural isomorphism).

Let $M_{i,1}^{**}=\sum_t R_iy_t^{(i)}\subseteq M_{i,1}$. We assume that Λ is hereditary. Then we obtain $M_{i,1}=P_i\oplus(M_{i,1}N_1+\cdots+M_{i,i-1}M_{i-1,1})$ for some R_i-module P_i from Lemma 3. Hence $M_{i,1}N_1+\cdots+M_{i,i-1}M_{i-1,1}+N_iM_{i,1}=N_iP_i\oplus(M_{i,1}N_1+\cdots+M_{i,i-1}M_{i-1,1})$. Therefore, we may assume that $y_t^{(i)}\in P_i$. Then $P_i=M_{i,1}^{**}=M_{i,1}$ since N_i is nilpotent. Now, 4) is equivalent to facts that $\bar{M}_{i,1}=\sum_t\oplus R_iy_t^{(i)}$ and that $\sum_t M_{l,i}y_t^{(i)}$ and $\sum_{i=1}^{l-1}M_{l,i}\bar{M}_{i,1}+\bar{M}_{l,1}$ are direct sums, respectively. The above arguments are true by replacing 1 by any h. Furthermore, we obtain from (2)

(6) $$M_{k,i}M_{i,j}M_{i,h}\subseteq M_{k,j}M_{j,h}\ \text{for all}\ k\geq i\geq j\geq h.$$

Since $M_{k,i}\otimes M_{i,j}=M_{k,i}\otimes \overline{M}_{i,j}\oplus M_{k,i}\otimes(\sum_{t=j}^{i-1}M_{i,t}M_{t,j})$, we obtain from (6)

$$M_{k,i}M_{i,j}\subseteq M_{k,i}\overline{M}_{i,j}+\sum_{t=j}^{i-1}M_{k,t}M_{t,j}\subseteq M_{k,i}\overline{M}_{i,j}+ M_{k,i-1}\overline{M}_{i-1,j}+\sum_{t=j}^{i-2}M_{k,t}M_{t,j}\subseteq\sum_{s=j+1}^{i}M_{k,s}M_{s,j}+M_{k,j}N_j.$$

Therefore,

$$M_{k,j}N_j+M_{k,j+1}M_{j+1,j}+\cdots+M_{k,k-1}M_{k-1,j}\subseteq\sum_{s=j+1}^{k-1}M_{k,s}\overline{M}_{s,j}+M_{k,j}N_j. \tag{7}$$

Hence $M_{k,j}=\sum\oplus M_{k,s}\overline{M}_{s,j}\oplus M_{k,j}N_j\oplus\overline{M}_{k,j}$ from the above observation. Which shows d), a) and c) by Lemmas 2 and 3. Furthermore, from (5) we obtain $f^{(1)}_{l(t)}y^{(1)}_t=y^{(1)}_t$. Hence (4) and (5) implies $\varphi^j_{i,1}$ is monomorphic on $M_{i,j}\otimes\overline{M}_{j,1}$. Using this argument we shall show that $\varphi^j_{i,k}$ is monomorphic for all $i\geq j\geq k$. Let $\Gamma=T_{n-j+1}(R_j,\ldots,R_n\,;\,M_{l,s})$ and $L=\begin{pmatrix}0\\M_{j,k}\\M_{n,k}\end{pmatrix}$. Since Γ is hereditary by lemma 4, we know from the above argument that $\varphi^j_{i,k}$ is monomorphic on $M_{i,j}\otimes\overline{M}_{j,k}$. However, $\overline{M}_{j,k}=M_{j,k}$ in this case, and hence, $\varphi^j_{i,k}$ is monomorphic on $M_{i,j}\otimes M_{j,k}$.

Conversely we assume that a)-d) are satisfied. From a) we have $\sum_t\oplus R_1y^{(1)}_t=N_1$. From a), c) and the remark after (5), we obtain $M^{**}_{i,1}=\overline{M}_{i,1}=\sum_t\oplus R_iy^{(i)}_t$. Since $\varphi^i_{l,1}$ is monomorphic on $M_{l,i}\otimes\overline{M}_{i,1}$ by b), we have (5) from d). Furthermore, d) and a) imply (4).

Corollary 2. *Let Λ be a g.t.a. matrix ring over semi-simple rings. If $\varphi^j_{i,k}$ is isomorphic, then Λ is hereditary.*

Proof. From the assumption, we have $\overline{M}_{i,i-1}=M_{i,i-1}$ for all i and $\overline{M}_{i,j}=(0)$ for all $i>j+1$. Hence Λ satisfies a)-d).

Remark 3. An usual tri-angular matrix ring over a semi-simple ring R is a special case of Corollary 2.

Theorem 2. *Let R_i be semi-primary rings and $M_{i,j}$ R_i-R_j modules. Let $\Lambda_i=T_{n-i+1}(R_i,\ldots,R_n\,;\,M_{l,j})$. Then* gl.dim $\Lambda_{i+1}\leq$gl.dim $\Lambda_i\leq$gl.dim $\Lambda_{i+1}+$ gl.dim R_i+1.

Proof. From Lemma 4, we obtain gl.dim $\Lambda_{i+1}\leq$gl.dim Λ_i. We denote Λ_i, Λ_{i+1} by Λ, Γ. Then $N=N(\Lambda)=T_{n-i+1}(N_i,\ldots,N_n\,;\,M_{i,j})$, where $N_i=N(R_i)$.

$N=L_i\oplus N(\Gamma)$ as a left Λ-module, where L_i is the same as in the proof of Theorem 1. From Lemma 4 we have gl.dim $\Lambda=1+\text{l.dim}_\Lambda N=1+\sup(\text{l.dim}_\Lambda L_i, \text{l.dim}_\Gamma N(\Gamma))$. If we use the same notations as in the proof of Theorem 1, then

$$0\to\varphi^{-1}(0)\to\sum_{j=i}^{n}\sum_{s}\Lambda\tilde{f}_s^{(j)}=\sum\oplus P_0^j\to L_i\to 0$$

is exact, where $P_0^j=\sum_s\Lambda f_s^{(j)}$. We consider $P^{(i)}=\sum_s\oplus\Lambda f_s^{(i)}\xrightarrow{\varphi^{(i)}}\Lambda\begin{pmatrix}N_i\\0\\ \cdot\\ \cdot\\ \cdot\\0\end{pmatrix}\to 0$. Then $\varphi^{(i)-1}(0)=\begin{pmatrix}K_i\\N'_{i+1,i}\\ \cdot\\ \cdot\\ \cdot\\N'_{n,i}\end{pmatrix}$, where $0\to K_i\to\sum_s\oplus R_if_s^{(i)}\to N_i\to 0$ is exact and $N'_{j,i}$ is an R_j-module. Hence we can see directly that $\varphi^{-1}(0)=\begin{pmatrix}K_i\\N_{i+1,i}\\ \cdot\\ \cdot\\ \cdot\\N_{n,i}\end{pmatrix}$; $N_{j,i}$ is an R_j-module. If we repeat this argument on K_i we have a minimal Λ-projective resolution of L_i:

$$\longrightarrow\sum\oplus P_k^j\xrightarrow{d_k}\cdots\longrightarrow\sum\oplus P_1^j\xrightarrow{d_1}\sum\oplus P_0^j\xrightarrow{d_0}L_i\longrightarrow 0$$

and the first row of each P^i forms a minimal R_i-projective resolution of N_i. Hence if gl.dim $R_i=m$, then $d_{m-1}^{-1}(0)$ is a Γ-module. Hence $\text{l.dim}_\Lambda d_m^{-1}(0)\le$ gl.dim Γ by Lemma 4. Therefore, $\text{l.dim}_\Lambda N\le$ gl.dim $R+$ gl.dim Γ. Thus, we obtain gl.dim $\Lambda\le$ gl.dim R_i+ gl.dim $\Gamma+1$.

COROLLARY 3. *We assume that all R_i are semi-simple rings in the above. Then* gl.dim $\Lambda_{i+1}\le$ gl.dim $\Lambda_i\le 1+$ gl.dim Λ_{i+1} *and* gl.dim $\Lambda\le n-1$.

Let Λ be a g.t.a. matrix ring $T_n(R_i\,;\,M_{i,j})$ and e_i be as in Lemma 4. For an element a in a two-sided ideals $\mathfrak{A}$ in Λ $e_iae_j\in\mathfrak{A}$. Hence $\mathfrak{A}=T_n(S_i\,;\,N_{i,j})$, where S_i is an ideal in R_i and $N_{i,j}$ is an R_i-R_j module in $M_{i,j}$. Hence we obtain from Corollary 3.

THEOREM 3. *Let Λ be a g.t.a. matrix ring $T_n(R_i\,;\,M_{i,j})$ over semi-simple* *rings* R_i. *Then for any two-sided ideal* A *in* Λ *we have*

$$\text{gl.dim } \Lambda/\mathfrak{A} \leq n-1.$$

Furthermore, we assume that all R_i *are simple rings and* $\mathfrak{A}+N/N \approx \sum_{t=1}^{s} \oplus R_{i_t}$. *Then* gl.dim $\Lambda/\mathfrak{A} \leq n-s-1$, *where* N *is the radical of* Λ.

3. Hereditary semi-primary rings

In this section we shall determine a type of hereditary semi-primary rings. Let Λ be such a ring and N the radical. By virtue of Remark 2, we shall first restrict ourselves to the case where Λ/N is a direct sum of division rings. Then we obtain $1=\sum e_i$, where $\{e_i\}$ is a set of non-isomorphic primitive idempotent elements in Λ. Since $N^t=(0)$ for some t, we have an integer s for e_i such that $N^s e_i \neq (0)$, $N^{s+1}e_i=(0)$. We denote such an integer s by $n(e_i)$. Then we can rearrange e_i as follows: $n(e_i) \geq n(e_{i+1})$ for all i.

We quote here a well known results by [3] and [12].

LEMMA 5. *Let* Λ *be an hereditary semi-primary ring. Then every* Λ*-projective module is isomorphic to* $\sum \oplus (\Lambda e_i)^{s_i}$[1].

LEMMA 6. *Let* Λ *be as above. If* e *is an idempotent element in* Λ *such that* $n(e)$ *is minimal among* $n(e_i')$, *where* e' *runs through all primitive idempotent elements in* Λ. *Then* $n(e)=0$.

Proof. If $Ne \neq (0)$, then $Ne=\sum \oplus (\Lambda e_i)^{s_i}$ by Lemma 5. Hence $n(e)>n(e_i)$, which is a contradiction.

LEMMA 7. *Let* Λ *be an hereditary semi-primary ring such that* $\Lambda/N \approx \sum \oplus \Delta_i$; Δ_i *division ring. Then* $e_i N e_j=(0)$ *for* $i \leq j$ *and* $e_i \Lambda e_i \approx \Delta_i$.

Proof. From Lemma 6 we know that $Ne_n=(0)$. Therefore, $e_i N e_n=(0)$ for all i. We assume that $e_i N e_j=(0)$ for $i \leq j$ and $j>k$. Since Ne_k is Λ-projective, we have from Lemma 5 and the assumption $n(e_i) \geq n(e_{i+1})$

$$Ne_k \approx \sum_{l>k} \oplus (\Lambda e_l)^{s_l}.$$

Hence, $e_i N e_k \approx \sum \oplus (e_i \Lambda e_l)^{s_l} = \sum \oplus (e_i N e_l)^{s_l}=(0)$ for $i \leq k$ from the above assumption. Therefore, $e_i N e_j=(0)$ for all $i \leq j$. Since $e_i N e_i=(0)$, $\Delta_i \approx e_i/e_i N e_i = e_i \Lambda e_i$.

---

[1] M^n means a directsum of n copies of M.

https://doi.org/10.1017/S0027763000026313 Published online by Cambridge University Press

THEOREM 4′. *Let Λ be an hereditary semi-primary ring such that $\Lambda/N \approx \sum_{i=1}^{n} \oplus \Delta_i$. Then Λ is isomorphic to a g.t.a. matrix ring over Δ_i, where N is the radical of Λ and the Δ_i's are division rings*, ([2] and [14]).

Proof. Since $1 = \sum_{i=1}^{n} e_i$, $\Lambda = \sum_{i,j} \oplus e_i \Lambda e_j$ and $e_i \Lambda e_j = (0)$ if $i<j$. If we put $M_{i,j} = e_i \Lambda e_j$ and $\varphi^j_{i,k}$ is a product of $M_{i,j}$ and $M_{j,k}$, then $\Lambda = T_n(\Delta_1, \ldots, \Delta_n ; M_{i,j})$ by Lemma 7.

We shall generalize the above theorem.

Let $\Lambda = T_n(R_i ; M_{i,j})$ be a g.t.a. matrix ring over rings R_i. Let

$$\mathfrak{M}_{i,j} = \left.\begin{pmatrix} \overbrace{M_{i,j} \cdots M_{i,j}}^{s_j} \\ \cdots \\ \cdots \\ M_{i,j} \cdots M_{i,j} \end{pmatrix}\right\} s_i \equiv M_{i,j}(s_i \times s_j).$$

Then we can define a natural operation of elements in $(R_i)_{s_i}$ (resp. $(R_j)_{s_j}$) from the left side (resp. right side) on $\mathfrak{M}_{i,j}$. We put $\Gamma = T_n((R_1)_{s_1}, \ldots, (R_n)_{s_n} ; \mathfrak{M}_{i,j})$, then Γ is also a g.t.a. matrix rings over $(R_i)_{s_i}$ with naturally extended bi-linear mapping $\Phi^j_{i,k} : ((x_{t,p})) \underset{(R_p)s_p}{\otimes} ((y_{r,q})) \to ((\sum \varphi^j_{i,k}(x_{t,p} \otimes y_{p,q}))$, $x_{t,p} \in M_{i,j}$, $y_{r,q} \in M_{j,k}$.

Let $e^{(i)}_{l,m}$ be the matrix units in $(R_i)_{s_i}$ and $E_i = T_n(0, \ldots, 0 \overset{i}{e_{1,1}}, 0 \cdots 0 ; 0)$. If we put $E = \sum E_i$ then $\tau_\Gamma(\Gamma E) = \Gamma$ and $E\Gamma E \approx \Lambda$. Hence we have from Proposition 2

$$\text{gl.dim}\, \Lambda = \text{gl.dim}\, \Gamma.$$

We call Γ an *induced g.i.a. mairix ring* from Λ.

THEOREM 4″. *Let Λ be an hereditary semi-primary ring. Then Λ is isomorphic to an induced g.t.a. matrix ring $T(R_i ; \mathfrak{M}_{i,j})$ over simple rings R_i from a g.t.a. matrix ring as in Theorem 4′. $\Phi^j_{i,k}$ is monomorphic for all $i \geq j \geq k$ and $\mathfrak{M}_{i,j+1}\overline{\mathfrak{M}}_{j+1,j} + \cdots + \mathfrak{M}_{i,i-1}\overline{\mathfrak{M}}_{i-1,j}$ is a directsum in $\mathfrak{M}_{i,j}$ as a left R_i-module, where $\mathfrak{M}_{i,j} = \overline{\mathfrak{M}}_{i,j} \oplus \sum_{t=j+1}^{i-1} \mathfrak{M}_{i,t}\mathfrak{M}_{t,j}$ as a left R_i-module.*

Proof. Let $\Lambda/N = \sum \oplus R_i$ and $e = \sum e_i$ as in § 1. Then $\Lambda = \text{Hom}_{e\Lambda e}(\Lambda e, \Lambda e)$ and $e\Lambda e$ is an hereditary semi-primary ring by Propositions 1 and 2. Furthermore, Λe is a finitely generated $e\Lambda e$-projective module, and hence,

$$\Lambda e_i \approx \sum \oplus (f_i \Gamma)^{s_i},$$

https://doi.org/10.1017/S0027763000026313 Published online by Cambridge University Press

where $\Gamma = e\Lambda e$, and $\{f_i\}$ is a set of non-isomorphic primitive idempotent elements in Γ and all s_i are finite. We may assume $n(f_i) \geq n(f_{i+1})$ for all i. Hence it is clear that $\Lambda = T_n((f_1\Gamma f_1)_{s_1}, \ldots, (f_n\Gamma f_n)_{s_n} ; (f_i\Gamma f_j(s_i \times s_j))$. The second part is clear from Theorem 1.

We shall modify slightly the above theorem.

LEMMA 8. *If* $n(e_i) = n(e_{i+1}) = \cdots = n(e_{i+s})$, *then* $e_k\Lambda e_p = (0)$ *for* $k \leq i+s$, $i \leq p \leq i+s-1$ *and* $k \neq p$.

Proof. Since $Ne_p = \sum \oplus (\Lambda e_l)^{q_l}$ for $i \leq p \leq i+s$ from the assumption, $e_kNe_p = \sum_{l \geq i+s+1} \oplus (e_k\Lambda e_l)^{q_l}$. Hence if $k \leq i+s$, $e_k\Lambda e_l = (0)$ by Lemma 7.

LEMMA 9. $n(e_{i+1}) \leq n(e_i) \leq n(e_{i+1}) + 1$.

Proof. Let $t = n(e_{i+1})$. $Ne_i = \sum_{j \geq i+1} \oplus (\Lambda e_j)^{s_j}$. Then $N^{t+2}e_i = \sum_{j \geq i+1} (N^{t+1}e_j)^{s_j} = (0)$. Hence $n(e_i) \leq t+1$.

First we assume that Λ is an hereditary semi-primary ring such that $\Lambda/N = \sum \oplus \Delta_i$. We assume $N^{n-1} \neq (0)$, $N^n = (0)$. Then it is clear that $n(e_1) = n-1$. If we classify e_i's by a relation $e \sim e' \Longleftrightarrow n(e) = n(e')$, then we have $(n-1)$ classes by Lemma 9. Furthermore, if $e_i, \ldots, e_{i+t-1}$ are in a class then we put $R_i = e_i\Lambda e_i \oplus \cdots \oplus e_{i+t-1}\Lambda e_{i+t-1}$. Then $\Lambda = T_n(R_i ; \mathfrak{M}_{i,j})$, where

$$\mathfrak{M}_{i,j} \equiv \begin{pmatrix} M_{i,j}M_{i,j+1} \cdots M_{i,j+t-1} \\ M_{i+1,j} \cdots\cdots M_{i+1,j+t-1} \\ \cdots\cdots\cdots \\ M_{i+t-1,j} \cdots M_{i+t-1,j+t-1} \end{pmatrix} \tag{8}$$

and $N = \mathfrak{M}_{n,n-1}\mathfrak{M}_{n-1,n-2} \cdots \mathfrak{M}_{2,1} \neq (0)$. Therefore, $\mathfrak{M}_{i,j} \supseteq \mathfrak{M}_{i,i-1}\mathfrak{M}_{i-1,i-2} \cdots \mathfrak{M}_{j+1,j} \neq (0)$. Hence we have in general.

THEOREM 4‴. *Let* Λ *be an hereditary semi-primary ring such that* $N^{n-1} \neq (0)$, $N^n = (0)$. *Then* Λ *is isomorphic to an induced g.t.a. matrix ring* $T_n(R_1, \ldots, R_n ; \mathfrak{M}_{i,j})$ *over semi-simple rings such that all* $M_{i,j} \neq (0)$. *Furthermore,* $T_{n-i+1}(R_i, \ldots, R_n ; \mathfrak{M}_{i,j})$ *is also an hereditary semi-primary ring with radical* N_i *such that* $N_i^{n-i} \neq (0)$, $N_i^{n-i+1} = (0)$.

Remark 4. The expression of Λ in Theorem 4‴ is not unique. For example,

$$\Lambda = \begin{pmatrix} \Delta & & & \\ 0 & \Delta & & 0 \\ \Delta & 0 & \Delta & \\ \Delta & \Delta & \Delta & \Delta \end{pmatrix}$$

Then Λ is hereditary by Corollary 2. However we have two expressions as Theorem 4‴:

$$\left(\begin{array}{cc|c|c} \Delta & & & \\ 0 & \Delta & & 0 \\ \hline \Delta & 0 & \Delta & \\ \hline \Delta & \Delta & \Delta & \Delta \end{array}\right) \text{ and } \left(\begin{array}{c|cc|c} \Delta & & & 0 \\ \hline 0 & \Delta & & \\ \Delta & 0 & \Delta & \\ \hline \Delta & \Delta & \Delta & \Delta \end{array}\right).$$

The latter is as in Theorem 4‴.

COROLLARY 4. *Let Λ be as above. Then for any two-sided ideal $\mathfrak{A}$ in Λ*

$$\text{gl. dim}\, \Lambda/\mathfrak{A} \leq n-1.$$

We shall give one more remark for this expression.

From Theorem 4″, we do not lose a generality if we consider a case of $\Lambda/N=\sum\oplus\Delta_i$. If $n(e_i)=n(e_{i+1})=\cdots=n(e_{i+t})>n(e_{i+t-1})=\cdots=n(e_{i+t+t'})>\cdots$ then for any j such that $i\leq j\leq i+t$ we have $Ne_j=\Lambda e_s\oplus\cdots$, where $i+t+1\leq s\leq i+t+t'$. Hence $e_sNe_j\neq(0)$. Therefore, if $M_{i,j}$ is as in (8), then each column of $M_{i,i-1}$ is not zero. Since $M_{i,j}\supseteq M_{i,i-1}M_{i-1,i-2}\cdots M_{j+1,j}$, each column of $M_{i,j}$ is not zero. Conversely, in an expression in Theorem 4‴, if we assume that each column of $M_{i,i-1}$ is not zero, then each unit element e of simple components of R_i has the same $n(e)$ and converse. Under such an assumption we have a unique expression up to isomorphism.

We call such a representation of hereditary ring *a left normal representation as a g.t.a. matrix ring.*

If we start from properties of eN instead of Ne, we have the similar arguments as above.

By $n'(e)$ we denote an integer such that $eN^n\neq(0)$, $eN^{n+1}=(0)$. In general, there are no relations between $n(e)$ and $n'(e)$. For instance

$$\Lambda=\begin{pmatrix} \Delta & & & 0 \\ \Delta & \Delta & & \\ 0 & 0 & \Delta & \\ \Delta & \Delta & \Delta & \Delta \end{pmatrix}. \quad \text{Then } n(e_3)=1 \text{ and } n'(e_3)=0.$$

However from the above observation we have

PROPOSITION 3. *Let Λ be an hereditary semi-primary ring such that $N^{n-1}\neq(0)$, $N^n=(0)$. Then for any idempotent element e $n(e)=n-n'(e)$ if and only if Λ has a right and left normal representation as a g.t.a. matrix ring.*

Finally we give a characterization of a g.t.a. matrix ring over semi-simple rings. We recall the definition of connected sequence of primitive idempotents, (cf. [11], §1. We do not need an assumption $N^2=(0)$).

A sequence $(e_0, e_1, \dots, e_n)$ of primitive idempotents in Λ is called connected if $e_{i+1}Ne_i \neq (0)$ for $i=0, \dots, n-1$ and we denote the maximal length of connected sequence by $l(\Lambda)$.

Proposition 4. *Let Λ be semi-primary. Then* gl.dim $\Lambda/N^2 = l(\Lambda) = l(\Lambda/N^2)$ *and* $N^{l(\Lambda)+1}=(0)$.

Proof. It is clear that $l(\Lambda) \geq l(\Lambda/N^2)$. If $eNf \neq (0)$ and $eNf \subseteq N^2$, then we may assume $eNf \subseteq N^s$, $eNf \nsubseteq N^{s+1}$. Then $eNf = eN^sf$. Hence, there exist primitive idempotents $e_0 = f, e_1, \dots, e_s = e$ such that $e_{i+1}Ne_i \nsubseteq N^2$. Hence, $(f, e_1, \dots, e_{s-1}, e)$ is a connected sequence in Λ and Λ/N^2. Therefore, $l(\Lambda) \leq l(\Lambda/N^2)$. We know gl.dim $\Lambda/N^2 = l(\Lambda/N^2)$ by [11], Proposition 2.

Theorem 5. *Let Λ be a semi-primary ring with radical N. Then the following conditions are equivalent.*

1) *Λ is a g.t.a. matrix ring over semi-simple rings.*

2) $l(\Lambda) < \infty$.

3) gl.dim $\Lambda/N^2 < \infty$.

4) *Λ is a homomorphic image of an hereditary semi-primary ring Ω such that $l(\Lambda) = l(\Omega)$.* (cf. [2], Theorem 4.1 and [11], Theorem 5).

Proof. 1)→2). Let $\Lambda = T_n(R_i; M_{i,j})$; R_i semi-simple rings and $\Gamma = T_n(R_i; 0)$. Then $\Lambda = \Gamma \oplus N$. Since we can replace idempotents in a connected sequence by isomorphic ones, we may assume that idempotents in a sequence are in Γ. Then $e_{i+1}Ne_i \neq (0)$ implies $\rho(i) < \rho(i+1)$, where $e_i \in R_{\rho(i)}$. Hence, every length of connected sequence does not exceed n.

2)→3). It is clear from Proposition 4.

3)→4). Let $E_1, \dots, E_n$ be mutually orthogonal idempotents in Λ such that $E_i\Lambda E_i/E_iNE_i$ is a simple component of Λ/N. Since $l(\Lambda) < \infty$, $E_iNE_i = (0)$. Let $\Gamma = \sum \oplus E_i\Lambda E_i \subset \Lambda$. We use a similar argument to [11], §2. Put $\Omega = \Gamma \oplus N \oplus N \underset{\Gamma}{\otimes} N \oplus \cdots \oplus \overbrace{N \underset{\Gamma}{\otimes} N \otimes \cdots \underset{\Gamma}{\otimes} N}^{n} \oplus \cdots$. By the natural multiplication Ω becomes a ring. If $\overbrace{N \otimes N \cdots \otimes N}^{s} \neq (0)$, then there exist idempotents e_i, f_i in

https://doi.org/10.1017/S0027763000026313 Published online by Cambridge University Press

Γ and n_i in N such that $e_s n_s f_s \otimes \cdots \otimes e_1 n_1 f_1 \neq 0$. Hence, $e_i N f_i \neq (0)$ and $f_{i+1} e_i \neq 0$ for all i, which means $f_{i+1} \approx e_i$ and $f_{i+1} N f_{i+1} \neq (0)$. Therefore, $\overbrace{N \otimes \cdots \otimes N}^{l(\Lambda)+1} = (0)$. It is clear that $N(\Omega) = N \oplus N \otimes N \oplus \cdots = \Omega \otimes_{\Gamma} N$. We have a natural epimorphism φ of Ω to Λ by setting $\varphi\ (r + n_1 + n_2 \otimes n_3 + \cdots) = r + n_1 + n_2 n_3 + \cdots)$. From the construction of Ω we know that Ω is hereditary and $l(\Omega) \geq l(\Lambda)$. However $N(\Omega)^{l(\Lambda)+1} = (0)$. Hence, $l(\Omega) \leq l(\Lambda)$ by the following proposition 5.

4) $\rightarrow$ 1). It is clear from Theorem 4‴.

Remark 5. From Proposition 4 and Theorems 2 and 5 we obtain $l(\Lambda) =$ gl.dim $\Lambda/N^2 \geq$ gl.dim $\Lambda/\mathfrak{A}$ for any twoside ideal $\mathfrak{A}$ of Λ.

4. Applications.

In this section we shall give some properties of hereditary semi-primary rings as applications of results in §§1–3.

From Theorems 4, 4″ and 4‴ we have

THEOREM 6. *Let Λ be an hereditary semi-primary ring with nilpotency n of the radical. Then for any two-sided ideal $\mathfrak{A}$ in Λ*

$$\text{gl.dim } \Lambda/\mathfrak{A} \leq n - 1.$$

Furthermore, if Λ/N is a directsum of m simple rings and $N + \mathfrak{A}/N$ is a directsum of s simple rings, then

$$\text{gl.dim } \Lambda/\mathfrak{A} \leq m - s + 1.$$

(cf. [4], Theorem 8 and [11], Proposition 7).

The following proposition shows that the first inequality in the theorem is best, which was given in [4], Corollary 11.

PROPOSITION 5. *Let Λ be an hereditary semi-primary ring. Then $l(\Lambda) = l(\Lambda/N^2)$ = gl.dim Λ/N^2 is equal to (the nilpotency of N) $- 1$.*

Proof. From Theorem 4‴ we know $l(\Lambda) + 1 =$ *the nilpotency of N.*

PROPOSITION 6. *Let Λ be an hereditary semi-primary ring. If Λ/N is a simple ring, then so is Λ.*

PROPOSITION 7. *The center of an hereditary semi-primary ring Λ is a directsum of fields. Especially Λ is indecomposable if and only if its center is a field.*

Proof. We may assume that Λ is isomorphic to a g.t.a. matrix ring $T_n(R_i : M_{i,j})$ over simple rings R_i. It is clear that each component of center C of Λ is contained in the center of R_i. We denote the center of R_i by C_i. For $M_{i,j} \neq (0)$ we put $L_{i,j} = \{c \mid \in C_i$, there exists a unique element $c'(c)$ in C_j such that $cm = mc'(c)$ for all $m \in M_{i,j}\}$, and $R_{j,i} = \{c' \mid \in C_j$, there exists a unique element $c(c')$ in C_i such that $mc' = c(c')m$ for all $m \in M_{i,j}\}$. It is clear that $L_{i,j}$ and $R_{j,i}$ are fields and are isomorphic each other. We consider a path from the index 1 to i ; $i_1 = 1, i_2, \ldots, i_r = i$ such that $M_{i_k, i_{k+1}} \neq (0)$ or $M_{i_{k+1}, i_k} \neq (0)$ for all k. By I we denote all indexes which is connected to 1 by the above path. Then $\Lambda = \Lambda_I \oplus \Lambda_{I^c}$ where Λ_I consists of all elements in Λ whose (i, j)-components are zero for $i, j \in I^c$. If Λ is indecomposable, then every index is connected to 1. Therefore, in this case we know from the above observation that C is isomorphic to a subfield of $\bigcap_i R_{1,k_i}$. Hence in general C is a directsum of fields.

Proposition 8. *Let Λ be an hereditary semi-primary ring. We assume Λ is indecomposable and K is the center of Λ. $\Lambda \otimes_K L$ is hereditary and semi-primary for all extension field L of K if and only if Λ/N is separable over K,* (K-dim $\Lambda/N = 0$).

Proof. By Theorem 4″ Λ is isomorphic to a g.t.a. matrix ring $T_n(R_i ; M_{i,j})$ over simple rings R_i. If $\Lambda \otimes_K L$ is hereditary, then $R_i \otimes_K L$ is hereditary by Lemma 2, since $\Lambda \otimes_K L = T_n(R_i \otimes_K L ; M_{i,j} \otimes_K L)$. Since $\Lambda \otimes_K L$ is semi-primary, so is $R_i \otimes_K L$. Let C_i be the center of R_i. Then $C_i \otimes_K L$ is the center of $R_i \otimes_K L$. Hence $C_i \otimes_K L$ is a directsum of fields by Proposition 7. Therefore, $R_i \otimes_K L$ is a semi-simple ring with minimum conditions for any L by [10], p. 114, Theorem 1. Thus, we obtain from [6], Theorem 1 that $[R_i : K] < \infty$. Since C_i is separable over K, so is R_i. Conversely, if Λ/N is separable over K, then $[\Lambda/N : K] < \infty$ by [13], Theorem 1. Hence, $N \otimes_K L$ is the radical of $\Lambda \otimes_K L$. Since N is Λ-projective, $N \otimes_K L$ is $\Lambda \otimes L$-projective. Therefore, $\Lambda \otimes_K L$ is semi-primary and hereditary.

Lemma 10. *Let Λ be a g.t.a. matrix ring $T_n(\Delta_k ; M_{i,j})$ and $\Gamma = T_n(\Delta_1, \ldots, \overset{i}{0}, \ldots, \Delta_n ; M_{k,j} = (0)$ if k or $j = i)$. If Λ is hereditary then so is Γ.*

Proof. Let $e = T_n(0, \ldots, \overset{i}{1_i}, 0 \ldots ; 0)$ and $E = 1 - e$. Then $\Gamma = E\Lambda E$. $E\Lambda$

https://doi.org/10.1017/S0027763000026313 Published online by Cambridge University Press

$= E\Lambda E \oplus E\Lambda e$. $E\Lambda e$ is a left ideal of $\Gamma' = T_{n-i+1}(\Delta_i, \ldots, \Delta_n ; M_{k,j})$. Hence, $E\Lambda e$ is Γ'-projective. Furthermore, by Lemma 3 we obtain $0 = \text{l.dim}_{\Gamma'} E\Lambda e = \text{l.dim}_{\Gamma''} E\Lambda e = \text{l.dim}_{\Gamma} E\Lambda e$, where $\Gamma'' = T_{n-i}(\Delta_{i+1}, \ldots, \Delta_n ; M_{k,j})$. Therefore, $E\Lambda$ is Γ-projective. Since $N(\Gamma) = ENE$, ENE is Γ-projective.

THEOREM 7. *Let Λ be an hereditary semi-primary ring and M a finitely generated projective Λ-module. Then* $\text{Hom}_\Lambda(M, M)$ *is hereditary and semi-primary and so is $e\Lambda e$ for any idempotent element e in Λ.*

Proof. $M \approx \sum_{j=1}^{k} \oplus (\Lambda e_{i(j)})^{s_j}$. Hence, $\text{Hom}_\Lambda(M, M) = T_k(e_{i(j)}\Lambda e_{i(j)})s_j ; e_{i(k)}\Lambda e_{i(p)} (s_k \times s_p))$, which is an induced g.t.a. matrix ring from $\Gamma = T_k(e_{i(j)}\Lambda e_{i(j)} ; e_{i(p)}\Lambda e_{i(q)})$. From Lemma 10 we know Γ is hereditary and semi-primary. Hence, so is Λ and $e\Lambda e = \text{Hom}_\Lambda(\Lambda e, \Lambda e)$ is hereditary.

PROPOSITION 9. *Let Λ be an hereditary semi-primary ring and M a projective left Λ-module. Then the annihilator ideal $\mathfrak{A}$ of M in Λ is a direct summand of Λ as a left module and $\Lambda/\mathfrak{A}$ is hereditary.*

Proof. We may assume that Λ/N is a directsum of division rings. Since $M \approx \sum_{i=1}^{t} (\Lambda e_i)^{s_i}$, $\mathfrak{A}$ is equal to the annihilator ideal of $\sum_{i=1}^{t} \Lambda e_i$. If we denote the annihilator ideal of Λe_i in Λ by $\mathfrak{A}_i$, then $\mathfrak{A} = \bigcap_i \mathfrak{A}_i$. Let $\Lambda = T_n(\Delta_i ; M_{i,j})$ and $e_i = T_n(0, 1_{\rho(i)}, 0 ; 0)$. Then $\Lambda e_i = T_n(0, \ldots, \Delta_{\rho(i)}, 0 ; M'_{k,l} = (0)$ for $l \neq \rho(i)$, $M'_{k,\rho(i)} = M_{k,\rho(i)})$. It is clear that $\mathfrak{A}_i e_k = \Lambda e_k$ if $M_{k,\rho(i)} = (0)$ and $\mathfrak{A}_i e_k = (0)$ if $M_{k,\rho(i)} \neq (0)$. Hence, $\mathfrak{A}_i = \sum \Lambda e_{l'}$. Therefore, $\mathfrak{A} = \sum_{s=1}^{p} \Lambda e_{l(s)}$. We note that $M_{l(p),k} = (0)$ if $k \neq$ some $l(q)$. Hence, $\Lambda/\mathfrak{A} = e\Lambda e$ for some idempotent element e. Therefore, $\Lambda/\mathfrak{A}$ is hereditary from Theorem 7.

PROPOSITION 10. *Let Λ be as above. Then there exists a minimal faithful left ideal L and every faithful left Λ-projective module contains an isomorphic image of L as a direct summand.*

Proof. Let $\Lambda/N \approx \sum \oplus \Delta_i$ and $T_n(R_i ; \mathfrak{M}_{i,j})$ a g.t.a. matrix ring as a normal right representation, namely $R_i = \sum_{k=\rho(i)}^{\rho(i+1)-1} \oplus \Delta_k$, $M_{p,q}$ is a $\Delta_p - \Delta_q$ module and $\mathfrak{M}_{i,j}$ is as in (8) and furthermore, each row of $\mathfrak{M}_{i,j}$ is non-zero. From this assumption we can see as above that $\Lambda E_1 = L$ is a faithful left ideal in Λ where $E_1 = T_n(1_1, 0 ; 0)$. Let M be a faithful projective Λ-module. Since $M \approx \sum \oplus (\Lambda e_i)^{s_i}$ and $e_t M \neq (0)$ for primitive idempotent e_t in ΛE_1, $M \approx \Lambda E_1 \oplus M'$.

https://doi.org/10.1017/S0027763000026313 Published online by Cambridge University Press

COROLLARY 5. *Every hereditary semi-primary ring is a subdirectsum of finite many of hereditary rings in the endomorphism ring of vector space over a division ring.*

Proof. Let L be a minimal faithful left ideal and $L = \Lambda e_1 \oplus \cdots \oplus \Lambda e_t$. Then it is clear that $\Lambda e_i (i = 1, \ldots, t)$ is a right module over a simple ring (Δ_i-module in the above). Hence, we have the corollary from Proposition 9.

As a related problem to Corollary 5, we consider hereditary rings in the endomorphism ring of finitely generated module M over a division ring Δ. Let Λ be such a ring and $N = N(\Lambda)$. Then we have a chain of Λ-module : $M \supset NM \supset \cdots \supset N^{t-1}M \supset N^t M = (0)$. We put $\tilde{\Lambda} = \{x | \in \mathrm{Hom}_\Delta(M, M),\ xN^iM \subseteq N^iM$ for all $i\}$. $M = M_1 \oplus M_2 \oplus \cdots \oplus M_t$ as a Δ-module such that $N^iM = M_{i+1} \oplus \cdots \oplus M_t$. Then it is clear that $\tilde{\Lambda} = T_t(\Delta_{n_1}, \ldots, \Delta_{n_t} ; \Delta(n_i \times n_j))$ where $[M_i : \Delta] = n_i$, and $N(\tilde{\Lambda})^t = (0)$, $N(\tilde{\Lambda})^{t-1} \neq (0)$.

LEMMA 11. $\Delta(p \times q) \underset{\Delta_q}{\otimes} (q \times s) \approx \Delta(p \times s)$.

Proof. Let $e_i = i\begin{pmatrix} & 0 & \\ 1, 0 & \cdot & \cdot 0 \\ & 0 & \end{pmatrix}$ in $\Delta(p \times q)$. Then $\Delta(p \times q) = \sum \oplus e_i \Delta_q$. Hence $\Delta(p \times q) \underset{\Delta_q}{\otimes} \Delta(q \times s) = \sum e_i \otimes \Delta(q \times s)$. It is clear that $e_i \otimes \Delta(q \times s) \approx i\begin{pmatrix} & 0 & \\ \Delta & \cdot\cdot & \cdot\Delta \\ & 0 & \end{pmatrix}$.

From this lemma and Corollary 2 we know that $\tilde{\Lambda}$ is hereditary.

PROPOSITION 11. *Let Λ be an hereditary semi-primary ring in a simple ring Δ_n with nilpotency t of N and we assume that Λ has the unit element of Δ_n. Then there exists a maximal hereditary semi-primary ring in Δ_n with nilpotency t which contains Λ.*

Proof. Let $\tilde{\Lambda}$ be as above, Γ be an hereditary semi-primary ring containing $\tilde{\Lambda}$ and let its radical N' have the nilpotency t. We may assume $\Gamma = \tilde{\Gamma}$. We consider a chain; $M \supset N'M \supset \cdots \supset N'^{t-1}M \supset (0)$. It is clear that this chain is a composition series as a Γ-module. Furthermore, this is a chain as a $\tilde{\Lambda}$-module and M has a composition series of length t as a $\tilde{\Lambda}$-module. Hence, $N'M \supseteq NM$, $N'^2M \supseteq NN'M \supseteq N^2M \cdots$. Hence $N^iM = N'^iM$. Therefore, $\tilde{\Gamma} = \tilde{\Lambda}$.

PROPOSITION 12. *Let Δ_n be a simple algebra over a field K and Λ an hereditary algebra with $[\Lambda : K] < \infty$. We assume that Λ contains the unit element of Δ_n and its radical has the nilpotency t. Λ is a maximal hereditary ring with nilpotency* *t in* Δ_n *if and only if* $\Lambda/N \approx \sum_{i=1}^{t} \oplus \Delta_{n_i}$ *and* $\sum_{i=1}^{t} n_i = n$.

Proof. The "only if" part is clear from Proposition 11. We prove the "if" part. From the assumption and Theorem 4''' $\Lambda = T_t(R_i\ ;\ M_{i,j})$; T_i is a simple algebra and $M_{i,j} \neq (0)$. Let $1 = e_1 + e_2 + \cdots + e_t$, where e_i is the unit element in R_i. Further $e_i = \sum_{j=1}^{n(i)} e_{i,j}$ is a decomposition of e_i into the mutual orthogonal primitive idempotents in Δ_n. Then we may assume that those $e'_{i,j}s$ are a subset of matrix units in Δ_n. $e_i\Lambda e_i(\approx \Delta_{n_i}) \subseteq e_i\Delta_n e_i = \Delta_{n(i)}$. Hence, $n_i \leq n(i)$. However since $\sum n_i = n$, $n_i = n(i)$ for all i. Therefore, $e_i\Lambda e_i = e_i\Delta e_i$ since $[\Lambda : K] < \infty$. Furthermore, $e_i\Delta_n e_j = e_i\Delta_n e_i M_{i,j} e_j \Delta_n e_j = M_{i,j}$, since $M_{i,j} \neq (0)$.

5 Hereditary rings with minimum conditions

In this section we shall study hereditary rings with left or right minimum conditions. Such a ring is also semi-primary. Hence, all results in §§2 and 3 are valid for this ring. However, we give another approach to those results.

First we consider Λ such that $\Lambda/N = \sum \oplus \Delta_i$. Let $1 = \sum e_i$ as in §3. For any idempotent element e we define $l(e)$ as follows: $l(e)$ = the composition length of Λe as a left ideal. We can arrange $\{e_i\}$ as $l(e_i) \geq l(e_{i+1})$ for all i. Then we have similar results for $l(e_i)$ to $n(e_i)$. From Lemma 5, we have Lemma 7 for $l(e_i)$ and etc.

Lemma 12. *Let* Λ *be a g.t.a. matrix ring* $T_n(\Delta_i\ ;\ M_{i,j})$ *over division rings* Δ_i. *Then* $l(e_i) = \sum_{j=i+1}^{n} [M_{j,i} : \Delta_j]$, where $e_i = T_n(0, \ldots, \overset{i}{1_i}, 0, \ldots ; 0)$.

Thus, we have from Theorem 4'''

Theorem 8. Λ *is an hereditary ring with left minimum condition if and only if* Λ *is isomorphic to a g.t.a. matrix ring* $T_n(R_i\ ;\ M_{i,j})$ *over semi-simple rings* R_i, *which satisfies the conditions in Theorem 4'' and* $M_{i,j}$ *is a finitely generated* R_i*-module for all* $i > j$.

We note we do not have a relation as Lemma 8 for $l(e_i)$ and that there are no relations between $l(e_i)$ and $n(e_i)$ in general.

Proposition 13. *Let* Λ *be an hereditary g.t.a. matrix ring* $T_n(\Delta_i\ ;\ M_{i,j})$ *over division rings. Let* $t_k = [\overline{M}_{j,k} : \Delta_j]$. *Then* $l(e_i) = 1 + \sum_{k>i} t_k l(e_k)$, *where* $\overline{M}_{j,k} = M_{j,k}/\sum_l M_{j,l} M_{l,k}$.

Proof.

https://doi.org/10.1017/S0027763000026313 Published online by Cambridge University Press

$$Ne_i/N^2e_i = \begin{pmatrix} 0 \\ \overline{M}_{i+1,i} \\ \cdot \\ \cdot \\ \cdot \\ \overline{M}_{n,1} \end{pmatrix}.$$

and $\overline{M}_{j,i}$ is a left Δ_j-module. Hence, since Ne_i is Λ-projective, $Ne_i = \sum\oplus(\Lambda e_k)^{t_k}$ and Ne_i is a maximal Λ_i-module in Λe_i, $l(e_i) = \sum t_k l(e_k) + 1$.

Proposition 14. *Let Λ be an hereditary semi-primary ring such that $\Lambda/N = \sum_{i=1}^{n}\oplus \Delta_i$. Then $N^n = (0)$. Furthermore, $N^{n-1} \neq (0)$ if and only if $e_{i+1}\Lambda e_i \neq (0)$ for all i. In this case 1) $e_i\Lambda e_j \neq (0)$ for all $i \geq j$. 2) $[e_i\Lambda e_j : \Delta_i] \geq [e_i\Lambda e_{j'} : \Delta_i]$, $[e_i\Lambda e_j : \Delta_j] \geq [e_i\Lambda e_{j'} : \Delta_{j'}]$ (resp. $[e_i\Lambda e_j : \Delta_j] \geq [e_{i'}\Lambda e_j : \Delta_j]$, $[e_i\Lambda e_j : \Delta_i] \geq [e_{i'}\Lambda e_j : \Delta_{i'}]$) if $j' \geq j$ (resp. $i \geq i'$). 3) $l(e_i) > l(e_{i+1})$. 4) If Λ satisfies the left minimum condition, then $l(e) > l(e')$ is equivalent to $n(e) > n(e')$.*

Proof. We put $M_{i,j} = e_i\Lambda e_j$. Then Λ is a g.t.a. matrix ring $T_n(\Delta_i\ ;\ M_{i,j})$. Hence $N^n = (0)$. $N^{n-1} = M_{n,n-1}M_{n-1,n-2}\cdots M_{2,1}$. Since $\varphi^j_{i,k}$ is monomorphic by Theorem 1, $N^{n-1} \neq (0)$ if and only if $e_{i+1}\Lambda e_i = M_{i+1,i} \neq (0)$ for all i. We assume that all $M_{i+1,i} \neq (0)$. Then $M_{i,j} \supseteq M_{i,i-1}M_{i-1,i-2}\cdots M_{j+1,j} \neq (0)$. 2) If $j' \geq j$, then $M_{i,j} \supseteq M_{i,j'}M_{j',j}$. Hence $[M_{i,j} : \Delta_i] \geq [M_{i,j'} : \Delta_i]$ and $[M_{i,j} : \Delta_j] \geq [M_{i,j'} : \Delta_{j'}]$ since φ is monomorphic. 3) By Lemma 12 and 2) we obtain $l(e_i) = \sum_{j=i+1}^{n}[M_{j,i} : \Delta_j] > \sum_{j=i+2}^{n}[M_{j,i+1} : \Delta_j] = l(e_{i+1})$. 4) is clear from definition of $l(\ )$ and $n(\ )$.

Proposition 15. *Let Λ be as in Proposition 13. If N^{n-1} is a non-zero irreducible left Λ-module, then Δ_i is monomorphic to Δ_{i+1} and Λ satisfies the left minimum condition.*

Proof. $N^{n-1} = M_{n,n-1}\cdots M_{2,1}$. Since $\Lambda e_n = e_n\Lambda e_n = \Delta_n$, N^{n-1} is an irreducible Δ_n-module and hence, $[N^{n-1} : \Delta_n] = 1$. Furthermore, N^{n-1} contains an isomorphic image of $M_{n,n-1}\cdots M_{i+1,i}$ as a left Λ-module. Hence $[M_{i+1,i} : \Delta_{i+1}] = 1$, say $M_{i+1,i} = \Delta_{i+1}m_{i+1}$. Since $M_{i+1,i}$ is a right Δ_i-module, $\delta m_{i+1} = m_{i+1}\delta'$ for some $\delta' \in \Delta_i$. It is clear that a mapping $\delta \to \delta'$ is a monomorphism of Δ_{i+1} to Δ_i.

Lemma 13. *Let Λ be an hereditary g.t.a. matrix ring over Δ_i such that $[M_{i,j} : \Delta_i] = [M_{i,j} : \Delta_j] = 1$ for all i, j. Then Λ is isomorphic to a usual tri-angular matrix ring over Δ, where $\Delta \approx \Delta_i$.*

Proof. First we choose a generator $m_{i,i-1}$ of $M_{i,i-1}$ for $i=2, \ldots, n$; $M_{i,i-1}=m_{i,i-1}\Delta_{i-1}$. We define a generator $m_{i,j}$ of $M_{i\ j}$ as follows: $m_{i,j}=m_{i,i-1}\cdots m_{j+1,j}$. Since Λ is hereditary, $m_{i,j}\neq 0$. It is clear from the definition of $m_{i,j}$, $m_{i,j}m_{j,k}=m_{i,k}$. As in the proof of Proposition 15 we obtain an isomorphism $\alpha_{i,j}$ of Δ_i to Δ_j such that $xm_{i,j}=m_{i,j}x^{\alpha_{i,j}}$ for $x\in\Delta_i$. Since $m_{i,j}m_{j,k}=m_{i,k}$, we have $\alpha_{i,j}\alpha_{j,k}=\alpha_{i,k}$. Let Γ be a usual tri-angular matrix ring $T_n(\Delta ; \Delta)$ over Δ. We define a mapping Φ of Γ to Λ as follows: for an element $\gamma=T(\gamma_{i,j})$ $\Phi(\gamma)=T_n(\gamma_{i,j}^{\alpha_{n,i}}m_{i,j})$, where $m_{i,i}=1_i$. Then we can easily check, by noting the above observation, that Φ is isomorphic.

THEOREM 9. *Let Λ be an hereditary ring such that Λ/N is a directsum of n division rings. Then the following conditions are equivalent:*

1) *Λ is a usual tri-angular matrix ring over a division ring.*

2) *Λ is a general uniserial ring and $N^{n-1}\neq(0)$.*

3) $[e_n\Lambda e_1 : e_1\Lambda e_1]=[e_n\Lambda e_1 : e_n\Lambda e_n]=1$ *and* $N^{n-1}\neq(0)$.

4) $l(e_1)$, $r(e_n)\leq n$, *and* $N^{n-1}\neq(0)$.

5) $n(e_1)=l(e_1)=r(e_n)$ *and* $N^{n-1}\neq(0)$.

Where N is the radical of Λ and $\{e_i\}$ is the set of non-isomorphic primitive idempotent elements as the beginning of this section, and $r(e)$ is the composition length of $e\Lambda$ as a right Λ-module.

Proof. From the assumption and Theorem 4″, we know that $\Lambda=T_n(\Delta_i ; M_{i,j})$, where the Δ_i's are division rings.

1) → 2) is clear.

2) → 3). Since Λe_i has a unique composition series, we obtain that $[M_{i,j} : \Delta_i]=1$ (resp. $[M_{i,j} : \Delta_j]=1$) for all $i\geq j$.

3) → 4). By Proposition 14 we have $[M_{i,j} : \Delta_i]\leq[M_{n,1} : \Delta_n]=1$. Hence $l(e_1)$, $r(e_n)\leq n$.

4) → 5). 4) implies clearly that $l(e_i)=n-i+1$ for all i and hence $[M_{i,j} : \Delta_i]=1$. Therefore, $n(e_i)=l(e_i)$.

5) → 1). Since $n(e_i)=n(e_{i+1})+1$ by Lemma 8, $n(e_1)$, $r(e_n)=n$. Therefore, $[M_{i,j} : \Delta_i]=[M_{i,j} : \Delta_j]=1$ as above. Hence, we have 1) from Lemma 12.

Remark 7. In Theorem 9 if we replace the assumption "Λ/N is a directsum of n division rings" by simple rings, then Theorem 9 is true provided we replace 1) by 1′): Λ is a g.t.a. matrix ring $T_n((\Delta)_{s_i} ; M_{i,j}=(s_i\times s_j)$-matrices over $\Delta)$.

https://doi.org/10.1017/S0027763000026313 Published online by Cambridge University Press

Appendix.

Let R be a discrete rank one valuation ring with quotient field K and $\sum = K_n$. We know, by [8], Theorem 6.2, all types of hereditary orders over R in $\sum$. We shall give another proof by a similar argument to § 4.

By standard argument (cf. [8], § 1) we may assume that R is complete. In this case we can use Lemma 5. Let $\mathfrak{p}$ be a unique maximal ideal in R and $\{e_i\}$ the set of primitive idempotent elements as in § 1. Then $e\Lambda e$ is an hereditary order in $e\sum e$, where $e=\sum_i e_i$ and $e\Lambda e/eNe$ is a directsum of division rings.

First, we assume that $\Lambda = e\Lambda e$ and $\sum = e\sum e$. Since $\sum e_i$ is an irreducible left ideal in $\sum$ by [1], Proposition 2.8, we may assume that e_i is a matrix unit $e_{i,j}$ in $\sum$. Furthermore, we may assume that Λ is contained in a maximal order $\Omega = R_n$. By [8], Lemma 3.2 we know that $N(\Omega) = \mathfrak{p}\Omega \leq N$. Let $\overline{\Lambda} = \Lambda/N(\Omega)$.

Lemma A. *If* $Ne_i \approx \Lambda e_j$, *then* 1) $e_{j,i} \in \Lambda$, $Ne_i = \Lambda e_{j,i}$, $Ne_i/\mathfrak{p}\Omega e_i$ *is* $\Lambda/\mathfrak{p}\Omega$*-projective and* $l(\overline{e}_i) = 1 + l(\overline{e}_j)$ *or* 2) $Ne_i = \Lambda \mathfrak{p} e_{j,i}$.

Proof. Let $\varphi : \Lambda e_j \to Ne_i$ be an isomorphism. $\varphi(e_j) = ne_i = e_j ne_i$ for some $n \in N$. Since $p \in N$, there exists λ in Λ such that $\lambda e_j ne_i = \mathfrak{p} e_i = e_i \lambda e_j ne_i$. Hence, $p = e_i \lambda e_j ne_i$. Therefore, 1) $e_i \lambda e_j = p\varepsilon e_{i,j}$ and $e_j ne_i = \varepsilon' e_{j,i}$ or 2) $e_i \lambda e_j = \varepsilon$, and $e_j ne_i = p\varepsilon' e_{j,i}$, where ε, ε' are unit elements in R. Case 1). $Ne_i = \Lambda e_j ne_i = \Lambda e_{ji}$ and $\varphi(\mathfrak{p}\Omega e_j) = \mathfrak{p}\Omega e_{j,i} = \mathfrak{p}\Omega e_i$. Hence, $Ne_i/\mathfrak{p}\Omega e_i \approx \Lambda/\mathfrak{p}\Omega \overline{e}_j$. Since $\overline{\Lambda}\overline{e}_{i,i}$ is irreducible, $l(\overline{e}_i) = l(\overline{e}_j) + 1$.

Lemma B. $\Lambda/\mathfrak{p}\Omega$ *is an hereditary ring with minimum conditions.*

Proof. Since $N = \sum Ne_i$, $N/\mathfrak{p}\Omega = \sum \oplus N/\mathfrak{p}\Omega \overline{e}_i$ is $\Lambda/\mathfrak{p}\Omega$-projective by Lemma A and a fact that Ne_i is indecomposable.

Lemma C. *We assume* $l(\overline{e}_i) \geq l(\overline{e}_{i+1})$ *for all* i, *then* $l(\overline{e}_i) > l(\overline{e}_{i+1})$.

Proof. We assume that $l(\overline{e}_n) < l(\overline{e}_{n-1}) < \cdots < l(\overline{e}_{i+1})$. Since for $n < j \leq i+1$ $Ne_j \approx \Lambda e_{j+1}$ by Lemma A. Hence $e_{j+1}Ne_j \approx e_{j+1}\Lambda e_{j+1}$. Therefore, $\overline{e}\overline{\Lambda}\overline{e} = T_{n-i}(R/\mathfrak{p}\ ;\ M_{i,j} = R/\mathfrak{p})$ by Theorem 4′ and the assumption $\Omega \supseteq \Lambda$, where $e = \sum_{k=i+1}^{n} e_k$. If $l(\overline{e}_{i+1}) = l(\overline{e}_i)$, then $Ne_i \approx \Lambda e_{i+2}$ by Lemma A. Hence, $\overline{e}'\overline{\Lambda}\overline{e}' = T_{n-i+1}(R/\mathfrak{p}\ ;\ M_{i+1,i} = (0),\ M_{k,l} = R/\mathfrak{p}$ for $(k, l) \neq (i+1, i))$, where $e' = e_i + e$. Therefore, N has the same components on i^{th} and $i+1^{th}$ columns. Since $\Lambda = \mathrm{Hom}_\Lambda(N, N)$ by [7], Theorem 6.1, $\Lambda \ni e_{i+1,i}$, $e_{i,i+1}$, which is a contradiction.

From this lemma we have that $e_i\Lambda e_j = R$ for $i \geq j$. Therefore, we have

THEOREM. *Let R be a discret rank one valuation ring with quotient field K. An hereditary order over R in K_n is isomorphic to*

$$\begin{pmatrix} R & \mathfrak{p} & \mathfrak{p} \\ \cdots & \cdots & \cdots \\ R & R & \mathfrak{p} \\ \cdots & \cdots & \cdots \\ R & R & R \\ \cdots & \cdots & \cdots \end{pmatrix}$$

where $\mathfrak{p}$ is a unique maximal ideal in R.

REFERENCES

[1] M. Auslander and O. Goldman, *Maximal order* Trans. Amer. Math. Soc. **97** (1960), 1-24.

[2] S. U. Chase, *A generalization of the ring of triangular matrices*, Nagoya Math. J. **18** (1961), 13-25.

[3] S. Eilenberg, *Homological dimension and syzgies*, Ann Math. **64** (1956), 328-336.

[4] S. Eilenberg, H. Nagao and T. Nakayama, *On the dimension of modules and algebras*, IV, Nagoya Math. J. **10** (1956), 87-96.

[5] M. Harada, *Note on the dimension of modules and algebras* J. Inst. Polyt. Osaka City Univ. **7** (1956), 17-27.

[6] M. Harada, *On homological theorems of algebras*, ibid, **10** (1959), 123-127.

[7] M. Harada, *Multiplicative ideal theory in hereditary orders*, J. Math. Osaka City Univ. **14** (1963), 83-106.

[8] M. Harada, *Structure of hereditary orders over local rings*, ibid, **14** (1963), 1-22.

[9] M. Harada, *Hereditary orders*, Trans. Amer. Math. Soc. **107** (1963), 273-290.

[10] N. Jacobson, Structure of Rings, Amer. Math. Soc. Press. (1956).

[11] J. P. Jans and T. Nakayama, *On the dimension of modules and algebras*, VII, Nagoya Math. J. **11** (1957), 67-76.

[12] H. Nagao and T. Nakayama, *On the structure of M_0- and M_u-modules*, Math. Zeits. **59** (1953), 164-170.

[13] A. Rosenberg and D. Zelinsky, *Cohomology of infinite algebras*, Trans. Amer. Math. Soc. **82** (1956), 85-98.

[14] T. Nakano, *A nearly semi-simple ring*, Comm. Math. Univ. Sancti Pauli **7** (1959), 27-33.

Osaka City University

https://doi.org/10.1017/S0027763000026313 Published online by Cambridge University Press