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BOUNDING THE ORDER OF THE NILPOTENT RESIDUAL OF A FINITE GROUP

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Abstract

The last term of the lower central series of a finite group G is called the nilpotent residual. It is usually denoted by $\gamma_\infty(G)$. The lower Fitting series of G is defined by $D_0(G) = G$ and $D_{i+1}(G) = \gamma_\infty(D_i(G))$ for $i = 0, 1, 2, \ldots$. These subgroups are generated by so-called coprime commutators γ_k^* and δ_k^* in elements of G. More precisely, the set of coprime commutators γ_k^* generates $\gamma_\infty(G)$ whenever $k \geq 2$ while the set δ_k^* generates $D_k(G)$ for $k \geq 0$. The main result of this article is the following theorem: let m be a positive integer and G a finite group. Let $X \subset G$ be either the set of all γ_k^*-commutators for some fixed $k \geq 2$ or the set of all δ_k^*-commutators for some fixed $k \geq 1$. Suppose that the size of a^X is at most m for any $a \in G$. Then the order of $\langle X \rangle$ is (k, m)-bounded.

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1. Introduction

All groups considered in the present article are finite. The last term of the lower central series of a group G is called the nilpotent residual. It is usually denoted by $\gamma_\infty(G)$. The lower Fitting series of G is defined by $D_0(G) = G$ and $D_{i+1}(G) = \gamma_\infty(D_i(G))$ for $i = 0, 1, 2, \ldots$.

It was shown in [8] that these subgroups are generated by so-called coprime commutators γ_k^* and δ_k^* in elements of G. These were introduced in [8] with the purpose of studying properties of finite groups that can be expressed in terms of commutators of elements of coprime orders. The definition goes as follows. Every element of G is both a γ_1^*-commutator and a δ_0^*-commutator. Now let $k \geq 2$ and let S be the set of all elements of G that are powers of γ_{k-1}^*-commutators. An element g is a γ_k^*-commutator if there exist $a \in S$ and $b \in G$ such that $g = [a, b]$ and $(|a|, |b|) = 1$. For $k \geq 1$, let T be the set of all elements of G that are powers of δ_{k-1}^*-commutators. The element g is a δ_k^*-commutator if there exist $a, b \in T$ such that $g = [a, b]$ and $(|a|, |b|) = 1$. One can easily see that if N is a normal subgroup of G and x an element whose image in G/N

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is a γ_k^*-commutator (respectively a δ_k^*-commutator), then there exists a γ_k^*-commutator (respectively a δ_k^*-commutator) $y \in G$ such that $x \in yN$. It was shown in [8] that for every $k \geq 2$ the subgroup generated by γ_k^*-commutators is precisely $\gamma_\infty(G)$ and, for every $k \geq 0$, the subgroup generated by δ_k^*-commutators is precisely $D_k(G)$.

There are several results in the literature that show that in some situations the order of $\gamma_\infty(G)$ can be bounded (cf. [1, 3, 4, 6]). In particular, it was shown in [1] that if G contains at most m γ_k^*-commutators, then the order of $\gamma_\infty(G)$ is m-bounded and, if G contains at most m δ_k^*-commutators, then the order of $D_k(G)$ is m-bounded. Throughout the article we use the expression '$(m, n, \dots)$-bounded' to abbreviate 'bounded from above in terms of $m, n, \dots$ only'. It is interesting to note that the bounds in this result do not depend on k. In the present article we discover a new phenomenon that also implies bounds for the order of the subgroups $D_k(G)$.

If X is a nonempty subset of a group G and $a \in G$, we write a^X to denote the set $\{x^{-1}ax \mid x \in X\}$. By $\langle X \rangle$, we denote the subgroup generated by X.

Our goal in the present article is to prove the following theorem.

Theorem 1.1. *Let m be a positive integer and G a group. Let $X \subset G$ be either the set of all γ_k^*-commutators for some fixed $k \geq 2$ or the set of all δ_k^*-commutators for some fixed $k \geq 1$. Suppose that the size of a^X is at most m for any $a \in G$. Then the order of $\langle X \rangle$ is (k, m)-bounded.*

2. Proof of the theorem

Given a subset Y of a group G, we say that a subgroup $H \leq G$ has Y-index t if Y is contained in a union of precisely t right cosets of H in G. The next lemma is straightforward. It is similar to [5, Lemma 2.1].

Lemma 2.1. *Let Y be a subset of a group G and $H_1, \dots, H_s$ subgroups. Suppose that $H_1, \dots, H_s$ have Y-indexes $m_1, \dots, m_s$, respectively. Then the intersection $\bigcap_i H_i$ has Y-index at most $m_1 m_2 \cdots m_s$.*

The following observation is self-evident.

Lemma 2.2. *Assume the hypothesis of Theorem 1.1. For each $a \in G$, the X-index of $C_G(a)$ is at most m.*

We remark that whenever an element x is a δ_k^*-commutator in a group G, there exist at most 2^k elements $a_1, \dots, a_{2^k} \in G$ such that x is a δ_k^*-commutator in $\langle a_1, \dots, a_{2^k} \rangle$. Similarly, whenever x is a γ_k^*-commutator in a group G, there exist at most k elements $a_1, \dots, a_k \in G$ such that x is a γ_k^*-commutator in $\langle a_1, \dots, a_k \rangle$.

Lemma 2.3. *Assume the hypothesis of Theorem 1.1. There exists a (k, m)-bounded positive integer n such that $x^n \in Z(G)$ for every $x \in X$.*

Proof. We will prove the lemma in the case where X is the set of all δ_k^*-commutators in G. The case where X is the set of all γ_k^*-commutators can be proved in the same manner. Choose $x \in X$. There exist 2^k elements $a_1, \dots, a_{2^k} \in G$ such that x

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is a δ_k^*-coprime commutator in $\langle a_1, \ldots, a_{2^k}\rangle$. Let $a_0 \in G$ be any element and set $E = \langle a_0, a_1, \ldots, a_{2^k}\rangle$. Of course, x is a δ_k^*-commutator in E. For each i, we have $|a_i^X| \leq m$. By Lemma 2.2, $C_G(a_i)$ has X-index at most m. Since $Z(E) = \bigcap_i C_E(a_i)$, it follows from Lemma 2.1 that $Z(E)$ has X-index at most m^{2^k+1}. Therefore, there are at most m^{2^k+1} δ_k^*-commutators in the quotient $E/Z(E)$. The main result of [1] now tells us that the order of $\delta_k^*(E/Z(E))$ is (k, m)-bounded. Since x is a δ_k^*-commutator in E, we conclude that the image of x in the quotient $E/Z(E)$ has (k, m)-bounded order. Hence, there exists a (k, m)-bounded positive integer n such that $x^n \in Z(E)$. It is clear that x^n commutes with a_0, which was chosen in G arbitrarily. Therefore, $x^n \in Z(G)$. This completes the proof. □

Let a be any element of the group G and let $u_1, \ldots, u_{s-1}, u_s = 1$ be elements of X such that $a^X = \{a^{u_1}, \ldots, a^{u_{s-1}}, a\}$. Our hypotheses imply that the elements $u_1, \ldots, u_{s-1}, u_s = 1$ can be chosen with $s \leq m$. The next lemma and proposition mimic parts of the proof of [2, Theorem 1.2].

Lemma 2.4. *Let h be an element of $\langle X\rangle$ and write $h = x_1 \cdots x_l$, where $x_1, \ldots, x_l \in X$. Then*

$$a^h = a^{u_{i_1} u_{i_2} \cdots u_{i_l}}$$

for some $1 \leq i_1, i_2, \ldots, i_l \leq s$.

Proof. We argue by induction on l. If $l = 1$, then $h = x_1$ and $a^h = a^{x_1} = a^{u_{i_1}}$ for some $1 \leq i_1 \leq s$. Suppose that the lemma holds for all elements of $\langle X\rangle$ which can be written as products of at most $l - 1$ elements of X. We have $a^{x_1} = a^{u_i}$ for some $1 \leq i \leq s$. Write

$$a^h = a^{x_1 \cdots x_l} = a^{u_i x_2 \cdots x_l} = a^{c_2 \cdots c_l u_i},$$

where $c_j = u_i x_j u_i^{-1}$. Note that $c_j \in X$ since X is a normal set of G. By the inductive hypothesis,

$$a^{c_2 \cdots c_l} = a^{u_{i_1} u_{i_2} \cdots u_{i_{l-1}}}$$

for some $1 \leq i_1, \ldots, i_{l-1} \leq s$. Consequently,

$$a^h = a^{c_2 \cdots c_l u_i} = a^{u_{i_1} u_{i_2} \cdots u_{i_{l-1}} u_i},$$

as desired. □

Proposition 2.5. *Assume the hypothesis of Theorem 1.1. There exists a (k, m)-bounded positive integer t such that for each $a \in G$, the index $[\langle X\rangle : C_{\langle X\rangle}(a)]$ is at most t.*

Proof. Choose arbitrarily $a \in G$ and let $u_1, \ldots, u_{s-1}, u_s = 1$ be elements of X such that $a^X = \{a^{u_1}, \ldots, a^{u_{s-1}}, a\}$ with $s \leq m$.

Define an ordering $<$ on the set of all (formal) products $u_{i_1} u_{i_2} \cdots u_{i_l}$ for $l \geq 1$ and $1 \leq i_j \leq s$ as follows. Put

$$u_{i_1} u_{i_2} \cdots u_{i_l} < u_{j_1} u_{j_2} \cdots u_{j_{l'}}$$

if and only if one of the following conditions is satisfied:

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(i) $l < l'$; or
(ii) $l = l'$ and there is an index $r \leq l$ such that $i_r < j_r$ and $i_v = j_v$ for all $v > r$.

For an element $h \in \langle X \rangle$, let $u_{i_1} u_{i_2} \cdots u_{i_l}$ be the smallest product of the elements $u_1, \ldots, u_s$ such that $a^h = a^{u_{i_1} u_{i_2} \cdots u_{i_l}}$. Let us show that $i_1 \geq \cdots \geq i_l$.

Indeed, suppose that there exists n such that $i_n < i_{n+1}$. Then

$$a^h = a^{u_{i_1} \cdots u_{i_{n-1}} u_{i_n} u_{i_{n+1}} u_{i_{n+2}} \cdots u_{i_l}} = a^{u_{i_1} \cdots u_{i_{n-1}} u' u_{i_n} u_{i_{n+2}} \cdots u_{i_l}},$$

where $u' = u_{i_n} u_{i_{n+1}} u_{i_n}^{-1} \in X$. By Lemma 2.4,

$$a^{u_{i_1} \cdots u_{i_{n-1}} u'} = a^{u_{j_1} \cdots u_{j_{n-1}} u_{j_{n+1}}}$$

for some $1 \leq j_1, \ldots, j_{n-1}, j_{n+1} \leq s$. Consequently,

$$a^h = a^{u_{j_1} \cdots u_{j_{n-1}} u_{j_{n+1}} u_{i_n} u_{i_{n+2}} \cdots u_{i_l}}.$$

This is a contradiction with the choice of the smallest product $u_{i_1} u_{i_2} \cdots u_{i_l}$, since

$$u_{j_1} \cdots u_{j_{n-1}} u_{j_{n+1}} u_{i_n} u_{i_{n+2}} \cdots u_{i_l} < u_{i_1} \cdots u_{i_{n-1}} u_{i_n} u_{i_{n+1}} u_{i_{n+2}} \cdots u_{i_l}$$

(it was assumed that $i_n < i_{n+1}$).

Thus, for an arbitrary element $h \in \langle X \rangle$, $a^h = a^{u_{i_1} u_{i_2} \cdots u_{i_l}}$, where $i_1 \geq i_2 \geq \cdots \geq i_l$ or, equivalently,

$$a^h = a^{u_{s-1}^{m_{s-1}} \cdots u_2^{m_2} u_1^{m_1}}$$

for some nonnegative integers $m_1, m_2, \ldots, m_{s-1}$. By Lemma 2.3, there exists a (k, m)-bounded positive integer n such that $y^n \in Z(G)$ for each $y \in X$. Therefore, we may assume that $m_i \leq n$ for all $i = 1, 2, \ldots, s-1$. Consequently, $|a^{\langle X \rangle}| \leq (n+1)^m$. □

As usual, if a group H acts on a group V, we denote by $[V, H]$ the subgroup generated by all elements of the form $v^{-1} v^h$, where $v \in V$ and $h \in H$. We will require the following proposition (cf. [7, Lemma 2.3]).

Proposition 2.6. *Let p be a prime and V an abelian p-group acted on by a p'-group H. Suppose that the order of $[V, h]$ is at most t for all $h \in H$. Then the order of $[V, H]$ is t-bounded.*

Proof of Theorem 1.1. Recall that X is either the set of all γ_k^*-commutators for some fixed $k \geq 2$ or the set of all δ_k^*-commutators for some fixed $k \geq 1$ in G. By the hypothesis, the size of a^X is at most m for any $a \in G$. We wish to prove that the order of $\langle X \rangle$ is (k, m)-bounded. By Proposition 2.5, there exists a (k, m)-bounded positive integer t such that for each $a \in G$ the index $[\langle X \rangle : C_{\langle X \rangle}(a)]$ is at most t. Thus, a theorem of Wiegold tells us that the order of the commutator subgroup $\langle X \rangle'$ is (k, m)-bounded [9]. We pass to the quotient $G/\langle X \rangle'$ and without loss of generality assume that $\langle X \rangle$ is abelian. In particular, without loss of generality we assume that G is soluble. Let $\pi(\langle X \rangle) = \{p_1, \ldots, p_s\}$ be the set of prime divisors of the order of $\langle X \rangle$ and $P_1, \ldots, P_s$ be the corresponding Sylow subgroups of $\langle X \rangle$. Our hypotheses imply that $\langle X \rangle = D_k(G)$ is the kth term of the lower Fitting series of G with $k \geq 1$. For each $i = 1, \ldots, s$, choose a Hall p'-subgroup H_i in $D_{k-1}(G)$. If $a \in H_i$, we have $P_i = [P_i, a] \times C_{P_i}(a)$. Therefore, by Proposition 2.5, $|[P_i, a]|$ is (k, m)-bounded. It follows from Proposition 2.6 that $|[P_i, H_i]|$ is (k, m)-bounded as well. By [1, Lemma 2.4], $P_i = [P_i, H_i]$ for each $i = 1, \ldots, s$. Since the order of $\langle X \rangle$ is just the product of orders of its Sylow subgroups, the result follows. □

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