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L-Series of Certain Elliptic Surfaces

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Abstract. In this paper, we study the modularity of certain elliptic surfaces by determining their L-series through their monodromy groups.

1 Introduction

In this article, we will make an attempt to understand the modularity of elliptic surfaces. This problem is related to many interesting areas of mathematics such as number theory, the theory of elliptic modular surfaces, and the theory of modular forms. In this paper, we will present some preliminary results toward understanding why certain elliptic surfaces to be modular, and why some character sum calculations yield satisfactory results to this end. Basic standard results and notations in the literature which are relevant to our discussion are reviewed in the Sections 2 and 3. Then we use three explicit examples of elliptic surfaces illustrating our approach in establishing their modularity.

Let V be an irreducible smooth projective variety of dimension d over $\mathbb{Q}$. Let p be a prime, and $\bar{V}_p$ be the reduction of V mod p. Suppose $\bar{V}_p \otimes \bar{\mathbb{F}}_p$ is irreducible and smooth over $\bar{\mathbb{F}}_p$. Let N_n be the number of points of $\bar{V}_p$ over the degree n field extension of $\mathbb{F}_p$.

The *local Zeta* function $Z_{\bar{V}_p}(T)$ attached to $\bar{V}_p$ is defined as

$$Z_{\bar{V}_p}(T) = \exp\Big(\sum_{n\geq 1} N_n \frac{T^n}{n}\Big), \tag{1}$$

where T is a variable. By the weak Weil conjecture, proved by B. Dwork, $Z_{\bar{V}_p}(T)$ is a rational function of T [Dwo60]. We now define the L-series of V over $\mathbb{Q}$ as

$$L(V,s) \text{ “=” } \prod_p Z_{\bar{V}_p}(p^{-s})^{-1},$$

where the symbol “=” indicates equality up to finitely many factors related to those p's such that $\bar{V}_p$ is singular. For the full Weil conjecture and basic knowledge of modular forms, refer to the book by W. C. Li [Li96, Chapter 2].

Here, by *modularity* of a variety V defined over $\mathbb{Q}$, we mean the partial L-series of V defined in Section 3, up to p-factors, the product of the Dirichlet series of one or more Hecke eigenforms. In terms of local Zeta function, this means that the number

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$\#(\bar{V}_p/\mathbb{F}_p)$ of rational points of $\bar{V}_p$ over $\mathbb{F}_p$, is related to the Euler p-factor of some Heck eigenforms.

Establishing the modularity of a given variety over $\mathbb{Q}$ is, in general, a difficult question.

In the dimension one case, the proof of the Shimura-Taniyama-Weil conjecture for semi-stable elliptic curves over $\mathbb{Q}$ by Wiles and others is one of the most celebrated results in the last century. It asserts that an elliptic curve E defined over $\mathbb{Q}$ with conductor N is modular, that is, there exists a modular parameterization from the modular curve $X_0(N)$ to E, and the modularity of the modular curve is given by the Eichler-Shimura theory [Shi71, Chapter 7].

However, for higher dimensional varieties, there are only sporadic results. Indeed, for 2 dimensional varieties (*i.e.*, surfaces), the modularity can be realized only in very few special cases so far, for example, singular $K3$ surfaces by T. Shioda and H. Inose [SI77], and elliptic modular surfaces by T. Shioda [Shi72]. There are several different approaches for establishing the modularity of certain surfaces. One approach, employed by S. Ahlgren, and K. Ono [AO00] is to use explicit character sums and the trace formulae. Another approach is based on Serre's mod 2 Galois representation criterion formulated by Livné [Liv87]. This method was applied by H. Verrill to compute the L-series of several Calabi-Yau varieties constructed using toric varieties [Ver00]. For a survey article in modularity of Calabi-Yau manifolds, refer to the paper by N. Yui [Yui01].

Our approach is to establish the modularity of a certain surface through studying its monodromy group. Given an elliptic surface, a theorem of M. Nori [Nor85] gives a criterion for checking whether it is an elliptic modular surface or not. If it is modular, we can study its homological invariant (its monodromy group) by way of its special fibres and the group of global sections. In our cases, we can determine explicitly the monodromy groups up to conjugacy classes. Hence by the theory of elliptic modular surfaces [Shi72], the L-series of these surfaces are related to the weight 3 newforms of the monodromy groups. From this perspective, we also obtain some insight for certain character sum calculations which will be discussed in the Sections 4 and 5.

2 $\mathrm{SL}_2(\mathbb{Z})$, $\mathrm{PSL}_2(\mathbb{Z})$ and Their Subgroups

Let $\mathrm{SL}_2(\mathbb{Z})$ denote the group of all 2×2 integral matrices of determinant 1. The projective special linear group $\mathrm{PSL}_2(\mathbb{Z})$ is defined as the quotient group

$$\mathrm{SL}_2(\mathbb{Z})/\pm I.$$

The group $\mathrm{SL}_2(\mathbb{Z})$ acts naturally on the Poincaré upper-half plane $\mathfrak{H} = \{\tau \in \mathbb{C} \mid \operatorname{Im}\tau > 0\}$ by fractional linear transformations. Explicitly, for any $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$ and $\tau \in \mathfrak{H}$, γ maps τ to $\gamma \cdot \tau = \frac{a\tau+b}{c\tau+d}$. Since $\gamma \cdot \tau = -\gamma \cdot \tau$, the action of $\mathrm{SL}_2(\mathbb{Z})$ on $\mathfrak{H}$ induces an action of $\mathrm{PSL}_2(\mathbb{Z})$ on $\mathfrak{H}$.

Given an integer $N > 0$, the *level N principal congruence subgroup* $\Gamma(N)$ of $\mathrm{SL}_2(\mathbb{Z})$ is defined as

$$\Gamma(N) = \{\gamma \in \mathrm{SL}_2(\mathbb{Z}) \mid \gamma \equiv I \bmod N\}.$$

A subgroup Γ of $\mathrm{SL}_2(\mathbb{Z})$ is called a *congruence subgroup* if it contains $\Gamma(N)$ for some integer $N > 0$. Similarly, we can define the level N principal congruence subgroup and congruence subgroups of $\mathrm{PSL}_2(\mathbb{Z})$.

Given a finite index subgroup Γ of $\mathrm{PSL}_2(\mathbb{Z})$ or $\mathrm{SL}_2(\mathbb{Z})$, Γ acts freely on $\mathfrak{H}$ provided $-I \notin \Gamma$. The orbit space $\mathfrak{H}/\Gamma$ is a Riemann surface of complex dimension 1. The surface $\mathfrak{H}/\Gamma$ can be compactified by adding finitely many points, namely, the equivalent classes of Γ acting on $\mathbb{Q} \cup \{\infty\}$, and they are called the *cusps* of Γ. The compactified surface C_Γ is called the *modular curve associated to* Γ. The singularities of the compactified surfaces are the cusps and the *elliptic points*, which are the fixed points of the torsion elements of Γ.

Given a cusp v of $\Gamma \subset \mathrm{SL}_2(\mathbb{Z})$, the stabilizer of v is a cyclic subgroup of Γ generated by a matrix N conjugate to $\pm\left(\begin{smallmatrix} 1 & b \\ 0 & 1 \end{smallmatrix}\right)$ for some positive integer b. The integer b is called the *cusp width* of v. The cusp v is called the *first kind* if the matrix N is conjugate to $\left(\begin{smallmatrix} 1 & b \\ 0 & 1 \end{smallmatrix}\right)$, and is called the *second kind* otherwise.

A finite index subgroup Γ of $\mathrm{PSL}_2(\mathbb{Z})$ is said to be *torsion-free* if it does not contain any torsion elements and Γ is said to be *genus zero* if the associated modular curve C_Γ is of genus zero. A. Sebbar has classified all the torsion-free genus zero congruence subgroups of $\mathrm{PSL}_2(\mathbb{Z})$ [Seb01].

3 Basics of Elliptic Surfaces

Let C be a curve defined over an algebraically closed field K. An *elliptic surface* X over C is defined as a two dimensional variety with a morphism $\pi\colon X \to C$ and a finite collection of points Σ on C such that if $t \notin \Sigma$ then the fibre $X_t = \pi^{-1}(t)$ is a smooth curve of genus 1. If $t \in \Sigma$, X_t is called a *special fibre*. A global section σ is a morphism $\sigma\colon C \to X$ such that $\pi \circ \sigma$ is the identity map on C. Given an elliptic surface X, there are two important invariants associated to $\pi\colon X \to C$ [Kod63]. One is called the *functional invariant*, which is a rational function and we will also refer it as the j-function of the elliptic surface $\pi\colon X \to C$; another one is called the *homological invariant*, which is a sheaf over C, locally constant over $C - \Sigma$ with general stalk $\mathbb{Z} \oplus \mathbb{Z}$, determined by the monodromy representation $\rho\colon \pi_1(C - \Sigma) \to \mathrm{SL}_2(\mathbb{Z})$.

Subsequently we will be considering elliptic surfaces defined over number fields, or finite fields. We will say an elliptic surface $\pi\colon X \to C$ is defined over a number field or a finite field K if the curve C, the surface X, and the map $\pi\colon X \to C$ are all defined over K. In these cases, we will take models of elliptic surfaces in question defined over K. However, in this section we will assume that the elliptic surfaces are defined over algebraically closed fields unless we specify the field of definition.

Let $\pi_1\colon X_1 \to C$ and $\pi_2\colon X_2 \to C$ be two elliptic surfaces over the same curve C. We say that they are *birationally equivalent as elliptic surfaces over* C if there is a birational map: $f\colon X_1 \to X_2$ with $\pi_1 = \pi_2 \circ f$.

By the work of K. Kodaira, we will assume that the surfaces are *minimal* which means they do not contain any rational curves of self-intersection number -1. Sometimes, we will simply refer to the elliptic surface $\pi\colon X \to C$ as X without specifying the elliptic fibration map π and the base curve C.

Given an elliptic surface X, its minimal compact smooth model $\mathcal{X}$ is unique up to isomorphism, which is called the *Néron model* of X. Any two elliptic surfaces which are birationally equivalent as elliptic surfaces have the same Néron models. K. Kodaira [Kod63] and A. Néron [Nér64] independently classified all the possible types of the special fibres of minimal compact smooth elliptic surfaces. In this paper, we will use Kodaira's symbols of special fibres [Kod63]. J. Tate developed an algorithm for the determination of the special fibres of the Néron model of an elliptic surface given by a Weierstrass equation [Tat75].

The *Néron-Severi group* of an elliptic surface X, denoted by $NS(X)$, is the group of divisors on X modulo algebraic equivalence. It lies in the second cohomology group $H^2(X,\mathbb{Z})$ of X, and its rank, denoted by $\rho(X)$, is called the *Picard number of* X. An elliptic surface can also be viewed as an elliptic curve defined over the functional field of its base curve C. When the j-function of X is not a constant, by the Mordell-Weil Theorem [Lan62, P.71], the global sections of the elliptic surface with respect to the fixed elliptic fibration π form a finitely generated abelian group. This group is called the *Mordell-Weil group* of X, denoted by $MW(X)$. The rank of $MW(X)$ is called the *Mordell-Weil rank* of X, denoted by $r(X)$ in the sequel. T. Shioda explained that the Néron-Severi group of a given elliptic surface $\pi\colon X \to C$ is generated by its Mordell-Weil group and by the irreducible components of the fibres [Shi72]. He obtained the Shioda-Tate formula:

$$\rho(X) = 2 + r + \sum_{t\in\Sigma}(m_t - 1), \tag{2}$$

where m_t is the number of irreducible components of the fibre $X_t = \pi^{-1}(t)$ for $t \in \Sigma$. Using Kodaira's notation, if X_t is of the type I_b, then $m_t = b$, and if X_t is of the type I_b^*, $m_t = 5 + b$.

Now we consider some numerical invariants for a minimal compact smooth elliptic surface X. We will always assume that the j-function of X is not a constant. Let g denote the genus of the base curve C, p_g the geometric genus of X, and c_2 the Euler characteristic of X. K. Kodaira [Kod63] gave a formula to calculate c_2 as follows:

$$\begin{aligned} c_2 &= 12(p_g - g + 1) \\ &= d + 6\sum_{b\geq 0}\nu(I_b^*) + 2\nu(II) + 10\nu(II^*) + 3\nu(III) \\ &\quad + 9\nu(III^*) + 4\nu(IV) + 8\nu(IV^*), \end{aligned} \tag{3}$$

where d is the degree of the j-function, and $\nu(\cdot)$ represents the number of special fibres of X of the type indicated in $(\cdot)$.

Now we consider the elliptic surfaces X given by Weierstrass equations

$$y^2 + a_1(t)xy + a_3(t)y = x^3 + a_2(t)x^2 + a_4(t)x + a_6(t), \quad a_i \in K[t], \tag{4}$$

where K is a perfect field of characteristic different from 2 and 3. By the discussion in [SB85, part II], there exists a minimal compact smooth Néron model $\mathcal{X}$ of X defined over K. In particular, if $K = \mathbb{Q}$, we can investigate the local Zeta function $Z_{\tilde{\mathcal{X}}_p}(T)$ for every prime p. We are mainly interested in the factor $P_{2,p}(\mathcal{X}, T) \in \mathbb{Z}[T]$ of $Z_{\tilde{\mathcal{X}}_p}(T)$, which is the characteristic polynomial of the action of Frobenius on $H^2_{\text{crys}}(\bar{\mathcal{X}}/\mathbb{Z}_p)$. By a Theorem of Shioda [Shi72, Theorem 1.1], we know that, given an elliptic fibration on $\mathcal{X}/\bar{\mathbb{F}}_p$, the algebraic equivalence classes of the general fibre, of the zero section and of the components of the singular fibres which do not meet the zero section, are linearly independent elements in the Néron-Severi group $NS(\mathcal{X}/\bar{\mathbb{F}}_p)$. This gives a subspace V of $NS(\mathcal{X}/\bar{\mathbb{F}}_p) \otimes \mathbb{Q}$. The dimension of this subspace is given by

$$\dim V = 2 + \sum_{t \in \Sigma} (m_t - 1), \tag{5}$$

where m_t is the number of irreducible components of the special fibre X_t as before. We note that $\dim V$ is independent of the characteristic p. Since $NS(\mathcal{X}/\bar{\mathbb{F}}_p) \otimes \mathbb{Q}$ is a subspace of the crystalline cohomology group $H^2_{\text{crys}}(\bar{\mathcal{X}}/\mathbb{Z}_p)$ of $\bar{\mathcal{X}}_p$, so is V. The action of the Frobenius on this space V is p times the map induced by the endomorphism of the space $\mathcal{X}$ which raises coordinates to their p-th power. Let $P_{2,2,p}(\mathcal{X}, T)$ be the characteristic polynomial of the Frobenius on V. Then $P_{2,2,p}(\mathcal{X}, T) \in \mathbb{Z}[T]$. Moreover, $P_{2,p}(\mathcal{X}, T)$ can be decomposed as

$$P_{2,p}(\mathcal{X}, T) = P_{2,2,p}(\mathcal{X}, T) P_{2,1,p}(\mathcal{X}, T),$$

for some $P_{2,1,p}(\mathcal{X}, T) \in \mathbb{Z}[T]$ of degree $b_2(\bar{\mathcal{X}}_p \otimes \bar{\mathbb{F}}_p) - \dim V$. Here $b_2(\bar{\mathcal{X}}_p \otimes \bar{\mathbb{F}}_p)$ is the second Betti number of $\bar{\mathcal{X}}_p \otimes \bar{\mathbb{F}}_p$.

Given an elliptic surface X in the Weierstrass form with $a_i[t] \in \mathbb{Z}[t]$, in this paper, we define the *partial L-series* of its Néron model $\mathcal{X}$ over $\mathbb{Q}$ as

$$L(\mathcal{X}, s) \text{ “=” } \prod_{p \text{ prime}} P_{2,1,p}(\mathcal{X}, p^{-s})^{-1}. \tag{6}$$

Later, we will determine the partial L-series $L(X, s)$ defined by (6) for elliptic surfaces X over $\mathbb{Q}$, rather than the full Hasse-Weil L-series of X.

In [Shi72], T. Shioda defined the elliptic modular surfaces associated to finite index subgroups of $\mathrm{SL}_2(\mathbb{Z})$ which do not contain $-I$. An elliptic modular surface X_Γ associated to such a subgroup Γ can be considered as a universal elliptic curve over the modular curve associated to Γ. For a torsion-free finite index subgroup Γ of $\mathrm{SL}_2(\mathbb{Z})$, the Euler characteristic of X_Γ is given by [Shi72, formula (4.10)]

$$c_2 = \mu + 6t_2, \tag{7}$$

where $\mu = [\mathrm{SL}_2(\mathbb{Z}) : \Gamma \cup -\Gamma]$, and t_2 is the number of cusps of Γ of the second kind. The special fibres of such a surface X_Γ are either of the type I_b for cusps with cusp width b of the first kind or of the type I_b^* for cusp with cusp width b of the second kind. Let $S_3(\Gamma)$ denote the space of weight 3 cusp forms of Γ. Then

$$\dim S_3(\Gamma) = p_g(X_\Gamma). \tag{8}$$

Proposition 3.1 (**[Shi72]**) *The Hodge decomposition of the second cohomology of the surface X_Γ is given as follows:*

$$H^2(X_\Gamma, \mathbb{C}) = S_3(\Gamma) \oplus \bar{S}_3(\Gamma) \oplus \left(NS(X_\Gamma) \otimes \mathbb{C} \right).$$

Furthermore, if X_Γ satisfies an algebraic equation with coefficients in $\mathbb{Z}$, then by the result of P. Deligne [Del73], the partial L-series of X_Γ, defined by the formula (6) is given, up to a finite number of p factors, by

$$L(X_\Gamma, s) = \prod_i D(f_i, s). \tag{9}$$

Here f_i's are normalized cuspidal eigenforms of Γ which form a basis of the space $S_3(\Gamma)$ and $D(f_i, s)$ is the associated Dirichlet series of f_i. Given a normalized cuspidal eigenform $f = \sum_{n\geq 1} a_n q_d^n$, $q_d = e^{\frac{2\pi i z}{d}}$, we define the associated Dirichlet series as

$$D(f, s) = \sum_{n\geq 1} a_n n^{-s}.$$

Theorem 3.2 **(Nori [Nor85])** *Let X be a complex elliptic surface with non-constant j-function such that X which admits a global section. Suppose X satisfies all the following conditions:*

1. *it has no singular fibres of the type II^* or III^*;*
2. *the Hodge number $h^{1,1}$ of X is equal to the Picard number $\rho(X)$;*
3. *the Mordell-Weil rank $r(X) = 0$.*

Then X is an elliptic modular surface associated to a finite index subgroup of $\mathrm{SL}_2(\mathbb{Z})$.

4 The Legendre Elliptic Surface and Its Twist

We first consider the Legendre family of elliptic curves given by the equation $y^2 = x(x-1)(x-t)$, which gives rise to an elliptic surface. Consider the smooth minimal model $\mathcal{X}_1$ of this surface.

Theorem 4.1 *The Néron model $\mathcal{X}_1$ of the Legendre family $y^2 = x(x-1)(x-t)$ is an elliptic modular surface associated to an index* 12 *subgroup Γ_1 of* $\mathrm{SL}_2(\mathbb{Z})$. *The group Γ_1 is conjugate by an element in* $\mathrm{SL}_2(\mathbb{Z})$ *to*

$$\left\{\gamma \in \mathrm{SL}_2(\mathbb{Z}) \;\middle|\; \gamma \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 2 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}, \text{ or } \begin{pmatrix} -1 & 2 \\ 2 & -1 \end{pmatrix} \bmod 4 \right\}.$$

The partial L-series of $\mathcal{X}_1$ defined in (6) is 1.

Proof By the Tate algorithm [Tat75], we know that the special fibres of $\mathcal{X}_1$ are of the types I_2, I_2, I_2^*. By Kodaira's formula (3), $c_2(\mathcal{X}_1) = 12$. This means $\mathcal{X}_1$ a rational surface with geometric genus 0, and the Hodge number $h^{1,1}(X) = \dim H^{1,1}(X) = 10$. By the Shioda-Tate formula (2),

$$\rho(\mathcal{X}_1) = 2 + (2-1) + (2-1) + (5+2-1) + r(\mathcal{X}_1) = 10 + r(\mathcal{X}_1).$$

Since for any compact complex elliptic surface X over a field of characteristic 0, $\rho(X) \leq h^{1,1}(X)$, $r(\mathcal{X}_1) = 0$. By M. Nori's Theorem 3.2, $\mathcal{X}_1$ is elliptic modular. Its monodromy group Γ_1 is a finite index subgroup of $\mathrm{SL}_2(\mathbb{Z})$.

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Now we describe Γ_1 explicitly. The group Γ_1 is torsion-free (since it only has special fibres of the types I_b and I_b^*). It has three cusps, two of them with cusp width 2 of the first kind, and another one with cusp width 2 of the second kind. By the formula (7),

$$12 = 2\mu + 6.$$

Since $-I \notin \Gamma_1$,

$$[\mathrm{SL}_2(\mathbb{Z}) : \Gamma_1] = 2\mu = 12.$$

Furthermore, the Mordell-Weil group of $\mathcal{X}_1$ is isomorphic to $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ since $\mathcal{X}_1$ has 4 torsion sections and the $\mathcal{X}_1$ has a special fibre of the type I_2^*. This implies that for any $\gamma \in \Gamma_1$, and any $m, n \in \mathbb{Z}$, $(\gamma - I)(\frac{m}{2}, \frac{n}{2})^T \in \mathbb{Z}^2$. Hence $\Gamma_1 \subset \Gamma(2)$ of index 2. Now pick any three rational points as the cusps of Γ_1. In this case different choices of cusps determine different monodromy groups which are conjugate to each other in $\mathrm{SL}_2(\mathbb{Z})$. Suppose that the three cusps we pick are ∞, 0, 1 and the stabilizer of ∞ is generated by $\begin{pmatrix} -1 & -2 \\ 0 & -1 \end{pmatrix}$, and the stabilizers of 0, 1 are generated by elements conjugate to $\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$. The above information and assumption determine that the monodromy group Γ_1 is conjugate by an element in $\mathrm{SL}_2(\mathbb{Z})$ to the group:

$$\left\{ \gamma \in \mathrm{SL}_2(\mathbb{Z}) \;\middle|\; \gamma \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 2 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}, \text{ or } \begin{pmatrix} -1 & 2 \\ 2 & -1 \end{pmatrix} \bmod 4 \right\}.$$

Since the surface $\mathcal{X}_1$ is a rational surface, and the space of weight 3 cusp forms of Γ_1 is empty. Hence by the formula (9) the partial L-series of $\mathcal{X}_1$ is trivial. ■

In the Arizona winter school 2000, part of the project of N. Katz's group was to work on the following problem. Prove that

Theorem 4.2 *For any odd prime p*

$$\sum_{t,x \in \mathbb{F}_p} \left(\frac{t(t-1)x(x-1)(x-t)}{p} \right) = \begin{cases} 0 & \text{if } p \equiv 3 \bmod 4; \\ 2(b^2 - a^2) & \text{if } p \equiv 1 \bmod 4, \end{cases}$$

where integers a and b are given by $p = a^2 + b^2$ and $a \equiv 0 \bmod 2$.

One proof of this result can be found in [Eva81].

To get some geometric insight to Theorem 4.2, we consider the elliptic surface X_2, which is the Legendre family twisted by a factor $t(t-1)$:

$$\pi\colon X_2 \to C - \Sigma, \quad \{(x, y, t) \mid y^2 = t(t-1)x(x-1)(x-t)\} \mapsto t,$$

where $\Sigma = \{0, 1, \infty\}$. Let $\mathcal{X}_2$ be the Néron model of X_2.

Theorem 4.3 *The Néron model $\mathcal{X}_2$ of the surface $y^2 = t(t-1)x(x-1)(x-t)$ is a $K3$ surface. It is an elliptic modular surface associated to an index 12 subgroup Γ_2 of $\mathrm{SL}_2(\mathbb{Z})$, which is conjugate by an element in $\mathrm{SL}_2(\mathbb{Z})$ to the group*

$$\left\{ \gamma \in \mathrm{SL}_2(\mathbb{Z}) \;\middle|\; \gamma \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 2 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 2 & -1 \end{pmatrix}, \text{ or } \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \bmod 4 \right\}.$$

The L-series of the surface $\mathcal{X}_2$ is given by

$$L(\mathcal{X}_2, s) = D(f, s), \tag{10}$$

where

$$f(z) = q_2 \prod_{n\geq 1} \big(1 - (q_2)^{4n}\big)^6, \quad q_2 = e^{\frac{2\pi i z}{2}},$$

is the weight 3 *cusp form of* Γ_2 *with integral coefficients.*

Proof The surface $y^2 = t(t-1)x(x-1)(x-t)$ is birationally equivalent as an elliptic surface to the surface $y^2 = x\big(x - t(t-1)\big)\big(x - t^2(t-1)\big)$ by $(x, y) \mapsto \big(xt(t-1), yt(t-1)\big)$. The latter surface is a Weierstrass equation. The smooth minimal models are isomorphic, denoted by $\mathcal{X}_2$ as above. By the Tate algorithm, $\mathcal{X}_2$ have 3 special fibres of the types I_2^*, I_2^*, I_2^* respectively. By Kodaira's formula (3), $c_2(\mathcal{X}_2) = 24$. Hence $\mathcal{X}_2$ is a $K3$ surface and its geometric genus is 1. By the Shioda-Tate formula (2),

$$\rho(\mathcal{X}_2) = 2 + (5 + 2 - 1) + (5 + 2 - 1) + (5 + 2 - 1) + r(\mathcal{X}_2) = 20 + r(\mathcal{X}_2).$$

Since $\rho(\mathcal{X}_2) \leq h^{1,1}(\mathcal{X}_2) = 20$, we have $r(\mathcal{X}_2) = 0$. By Nori's theorem, $\mathcal{X}_2$ is an elliptic modular surface, and the associated subgroup of $\mathrm{SL}_2(\mathbb{Z})$ is denoted by Γ_2.

To determine Γ_2, we first note by the formula that (7)

$$24 = \mu + 6 \cdot 3.$$

Hence

$$[\mathrm{SL}_2(\mathbb{Z}) : \Gamma_2] = 2\mu = 12.$$

Since $MW(\mathcal{X}_2) \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, we have $\Gamma_2 \subset \Gamma(2)$ of index 2. Then by picking ∞, 0, 1 to be the three cusps of Γ_2 whose stabilizers all generated by elements conjugate to $\left(\begin{smallmatrix} -1 & -2 \\ 0 & -1 \end{smallmatrix}\right)$, we obtain that the group Γ_2 is uniquely determined up to conjugacy in $\mathrm{SL}_2(\mathbb{Z})$. It has a representation as

$$\left\{ \gamma \in \mathrm{SL}_2(\mathbb{Z}) \;\middle|\; \gamma \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 2 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 2 & -1 \end{pmatrix}, \text{ or } \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \bmod 4 \right\}.$$

By the formula (8),

$$\dim S_3(\Gamma_2) = p_g(\Gamma_2) = 1.$$

Note that $\Gamma(4) \subset \Gamma_2$, hence $S_3(\Gamma_2) \subset S_3\big(\Gamma(4)\big)$. Since $p_g\big(\Gamma(4)\big) = 1$, it has a generator given by

$$g(z) = q_4 \prod_{n\geq 1} \big(1 - (q_4)^{4n}\big)^6,$$

where $q_4 = e^{\frac{2\pi i}{4}}$ is the uniformizer of the modular curve associated to $\Gamma(4)$ at the cusp ∞. Hence $S_3(\Gamma_2)$ is generated by

$$q_2 \prod_{n\geq 1} \big(1 - (q_2)^{4n}\big)^6 = \sum_{n\geq 1} a_{4n+1} (q^{1/2})^{4n+1},$$

where $q_2 = e^{\frac{2\pi i}{2}}$ is the uniformizer of the modular curve associated to the group Γ_2 at the cusp ∞ and a_{4n+1} are the integers appeared in Theorem 4.2.

Since $\mathcal{X}_2$ is the Néron model for X_2 defined by an equation with coefficients in $\mathbb{Z}$ and $\mathcal{X}_2$ is elliptic modular, by the result of (9), its partial L-series is of the form of (10). ■

We return to consider how Theorem 4.2 is related to Theorem 4.3. By the definition of L-series in (6), the factor $P_{2,1,p}(\mathcal{X}_2, p^{-s})^{-1}$ of the local Zeta function $Z_{\tilde{\mathcal{X}}_p}(p^{-s})$ is related to the local Euler p-factor of f, which can be written as $(1-a_p p^{-s}+p^{2-s})^{-1}$. Hence a_p is related to the coefficients of $P_{2,1,p}(T)$. According to the definition of local Zeta functions (1), we need to consider the number of rational points of $\mathcal{X}_2$ over $\mathbb{F}_p$. For an odd prime p, this number is equal to $\#(X_2/\mathbb{F}_p)$, the number of rational points of X_2 over $\mathbb{F}_p$, plus the contributions from the compactified fibres over Σ. We note that the calculation for $\#(X_2/\mathbb{F}_p)$ is essential for finding the local Zeta function of Y.

We have

$$\#(X_2/\mathbb{F}_p) = \sum_{x,t\in\mathbb{F}_p, t\neq 0,1} \left(\left(\frac{t(t-1)x(x-1)(x-t)}{p} \right) + 1 \right) \tag{11}$$

$$= \sum_{x,t\in\mathbb{F}_p} \left(\frac{t(t-1)x(x-1)(x-t)}{p} \right) + p(p-2), \tag{12}$$

where $(\frac{\cdot}{p})$ is the Legendre symbol. Hence the value of the character sum

$$S_p = \sum_{x,t\in\mathbb{F}_p} \left(\frac{t(t-1)x(x-1)(x-t)}{p} \right)$$

is related to the p-th coefficient of f. It is easy to see that $S_p = 0$ when $p \equiv 3 \bmod 4$.

Indeed for any odd prime $p \equiv 1 \bmod 4$,

$$\sum_{x,t\in\mathbb{F}_p} \left(\frac{t(t-1)x(x-1)(x-t)}{p} \right) = a_p = 2(b^2 - a^2),$$

where $p = a^2 + b^2$ and $a \equiv 0 \bmod 2$.

5 The Elliptic Surface Related to Generalized Congruence Numbers

Another elliptic surface which is of similar flavor and related to the above two surfaces is give by

$$y^2 = x(x - 1/t)(x + t). \tag{13}$$

This surface arises from the study of a generalized congruent number problem by Long [Lon03].

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Let us briefly discuss how to obtain this surface. Let a, b, c be the lengths of three sides of a triangle with area 1 and suppose that the angle α between a and b satisfies $\sin(\alpha) = \frac{2t}{t^2+1}$ and $\cos(\alpha) = \frac{t^2-1}{t^2+1}$. Letting $x = c^2/4$ and $y = c(a^2 - b^2)/8$, then x and y satisfy the equation (13).

Theorem 5.1 *The Néron model $\mathcal{X}_3$ of the elliptic surface*

$$y^2 = x(x - 1/t)(x + t)$$

is a $K3$ surface. It is an elliptic modular surface associated to a congruence subgroup Γ_3 of $\mathrm{SL}_2(\mathbb{Z})$. Up to conjugation by an element in $\mathrm{SL}_2(\mathbb{Z})$, Γ_3 is given by

$$\Gamma_3 := \gamma^{-1}\big(\Gamma(4) \cup \alpha\Gamma(4)\big)\,\gamma,$$

where $\gamma = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 2 \end{smallmatrix}\right)$ and $\alpha = \left(\begin{smallmatrix} -1 & -2 \\ 0 & -1 \end{smallmatrix}\right)$.

The L-series of $\mathcal{X}_3$ is given by $L(\mathcal{X}_3, s) = D(g, s)$, where

$$g(z) := \sum_{n\geq 1} a_n \left(\frac{2}{n}\right) q_8^n, \quad q_8 = e^{\frac{2\pi i z}{8}}$$

is a weight 3 modular form associated to Γ_3, and a_n are the coefficients of the modular form f which is defined in Theorem 4.2.

Proof By the Tate algorithm, $\mathcal{X}_3$ has 4 special fibres of the types I_2, I_24, I_4^*, I_4^* respectively. By the formula (2), we obtain that it is a $K3$ surface with the Picard number is 20, and the Mordell-Weil rank of $\mathcal{X}_3$ is 0. By Nori's Theorem 3.2, it is an elliptic modular surface.

To obtain the modular group associated to this surface, we first note that it is relatively easy to get a torsion-free finite index subgroup Γ_3' of $\mathrm{SL}_2(\mathbb{Z})$ with 4 cusps: two of them with cusp width 4 of the first kind and another two with cusp width 2 of the second kind. Without loss of generosity, we many assume that 0 and 1 are two cusps with cusp width 4 of the first kind; ∞ and $1/2$ are two cusps with cusp width 2 of the second kind. This group Γ_3' is determined as

$$\Gamma_3' = \left\{ \gamma \in \mathrm{SL}_2(\mathbb{Z}) \;\middle|\; \gamma \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},\ \text{or}\ \begin{pmatrix} -1 & -2 \\ 0 & -1 \end{pmatrix} \bmod 4 \right\}.$$

We note that $\Gamma(4) \subset \Gamma_3'$ of index 2, and $\mu = [\mathrm{SL}_2(\mathbb{Z}) : \pm\Gamma_3'] = 12$. Let $X_{\Gamma_3'}$ the elliptic modular surface associated to Γ_3', by the formula (7), $c_2(X_{\Gamma_3'}) = 12 + 2\cdot 6 = 24$. Hence

$$\dim\big(S_3(\Gamma_3')\big) = \dim\Big(S_3\big(\Gamma(4)\big)\Big) = 1.$$

It follows that the normalized new form $f' = q_4 \prod_{n\geq 1}(1 - q_4^n)^6$ of $\Gamma(4)$ is induced from Γ_3'. In other words $S_3(\Gamma_3') = \langle f' \rangle$.

Now let $\Gamma_3 = \gamma^{-1}\Gamma_3'\gamma$, where $\gamma = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 2 \end{smallmatrix}\right)$. We can check that $\Gamma(8) \subset \Gamma_3$ of index 16, and the group Γ_3 has 4 cusps: ∞ and 0 with cusp width 4 of the second kind; 1

and -1 with cusp width 2 of the first kind. This group is determined by $\mathcal{X}_3$ up to conjugation by an element in $\mathrm{SL}_2(\mathbb{Z})$. Moreover,

$$\dim S_3(\Gamma_3) = \dim S_3(\Gamma_3') = 1,$$

and $S_3(\Gamma_3) = \langle f'(\gamma z) = f'(\frac{z}{2} + \frac{1}{2})\rangle$, where

$$\begin{aligned} f'\Big(\frac{z}{2}+\frac{1}{2}\Big) &= \sum_{n\geq 1} a_{4n+1} e^{\frac{2\pi i(4n+1)}{4}(\frac{z}{2}+\frac{1}{2})} \\ &= i\sum_{8n+1} a_{8n+1} e^{\frac{2\pi i(8n+1)}{8}z} - i\sum_{8n+5} a_{8n+5} e^{\frac{2\pi i(8n+5)}{8}z} \\ &= i\sum_{4n+1} a_{4n+1}\Big(\frac{2}{4n+1}\Big) e^{\frac{2\pi i(4n+1)}{8}z} \\ &= i\sum_{n\geq 1} a_n\Big(\frac{2}{n}\Big) e^{\frac{2\pi i n}{8}z} \\ &= i\sum_{n\geq 1} a_n\Big(\frac{2}{n}\Big) q_8^n \\ &= ig(z). \end{aligned}$$

Since $\mathcal{X}_3$ is the Néron model of X_3 defined by an algebraic equation with integral coefficients and it is elliptic modular, hence its L-series defined by (9) is

$$L(\mathcal{X}_3, s) = L(g, s). \qquad \blacksquare$$

Another method for establishing the last claim of Theorem 5.1 is by a calculation of the character sum. Consider another surface $y^2 = xt(xt-1)(x+t)$, which is birationally equivalent as elliptic surfaces to the surface $y^2 = x(x-1/t)(x+t)$ by sending $(x, y) \mapsto (x, y/t)$.

Lemma 5.2 *For any odd prime p*

$$\sum_{t,x\in\mathbb{F}_p}\Big(\frac{tx(xt-1)(x+t)}{p}\Big) = \begin{cases} 0 & \text{if } p \equiv 3 \bmod 4, \\ (\frac{2}{p})2(b^2-a^2) & \text{if } p \equiv 1 \bmod 4, \end{cases}$$

where integers a and b are given by $p = a^2 + b^2$ and $a \equiv 0 \bmod 2$.

Proof When $p \equiv 3 \bmod 4$, we have $(\frac{-1}{p}) = -1$. If we replace x by $-x$ and t by $-t$, the product is differed by -1. Hence the sum is 0 by symmetry.

Now we assume $p \equiv 1 \bmod 4$. The surface $y^2 = -itx(tx-1)(x+t)$ is birationally equivalent to the surface $y^2 = xt(x-1)(t-1)(x-t)$ by the following birational map:

$$(x, y, t) \to \Big(\frac{-it-1}{xt-1}, \frac{(x+i)(t-1)y}{(xt-1)^3}, \frac{ix-1}{xt-1}\Big),$$

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over any field containing i, for example, $\mathbb{F}_p$ when $p \equiv 1 \bmod 4$.

When $p \equiv 5 \bmod 8$, $(\frac{2}{p}) = -1$, $i = 2^{\frac{p-1}{4}}$ in $\mathbb{F}_p$; when $p \equiv 1 \bmod 8$, such i is a square in $\mathbb{F}_p$ and $(\frac{2}{p}) = 1$. In either case, we have $(\frac{i}{p}) = (\frac{2}{p})$.

Hence we have

$$\begin{aligned}\sum_{x,t\in\mathbb{F}_p, xt\neq 1}\left(\frac{tx(xt-1)(x+t)}{p}\right) &= \left(\frac{2}{p}\right)\sum_{x,t\in\mathbb{F}_p}\left(\frac{2tx(xt-1)(x+t)}{p}\right)\\ &= \left(\frac{2}{p}\right)\sum_{x,t\in\mathbb{F}_p}\left(\frac{-itx(xt-1)(x+t)}{p}\right)\\ &= \left(\frac{2}{p}\right)\sum_{x,t\in\mathbb{F}_p}\left(\frac{t(t-1)x(x-1)(x-t)}{p}\right).\end{aligned}$$

This lemma follows from Theorem 4.2. ■

Remark 5.3 The projections of Γ_1, Γ_2 and Γ_3' in $\mathrm{PSL}_2(\mathbb{Z})$ are all the same, *i.e.*

$$\pm\Gamma_1/\pm I = \pm\Gamma_2/\pm I = \pm\Gamma_3'/\pm I = \pm\Gamma(2)/\pm I.$$

This projection to $\mathrm{PSL}_2(\mathbb{Z})$ is in the list of Sebbar's classification of genus zero, torsion-free congruence subgroup of $\mathrm{PSL}_2(\mathbb{Z})$.

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