THE NILPOTENT LENGTH OF FINITE SOLUBLE GROUPS

Dedicated to the memory of Hanna Neumann

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1. Introduction

Throughout this paper a "group" will mean a "finite soluble group".

Carter, Fischer and Hawkes [1] call a group G *critical* for a class $\mathfrak{X}$ of groups if every proper subgroup of G, but not G itself, belongs to $\mathfrak{X}$. We shall say a group is *$\mathfrak{L}$-critical* if it is critical for $\mathfrak{L}(h)$ for some integer n (where $\mathfrak{L}(h)$ denotes the class of groups of nilpotent length at most n). Thus a group is $\mathfrak{L}$-critical if its nilpotent length is greater than that of each of its proper subgroups.

The class of $\mathfrak{L}$-critical groups is not closed under homomorphic images. For example, the group $G = AB$ where A is cyclic of order 3, B is cyclic of order 4, and a generator of B takes a generator of A to its inverse is $\mathfrak{L}$-critical, whereas G/A is not. Knowledge of when a homomorphic image of an $\mathfrak{L}$-critical group is $\mathfrak{L}$-critical can be useful when proving results on $\mathfrak{L}$-critical groups by induction methods. The main theorem of this paper implies that a homomorphic image of an $\mathfrak{L}$-critical group is $\mathfrak{L}$-critical if it possesses a complemented minimal normal subgroup. This result is then used in the last section of this paper, which is an attempt at giving as complete a description of the $\mathfrak{L}$-critical A-groups as possible. (A-groups are groups whose Sylow subgroups are all abelian.) The author was, however, unable to give a complete classification up to isomorphism classes of these groups.

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2. Main Theorem

We first give some notation and definitions and note down some results from [1] that will be needed to prove the main theorem.

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The Frattini subgroup of a group G will be denoted by $\Phi(G)$.

A chief factor H/K of a group G is said to be *complemented* in G if G has a subgroup M such that $HM = G$ and $H \cap M \leqq K$. If no such M exists, then $H/K \leqq \Phi(G/K)$ and in this case H/K is said to be *Frattini*.

When speaking of the kth complemented chief factor of a group G in a chief series

$$1 = G_0 < G_1 < \cdots < G_n = G$$

of G, the counting will be from left to right.

If G is a group then $L(G)$ will denote the smallest normal subgroup of G such that $G/L(G)$ is nilpotent. The *lower nilpotent* series of G is the series $\{L_i(G)\}$ defined by

$$L_0(G) = G,\ L_i(G) = L(L_{i-1}(G)),\ i = 1, 2, \cdots.$$

The Fitting subgroup of G will be denoted by $F(G)$. The *upper nilpotent series* $\{F_i(G)\}$ of G is the series defined by

$$F_0(G) = 1,\ F_i(G)/F_{i-1}(G) = F(G/F_{i-1}(G)),\ i = 1, 2, \cdots$$

The *nilpotent length* of G is the smallest integer n such that $L_n(G) = 1$ (or, equivalently, $F_n(G) = G$) and will be denoted by $l(G)$.

As in [1] the series $\{\Phi_i(G)\}$ is defined by

$$\Phi_i(G)/F_{i-1}(G) = \Phi(G/F_{i-1}(G)),\ i = 1, 2, \cdots.$$

Following [1], we say a group G is *extreme* if $F_i(G)/\Phi_i(G)$ is a chief factor of G for each $i = 1, 2, \cdots, l(G)$. From [1] Theorem 2.8 we have

LEMMA 2.1. *Every factor group of an extreme group is extreme.*

The number of complemented chief factors in a chief series of a group G will be denoted by $c(G)$. (Lemma 2.6 of [1] shows that this number is independent of the particular chief series.) The following characterization of extreme groups is given in [1] Theorem 2.9 (iii).

LEMMA 2.2. *A group G is extreme if and only if $l(G) = c(G)$.*

From [1] Theorem 5.1 follows

LEMMA 2.3. *If G is an $\mathfrak{L}$-critical group then*

(a) *G is extreme, and*

(b) *$G/F(G)$ is also $\mathfrak{L}$-critical.*

The main theorem will now be proved in the form

THEOREM 2.4. *If G is an $\mathfrak{L}$-critical group and M/N a complemented chief factor of G, then G/N is $\mathfrak{L}$-critical.*

Proof. Let $\mathfrak{C}$ be a chief series of G through M and N, and suppose M/N is the kth complemented factor of $\mathfrak{C}$. $(1 \leqq k \leqq c(G))$. If $k = 1$, then $N \leqq \Phi(G)$ and then $l(G/N) = l(G)$, so in this case G/N is $\mathfrak{L}$-critical since G is $\mathfrak{L}$-critical.

Now let $k > 1$ and assume that the theorem is true for the $(k-1)$th complemented factor A/B of $\mathfrak{C}$. Without loss of generality we may assume that $B = 1$. Then M/N is the second complemented factor of some chief series of G.

It follows from Lemma 2.3 that G as well as $G/F(G)$ are extreme. Therefore, by Lemma 2.2, every chief series of G has only one complemented factor below $F(G)$. If $F(G) \cap M \not\leqq N$ then $F(G) \cap M/F(G) \cap N$ would be a complemented chief factor of G and then G would have a series with two complemented chief factors below $F(G)$, which it does not. Therefore $F(G) \cap M \leqq N$. Let $n = l(G)$. Lemma 2.1 implies that G/M as well as G/N are extreme so, by Lemma 2.2,

$$l(G/N) = c(G/N) = n - 1$$

and

$$l(G/M) = c(G/M) = n - 2.$$

Let K/N be a maximal subgroup of G/N. Now $N \not\leqq \Phi(G)$ and $F(G)/\Phi(G)$ is a chief factor of G, so

$$K \geqq \Phi(G)N \geqq F(G).$$

But $G/F(G)$ is $\mathfrak{L}$-critical by Lemma 2.3 (b), hence

$$l(K/F(G)) \leqq n - 2,$$

so that

$$L_{n-2}(K) \leqq F(G) \cap M \leqq N$$

and hence

$$l(K/N) \leqq n - 2 < l(G/N).$$

Therefore G/N is $\mathfrak{L}$-critical.

3. On $\mathfrak{L}$-critical A-groups

The derived group of a group G will be denoted by G'. The *derived series* of G is the series $\{G^{(i)}\}$ defined by

$$G^{(0)} = G, \ G^{(i)} = (G^{(i-1)})' \ i = 1, 2, \cdots.$$

The *derived length* of G is the smallest integer n such that $G^{(n)} = 1$.

In the case of an A-group the lower nilpotent series and the derived series coincide, and the terms "derived length" and "nilpotent length" are identical.

The following result is Lemma 5.2 of [1].

Lemma 3.1. *If G is an extreme group then $L_{i-1}(G)/L_i(G)$ has prime power order for $i = 1, \cdots, l(G)$.*

(Note that our enumeration of the terms of the lower nilpotent series is different from that in [1].)

From [3] Theorem 8.3 follows

LEMMA 3.2. *If G is an A-group then the order of G is divisible by at least l(G) distinct primes.*

A group is said to be *homocyclic* if it is abelian of type $(p^t, p^t, \cdots, p^t)$ for some prime power p^t.

THEOREM 3.3. *Let G be an A-group of derived length n. Then G is $\mathfrak{L}$-critical if and only if the order of G is divisible by exactly n distinct primes and G has Sylow subgroups $M_1, M_2, \cdots, M_n$, corresponding to these primes which have the following properties.*

(i) *M_i normalizes M_j for $i > j$.*

(ii) $G^{(i)} = M_1 M_2 \cdots M_{n-1}$ *for* $i = 0, 1, \cdots, n-1$.

(iii) *Each M_i is homocyclic. Also, M_1 is elementary abelian and M_n is cyclic.*

(iv) *Let $p_i^{t_i}$ be the exponent of M_i $(i = 1, \cdots, n)$. Then the factors*

$$M_1 M_2 \cdots M_{i-1} M_i^{p_i^{k-1}} / M_1 M_2 \cdots M_{i-1} M_i^{p_i^k},\ k = 1, \cdots, t_i,\ i = 1, \cdots, n,$$

are all chief factors of G.

(v) $C_{M_{i+1}}(M_i) = \Phi(M_{i+1})$ *for* $i = 1, \cdots, n-1$.

PROOF. Let G be an $\mathfrak{L}$-critical A-group of derived length n. It follows from Lemma 3.1 and Lemma 3.2 that the order of G is divisible by exactly n distinct primes. Therefore it follows from Lemma 3.1 that $G^{(i-1)}/G^{(i)}$ is a Sylow subgroup of $G/G^{(i)}$ and hence, by the Schur-Zassenhaus Theorem, has a complement in $G/G^{(i)}$ for $i = 1, \cdots, n$. Now let $M_1 = G^{(n-1)}$ and H_2 be a complement for M_1 in G, then let $M_2 = H_2^{(n-2)}$ and H_3 be a complement for M_2 in H_2, let $M_3 = H_3^{(n-3)}$ etc. Choosing the M_i in this way, we see that (i) and (ii) are satisfied.

(iii) Let $i \in \{1, \cdots, n\}$. By [3] Lemma 6.1, M_i can be written as

$$M_i = A_1 \times A_2 \times \cdots \times A_r$$

where each A_i is an indecomposable homocyclic H_{i+1}-group. (Take $H_{n+1} = 1$.) By [3] Corollary 6.2, each factor $A_j/A_j^{p_i}$ (where p_i is the prime divisor of the order of M_i) is an irreducible H_{i-1}-group, and hence the factors

$$M_1 M_2 \cdots M_{i+1} A_j / M_1 M_2 \cdots M_{i-1} A_j^{p_i},\ j = 1, \cdots, r,$$

are all complemented chief factors of G. However, it follows from Lemma 2.2 that every chief series of G has only one complemented factor between $M_1M_2\cdots M_{i-1}$ and $M_1M_2\cdots M_{i-1}M_i$. Therefore $r = 1$ and hence M_i is an indecomposable homocyclic H_{i+1}-group. As we have taken $H_{n+1} = 1$, it follows that M_n is cyclic.

Let $q = p_1^{t_1} - 1$. Then the map $\theta : M_1 \to M_1^q$ defined by

$$m\theta = m^q \quad \text{for all} \quad m \in M_1$$

is an H_2-homomorphism of M_1 onto M_1^q with kernel $M_1^{p_1}$, so $M_1/M_1^{p_1}$ is H_2-isomorphic to M_1^q, and hence $G/M_1^{p_1}$ is isomorphic to $M_1^qM_2\cdots M_n$. But

$$M_1^{p_1} = \Phi(M_1) \leqq \Phi(G),$$

so $l(G/M_1^{p_1}) = n$. But G is $\mathfrak{L}$-critical, so $t_1 = 1$ and hence M_1 is elementary abelian.

(iv) It was shown in (iii) that each M_i is an indecomposable homocyclic H_{i+1}-group. Therefore it follows from [3] Corollary 6.2 that each factor

$$M_i^{p_i^{k-1}} \,/\, M_i^{p_i^k}, \quad k = 1,\cdots,t_i,$$

is an irreducible H_{i+1}-group, and (iv) follows.

(v) By (iv) the factors

$$M_1M_2\cdots M_{i-1}M_i/M_1M_2\cdots M_{i-1}\,\Phi(M_i),\ i = 1,\cdots,n,$$

are all complemented chief factors of G. Therefore, in view of Theorem 2.4 and the fact that

$$C_{M_i}(M_{i-1}) = C_{M_i}(M_{i-1}/\Phi(M_{i-1})),\ i = 2,\cdots,n,$$

(see [2] Theorem 5.1.4), it is sufficient to show that $C_{M_2}(M_1) = \Phi(M_2)$. But $G/F(G)$ is $\mathfrak{L}$-critical by Lemma 2.3 (b) so (iii) implies that $M_2F(G)/F(G)$ is elementary abelian. Therefore

$$M_2^{p_2} \leqq F(G) \leqq C_G(M_1),$$

so $\Phi(M_2) \leqq C_{M_2}(M_1)$. But $M_2/\Phi(M_2)$ is a chief factor of H_2, so $C_{M_2}(M_1) = \Phi(M_2)$.

Conversely, suppose G is an A-group of derived length n which satisfies the conditions of the theorem. Let K by a maximal subgroup of G. Let i be the smallest integer such that $G^{(i)} \nleqq K$ and set $H = M_{n-i}M_{n-i+1}\cdots M_n$. Then $H \cap K$ is a maximal subgroup of H, so

$$\Phi(M_{n-i}) \leqq \Phi(H) \leqq K.$$

It follows from (ii) and (iv) that $G^{(i)}/G^{(i+1)}\Phi(M_{n-i})$ is a minimal normal subgroup of $G/G^{(i+1)}\Phi(M_{n-i})$, so $K/G^{(i+1)}\Phi(M_{n-i})$ complements $G^{(i)}/G^{(i+1)}\Phi(M_{n-i})$ in $G/G^{(i+1)}\Phi(M_{n-i})$. Hence

$$K^{(i)} \leqq K \cap G^{(i)} \leqq G^{(i+1)}\Phi(M_{n-i}).$$

But

$$l(G^{(i+1)}\Phi(M_{n-i})) < n - i$$

by (v). Therefore $l(K) < n$, and hence G is $\mathfrak{L}$-critical.

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