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DRIVING FROM DEGENERACY

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Abstract

A method is proposed for driving degenerate feasible solutions to linear programming problems away from essential degeneracy and in particular for identifying essentially degenerate optimal solutions. An essentially degenerate cycling example is also given, so answering a question raised earlier.

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1. Introduction

In this paper a method is proposed for driving an essentially degenerate optimal solution of a linear programming problem towards a basis where the optimal nature of the solution can be recognised. The method appears to be effective too in driving a non-optimal feasible solution away from essential degeneracy. An example is also provided of a problem with essential degeneracy which can cycle, so answering a question raised in Cameron [2]; however the 'driving method' forces out the essential degeneracy from that problem and so avoids cycling.

In Cameron [2] the lexicographic refinement is mentioned as a technique which can be used to guarantee no cycling when at an essentially degenerate iteration. Ryan and Osborne [6] apply instead Wolfe's ad hoc refinement (see Wolfe [7]), not just to prevent cycling but, at insignificant computational cost, to improve performance when applying the simplex method to highly degenerate problems. The interpretation of Wolfe's technique in terms of finding directions of recession is well expounded in Osborne [5]. Fletcher [4] addresses the important question of how to deal confidently with degeneracy in the practical computational setting where floating point (as opposed to exact) arithmetic is used. Like Wolfe's method, Fletcher's method is recursive and both methods deal locally with degeneracies as they arise, so either of these methods may be more appropriate to use than the lexicographic method in an essentially degenerate situation.

For notation used in this paper we refer to Cameron [2]. In particular, the standard minimum LP problem has the form

$$\min c^T x \text{ subject to } Ax = b \text{ and } x \geq 0$$

and if B is a basis matrix such that $B^{-1}b \geq 0$ then $x_B = B^{-1}b$, $x_F = 0$ is the corresponding basic solution. Furthermore, if $t^T = c_B^T B^{-1} A - c^T$ then $t \leq 0$ guarantees that the basic solution is optimal (the so-called test vector end-criterion).

2. Cycling example

We look at an example, virtually identical with one due to H. W. Kuhn which appeared in Balinski and Tucker [1], namely

$$\begin{aligned} &\min(-2x_1 - 3x_2 + x_3 + 12x_4) \\ \text{subject to} \quad & -2x_1 - 9x_2 + x_3 + 9x_4 \leq 0, \\ & x_1 + 3x_2 - x_3 - 6x_4 \leq 0, \\ & 2x_1 + 3x_2 - x_3 - 12x_4 \leq 2, \\ \text{and} \quad & x \geq 0. \end{aligned}$$

After introducing slack variables, the initial tableau, essentially degenerate, is

$$\begin{array}{c|ccccccc|c} & -2 & -9 & 1 & 9 & 1 & 0 & 0 & 0 \\ & 1 & 3^* & -1 & -6 & 0 & 1 & 0 & 0 \\ & 2 & 3 & -1 & -12 & 0 & 0 & 1 & 2 \\ \hline t^T & 2 & 3 & -1 & -12 & 0 & 0 & 0 & 0 \\ \hline \end{array}$$

The conventional choice of largest $t_l = t_2$ gives the $(2,2)$ pivot, as starred, leading to the second tableau, also essentially degenerate, with only $t_1 = 1$ positive and with first column $(1^*, 1/3, 1)^T$. If (against the lexicographic criterion) we select the first entry as pivot, rather than the second, the next tableau is similar to the first with positive $t_3 = 2$ and $t_4 = 3$. We pivot conventionally, allowing x_4 to become basic and the fourth tableau, similar to the second, has $t_3 = 1$ with third column $(1^*, 1/3, 1)^T$. Pivoting as shown (this time, too late, satisfying the lexicographic requirement) leads to a tableau with $t_5 = 2$

and $t_6 = 1$, fifth column $(-2, 1/3, 2)^T$ and sixth column $(-3, 1/3^*, 1)^T$. If we pivot, this time against convention, at the $(2, 6)$ entry we have for the sixth tableau $t_5 = 1$ and fifth column $(1^*, 1, 1)^T$. Pivoting as shown returns us to the initial tableau and cycling has occurred.

3. Driving method

It is possible, as in Hoffman's example (see Dantzig [3]) to have an essentially degenerate optimal solution which cycles without satisfying the test vector end-criterion. Such a solution may be driven to a basis where $t \leq 0$ by selecting the entering variable x_l so that, for some k where $(B^{-1}b)_k = 0$, t_l and $(B^{-1}A)_{kl}$ are positive and

$$\frac{t_l}{(B^{-1}A)_{kl}} = \max\left\{\frac{t_j}{(B^{-1}A)_{kj}} | t_j > 0 \text{ and } (B^{-1}A)_{kj} > 0\right\}.$$

From the updating formula

$$t(1)_j = t_j - t_l\frac{(B^{-1}A)_{kj}}{(B^{-1}A)_{kl}}$$

we have $t(1)_j \leq 0$ whenever $(B^{-1}A)_{kj} > 0$. Furthermore, if $t(1)_j > 0$ then $(B^{-1}A)_{kj} \leq 0$ so at the next stage $(B(1)^{-1}b)_k = 0$ is no longer *essentially* degenerate.

This technique removes the cycling possibility in Hoffman's example. If used on a non-optimal essentially degenerate tableau, then of course $t(1)_j > 0$ for at least one j. Indeed applying the technique to Kuhn's example above, since $t_1/(B^{-1}A)_{21} = 2 > 1 = t_2/(B^{-1}A)_{22}$ the $(2, 1)$ entry in the initial tableau is chosen. In the second tableau only $t_3 = 1$ is positive, the third column is $(-1, -1, 1^*)^T$ so the situation is no longer essentially degenerate. Using the unique pivot shown, the next tableau gives the (non-degenerate) optimal solution, with basic variables $x_1 = 2 = x_3 = x_5$.

If instead Wolfe's method is used in Kuhn's example (with $(0, 0)$ perturbed to $(1, 1)$ in the second tableau) six tableaux are needed to reach optimality, while if Fletcher's method is used five tableaux are needed with eight level changes, fourth level variables being reached.

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