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ON GROUPS WITH SMALL ENGEL DEPTH

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Every finite group G satisfies a law $[x, {}_r y] = [x, {}_s y]$ for some positive integers $r < s$. The minimal value of r is called the depth of G. It is well known that groups of depth 1 are abelian. In this paper we prove the following. Let G be a finite group of depth 2. Then $G/F(G)$ is supersoluble, metabelian and has abelian Sylow p-subgroups for all odd primes p. Moreover, $l_p(G) \leq 1$ for p odd and $l_2(G^2) \leq 1$.

1. Introduction

If G is a finite group, then there exist positive integers $r < s$ such that for all $x, y \in G$ the following holds: $[x, {}_r y] = [x, {}_s y]$. If r is chosen minimal with respect to this property, we call r the (Engel-) depth of G. Let V_r be the class of all finite groups of Engel depth less than or equal to r. Obviously, a finite nilpotent group belongs to V_r if and only if it satisfies the rth Engel condition.

In [7, Theorem 3.2] it has been proved that groups in V_1 are abelian. By contrast, the groups PSL(2, 5) and PSL(2, 8) are of depth 3 (D. Nikolova, Personal Communication).

Here we consider groups of depth 2. It turns out that these groups are soluble. More precisely, we shall prove

THEOREM. *Let* G *be a finite group of depth* 2. *Then*

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(a) $G/F(G)$ *is supersoluble, metabelian and for all odd primes* p *the Sylow* p*-subgroups of* $G/F(G)$ *are abelian,*

(b) *if* p *is an odd prime,* $l_p(G) \leq 1$ *; also* $l_2(G^2) \leq 1$.

Unless otherwise stated, all groups considered in this paper are finite.

2. The structure of groups in V_2

This section is devoted to a proof of the main theorem mentioned in the introduction. We first note a simple observation that turns out to be very useful in the proofs.

LEMMA 1. *Let* $H \in V_2$ *and let* A *be a nilpotent normal subgroup of* H . *Then for each* $a \in A$ *the normal closure* $\langle a^H \rangle$ *is abelian.*

Proof. Let $b \in H$. By assumption, we have $[b, {}_2a] = [b, {}_{2+k}a]$ for some k . So $[b, {}_2a] = [b, {}_{2+kt}a] \in \gamma_{kt+2}(A)$ for all positive integers t . As A is nilpotent, we get $[b, {}_2a] = 1$ and so $[a, a^b] = 1$. This implies that $\langle a^H \rangle$ is abelian.

We now prove that all groups in V_2 are soluble (this fact has been found independently by D. Nikolova). In order to do this, we examine the minimal simple groups (see [*11*]).

LEMMA 2. *The Suzuki groups* $\mathrm{Sz}(q)$ *and* $\mathrm{SL}(3, 3)$ *do not belong to* V_2 .

Proof. Let $G = \mathrm{Sz}(q)$, let A be a Sylow 2-subgroup of G and let $H = N_G(A)$. Any element in H of order $q - 1$ acts transitively on $(A/\Phi(A))^{\#}$ and so for any $a \in A \backslash \Phi(A)$ we have $A = \langle a^H \rangle$. But A is non-abelian and so $H \notin V_2$ by Lemma 1. This proves $G \notin V_2$.

The group $\mathrm{SL}(3, 3)$ contains a subgroup H isomorphic with $\mathrm{SL}(2, 3)$. The same argument yields $\mathrm{SL}(3, 3) \notin V_2$.

We now deal with the remaining minimal simple groups $G = \mathrm{PSL}(2, q)$.

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The search for suitable elements proving $G \notin V_2$ has been eased considerably by computer calculations performed on a TR440 at the Rechenzentrum der Universität Würzburg.

LEMMA 3. *Let* $q \geq 4$ *be a prime power. Then* $\mathrm{PSL}(2, q) \notin V_2$.

Proof. Because of the isomorphism $\mathrm{PSL}(2, 5) \cong \mathrm{PSL}(2, 4)$ we may assume $q \neq 5$. Let $e \in \mathrm{GF}(q)$ with $e^2 \neq \pm 1$. Let

$$x = \begin{pmatrix} -e^2(e^2+1)(e^2-1)^{-1} & (e^2-1)^{-3} \\ -e^2(e^2-1)^2 & e^{-2}(e^2+1)^{-1} \end{pmatrix}$$

and

$$y = \begin{pmatrix} e^{-1} & 0 \\ 0 & e \end{pmatrix} .$$

A straightforward computation yields

$$[x, {}_2y] = \begin{pmatrix} e^{-2} & e^{-2}(e^2-1)^{-1} \\ 0 & e^2 \end{pmatrix} .$$

So for any $k \geq 3$, we have $[x, {}_ky] = \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}$.

As $e^2 \neq \pm 1$ we have shown $[x, {}_2y] \neq \pm[x, {}_ky]$ for all $k \geq 3$. Hence $\mathrm{PSL}(2, q) \notin V_2$.

We now prove the first part of our main theorem.

THEOREM A. *Let* $G \in V_2$. *Then* $G/F(G)$ *is supersoluble.*

Proof. Let G be a minimal counterexample. Lemma 2, Lemma 3 and [11] imply that G is soluble. By [2, 2.9] we know that G is a split extension of its unique minimal normal subgroup N by a complement Q and all proper subgroups of Q are supersoluble. From [5] we infer that Q has a unique normal Sylow subgroup A possessing a complement B in Q . Moreover, $A/\Phi(A)$ is an irreducible B-module and A is noncyclic. Also $\Phi(A) \leq Z(A)$.

We first show that A is elementary abelian. Let $a \in A \backslash \Phi(A)$. By

Lemma 1 we know that $\langle a^B\rangle$ is abelian. As B acts irreducibly on $A/\Phi(A)$, we have $A = \langle a^B\rangle \cdot \Phi(A) = \langle a^B\rangle$ and so A is abelian. The proof of part (f) of [5, Satz 1] now yields that A is elementary.

Let $1 \neq a \in A$ and let $n \in N$ and $b \in B$ be arbitrary. Then $[b, na] = [b, a][b, n]^a$ and so

$$\begin{aligned}[b, {}_2na] &= [[b, a][b, n]^a, na] \\ &= [b, a, na]^{[b,n]^a}[[b, n]^a, na] \\ &= \left([b, a, a][b, a, n]^a\right)^{[b,n]^a}[[b, n]^a, a] \\ &= [b, a, n]^a[b, n, a]^a\end{aligned}$$

as $[b, a, a] = 1$.

From $[b, {}_2na] \in N$ we obtain by a straightforward computation

$$[b, {}_{2+k}na] = [b, a, n, {}_ka]^a[b, n, a, {}_ka]^a .$$

As $G \in V_2$, there exists some k with

$$[b, a, n] \cdot [b, n, a] = [b, a, n, {}_ka][b, n, a, {}_ka] .$$

In particular, we get

$$[b, a, n][b, n, a] \in [N, a]$$

and so

$$[b, a, n] \in [N, a] .$$

Hence $[n, [b, a]] = [n, a^{-b}a] \in [N, a]$ and finally $[n, a^{-b}] \in [N, a]$.

As $n \in N$ has been chosen arbitrarily, we get $[N, a^{-b}] \leq [N, a]$. The latter holds for any $b \in B$ and so $[N, a^{-b_1} \ldots a^{-b_t}] \leq [N, a]$ for all choices $b_i \in B$. As B acts irreducibly on A, we have $A = \langle a^B\rangle$ and so we arrive at $N = [N, A] \leq [N, a]$. This implies $C_N(a) = 1$. Hence every nonidentity element of A acts fixed point freely on N and so A is cyclic. This, however, contradicts the structure of A.

Using Theorem A, we can now prove

THEOREM B. *Let $G \in V_2$. Then for all odd primes p, the quotient $G/F(G)$ has abelian Sylow p-subgroups.*

Proof. Let p be an odd prime and let G be a counterexample of least possible order. From [2, 2.9] we infer that G is a split extension of a uniquely determined minimal normal subgroup $N = F(G)$ by a complement Q. Moreover, all proper subgroups of Q have abelian Sylow p-subgroups. This implies that Q is a nonabelian p-group all of whose proper subgroups are abelian. So Q is nilpotent of class two by a result of Redei [8, p. 309]. Also, N is a p'-group.

We claim that every nonidentity element of Q acts fixed point freely on N. Indeed, let $1 \neq b \in Q$ with $C_N(b) \neq 1$ be given. As Q acts faithfully and irreducibly on N, we have $b \notin Z(Q)$. So there exists $a \in Q$ with $z = [a, b] \neq 1$. Moreover, $z \in Z(Q)$.

Let $n \in C_N(b)$. We now compute $[a, {}_k nb]$. First

$$[a, nb] = zn_1 \quad \text{for some} \quad n_1 \in N .$$

As Q is nilpotent of class two, we have $[a, {}_k nb] \in N$ for all $k \geq 2$. As $G \in V_2$, there exists some positive integer d such that $[a, {}_2 nb] = [a, {}_{2+d} nb]$. Let $n_2 = [a, {}_{1+d} nb]$. Then $[zn_1, nb] = [n_2, nb]$. Hence $zn_1n_2^{-1} \in C_G(nb)$.

As $nb = bn$ and the orders of n and b are coprime, we have $n \in \langle nb \rangle$. So $zn_1n_2^{-1}$ centralizes n. From this we finally get $n \in C_N(z)$. This implies $C_N(b) \leq C_N(z) = 1$ which contradicts the choice of b.

From [6, Theorem 10.3.1, p. 339] we conclude that Q is cyclic. This contradicts the structure of Q.

COROLLARY. *Let $G \in V_2$. Then $G/F(G)$ is metabelian.*

Proof. Theorem A implies that $Q = G/F(G)$ is supersoluble, and so Q' is nilpotent. By Theorem B, all Sylow subgroups of odd order of Q'

are abelian. Let S be a Sylow 2-subgroup of Q. As $G \in V_2$, S satisfies the second Engel condition and so is nilpotent of class two. Hence S' is abelian. As Q is 2-nilpotent, S' is a Sylow 2-subgroup of Q'. So Q' is abelian and the result follows.

From this we can deduce a property of infinite soluble groups of depth two.

COROLLARY. *Let G be poly- (abelian or finite). Assume that for any $x, y \in G$ there exists some positive integer $s = s(x, y) > 2$ such that $[x, {}_2y] = [x, {}_sy]$. Then G is (2-Engel)-by-metabelian.*

Proof. Let U be a finitely generated subgroup of G. From [4, Theorem B] we infer that U is finite-by-nilpotent, and so U is residually finite. Every finite quotient of U belongs to the variety $\underline{\underline{V}}$ of all (2-Engel)-by-metabelian groups. This implies $U \in \underline{\underline{V}}$ and so $G \in \underline{\underline{V}}$.

The remainder of our main theorem now follows from

THEOREM C. *Let $G \in V_2$. Then*

(a) $l_p(G) \leq 1$ *for all odd primes* p,

(b) $l_2(G^2) \leq 1$.

Proof. *(a)* Let G be a counterexample of least possible order. By [8, p. 693], G is a split extension of its unique minimal normal subgroup $N = F(G)$, which is a p-group, by a complement Q. By the Hall-Higman reduction (see [1, p. 258]), Q is a split extension of a normal Sylow q-subgroup A of Q by a p-group B acting irreducibly on $A/\Phi(A)$. From Theorem A we infer that Q is supersoluble and hence A is cyclic. As all nilpotent subgroups of G satisfy the second Engel condition, every p-element of Q acts as a linear map on N with minimal polynomial dividing $(-1+X)^2$. The result now follows from [6, Theorem 11.1.1, p. 359] as G has abelian Sylow r-subgroups for all primes $r \neq p$.

(b) Let F be the class of all extensions of groups having 2-length one by elementary abelian 2-groups. As the product of a subgroup closed saturated formation containing all nilpotent groups with any formation is

saturated, we see that F is saturated.

Let G be a minimal counterexample. Again G is a split extension of a minimal normal subgroup $N = F(G)$ by a complement Q acting faithfully on N. Clearly N is an elementary abelian 2-group. From Theorem A we infer that Q is supersoluble so that, in particular, Q is 2-nilpotent. Let $x \in Q$ be a 2-element. Then $\langle N, x\rangle$ is a second Engel group and so a straightforward computation shows that x is an involution. This proves $l_2(G^2) = 1$ contradicting our choice of G.

3. Some groups of small depth

In the sequel a collection of examples may be found which illustrate that some stronger versions of the above theorems cease to be true. For example, the class V_2 does not contain all metabelian groups as there are metabelian p-groups of arbitrary Engel length. However

PROPOSITION 1 ([9]). *Let G be an extension of an abelian normal subgroup N by an abelian group Q. If the orders of N and Q are coprime, then $G \in V_2$.*

Proof. Let $x, y \in G$. Then $N = C_N(y) \times [N, y] = N_1 \times N_2$. We have $[x, y] = n_1n_2$ for some $n_i \in N_i$. So $[x, {}_2y] = [n_2, y] \in N_2$. As y acts fixed point freely on N_2, we infer from [3, Lemma 4] that there exists some positive integer $d = d(x, y)$ with $n_2 = [n_2, {}_dy]$. Hence $[x, {}_2y] = [x, {}_{2+d}y]$. Let D be the least common multiple of all $d(x, y)$. Then $[x, {}_2y] = [x, {}_{2+D}y]$ for all $x, y \in G$.

An obvious generalization of Proposition 1 to groups of higher derived length does not seem to be at hand as is shown by the following example which has been computed on a TR 440 at the Rechenzentrum der Universität Würzburg.

EXAMPLE. Let G be generated by elements $n_1, \ldots, n_5$, $a_1, \ldots, a_5$, b subject to the following defining relations:

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$n_i^3 = a_i^2 = b^5 = [n_i, n_j] = [a_i, a_j] = [n_i, a_i] = 1$ for all i, j ;

$n_i^{a_j} = n_i^{-1}$ for all $i \neq j$;

$a_i^b = a_{i+1}$, $n_i^b = n_{i+1}$ for $i = 1, \ldots, 4$;

$a_5^b = a_1$;

$n_5^b = n_1$.

Let $x = b$ and $y = n_1 a_1 b$. Then $[x, {}_5y] = [x, {}_{50}y]$ but $[x, {}_4y] \neq [x, {}_ky]$ for all $k > 4$. So the depth of G is at least 5 , but G has derived length 3 .

Another series of groups of depth 2 may be found among Frobenius groups.

PROPOSITION 2. *Let G be a Frobenius group with kernel N and complement Q . If N is abelian and Q is metacyclic then $G \in V_2$.*

Proof. This follows from [3, Lemma 4].

A similar sort of argument proves that any extension of an elementary abelian 2-group by the dihedral group of order $2p$, where p is any odd prime, has depth 2 . So groups in V_2 need not be metanilpotent.

We end with some speculations concerning the general situation. In view of the first corollary to Theorem B one might ask whether there is a bound $f(r)$ depending on r such that for any soluble group in V_r the quotient $G/F(G)$ has derived length less than or equal to $f(r)$. Or, in view of Theorem A, are the ranks of the chief factors of $G/F(G)$ bounded by some function of r ? The answer to both questions, however, is negative in general.

EXAMPLE. Let n be any positive integer. By [10] there exist finite groups of exponent 4 and derived length n . Let Q be such a group of least possible order. Then $Z(Q)$ is cyclic and so there exists a faithful and irreducible $GF(p)$ Q-module N (p denotes any odd prime). Let G

be the split extension of N by Q. Now [12] implies that Q satisfies the 4th Engel condition and so an argument similar to that one used in the proof of Proposition 1 shows $G \in V_5$.

By an analogous construction using a split extension of some faithful and irreducible GF(q) G-module M by G it is possible to disprove the second statement.

Presumably it is essential in this example that the groups under consideration are not generated by two elements. A positive answer to any of these questions for two-generator groups would establish the following

CONJECTURE. There exists a function F such that every soluble group in V_r has Fitting length at most $F(r)$.

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