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HOMOMORPHISMS BETWEEN ALGEBRAS OF CONTINUOUS FUNCTIONS

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Introduction: preliminaries and notations. We are concerned in this paper with the study of homomorphisms between different algebras of continuous functions, especially the algebras of real functions which are either weakly continuous on bounded sets or weakly uniformly continuous on bounded sets on a Banach space (see definitions below).

These spaces of weakly [uniformly] continuous functions appeared in relation with some questions in Infinite-dimensional Approximation Theory (see [**4**], [**6**], [**11**], [**12**], [**13**] and [**16**]); and since the structure of these function spaces is closely related with properties of different weak topologies (the bounded-weak and bounded-weak* topologies, respectively) and with the structure of Banach spaces on which they are defined, their study also presents interest from the point of view of Banach space theory, as can be seen in [**2**], [**12**] or [**17**].

On the other hand, some aspects of the topological algebra structure of these function algebras have been studied in [**17**] and [**14**]; and homomorphisms were also studied, for the special case of algebras of weakly uniformly continuous functions on bounded sets, in [**5**].

Now we start with some notations and definitions:

Let E be a real Banach space; E^* will denote the dual space and E^{**} the bidual. The space E, endowed with its weak topology $w = \sigma(E, E^*)$, will be denoted E_w, and E^{**}, endowed with its weak* topology $w^* = \sigma(E^{**}, E^*)$, will be denoted $E^{**}_{w^*}$. Analogous notations will be used for different topologies on E or E^{**}.

For a topological space T, we denote $C_w(E, T)$ the space of all continuous maps $f : E_w \longrightarrow T$; and we denote $C_{wb}(E, T)$ the space of all maps $f : E \longrightarrow T$ such that, for every bounded subset $B \subset E$, the restriction

$$f|_B : (B, w) \longrightarrow T$$

is continuous. For a locally convex space V, $C_{wbu}(E, V)$ denotes the space of all maps $f : E \longrightarrow V$ such that, for every bounded subset $B \subset E$,

$$f|_B : (B, w) \longrightarrow V$$

is uniformly continuous. For real-valued functions, we denote

$$C_{wb}(E) = C_{wb}(E, \mathbf{R}) \quad \text{and} \quad C_{wbu}(E) = C_{wbu}(E, \mathbf{R}).$$

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The algebra $C_{wb}(E)$ is endowed with the topology τ_{wc} of uniform convergence on weakly compact subsets of E, that is, topology generated by all seminorms of the form

$$f \longrightarrow \sup\{|f(x)| : x \in K\},$$

where K ranges over the weakly compact subsets of E; and the algebra $C_{wbu}(E)$ is endowed with the topology τ_b of uniform convergence on bounded subsets of E, that is, topology generated by all seminorms of the form

$$f \longrightarrow \sup\{|f(x)| : x \in B\},$$

where B ranges over the bounded subsets of E; note that $C_{wbu}(E)$ is a Fréchet Algebra (see [**20**], Chapter 4, as a reference about these spaces).

For each topological space X we denote by $C(X)$ the algebra of all continuous real functions defined on X, endowed with its compact-open topology.

Now let F be a real Banach space. We are interested in the unital algebra *homomorphisms* (i.e., multiplicative linear operators carrying the unit to the unit) defined on $C_{wb}(E)$ or $C_{wbu}(E)$ and with values in $C_{wb}(F)$ or $C_{wbu}(F)$. As an application, we will also study the following interpolation problem: given a bounded sequence $(a_n)_{n\in\mathbf{N}} \subset E$ and a sequence $(\lambda_n)_{n\in\mathbf{N}}$ of real numbers, in which cases does there exist $f \in C_{wbu}(E)$ such that $f(a_n) = \lambda_n$?

For the treatment of these problems, we will use the technique of representing $C_{wb}(E)$ and $C_{wbu}(E)$ as algebras of continuous real functions defined on a completely regular topological space, by means of the following topologies (see also [**20**], Chapter 4, as a reference):

Topology bw on E is the finest topology which agrees with weak topology on each bounded subset of E; it is not difficult to check that $C \subset E$ is bw-closed if, and only if, $C \cap B$ is w-closed for every closed ball B of E. The space E, endowed with topology bw, will be denoted E_{bw}; a function $f : E_{bw} \longrightarrow \mathbf{R}$ is continuous if, and only if, $f \in C_{wb}(E)$; note that topologies w and bw have the same compact subsets. Therefore, in order to represent $C_{wb}(E)$, it is natural to consider the topological algebra equality $C_{wb}(E) = C(E_{bw})$; but it is not known if E_{bw} is in general completely regular, and for this reason we consider *topology* τ_I on E which is the initial topology for the family $C_{wb}(E)$. (E, τ_I) is completely regular and, since $w \leqq \tau_I \leqq bw$, τ_I have the same compact subsets that E_w and E_{bw}. We will use the topological algebra equality $C_{wb}(E) = C(E, \tau_I)$.

Topology bw^* on E^{**} is the finest topology which agrees with weak* topology on each bounded subset of E^{**}. The space E^{**}, endowed with bw^* topology will be denoted $E^{**}_{bw^*}$; $C \subset E^{**}$ is bw^*-closed if, and only if, $C \cap B$ is w^*-closed for every closed ball $B \subset E^{**}$; and topologies w^* and bw^* have the same compact subsets. We have that topology bw^* is always locally convex. It is not difficult to check that each function $f \in C_{wbu}(E)$ admits a unique extension $e(f) \in C(E^{**}_{bw^*})$, and for every bounded subset $B \subset E$ we have that

$$\sup\{|f(x)| : x \in B\} = \sup\left\{|e(f)(x)| : x \in \bar{B}^{bw^*} = \bar{B}^{w^*}\right\};$$

moreover, if $f \in C(E^{**}_{bw^*})$ then

$$f|_E \in C_{wbu}(E);$$

in this way we obtain the Fréchet algebra isomorphism:

$$C_{wbu}(E) \underset{r}{\overset{e}{\rightleftarrows}} C(E^{**}_{bw^*})$$

where $r(f) = f|_E$.

Then in Section 1 we study the general situation of homomorphisms between $C(Y)$ and $C(X)$, for X and Y completely regular Hausdorff spaces. We apply these results to the case of homomorphisms between algebras of (uniformly) continuous functions on bounded sets in Sections 2 and 4, respectively; and interpolation problems are studied in Section 3.

1. Homomorphisms between $C(Y)$ and $C(X)$. Let X and Y be topological spaces. For each continuous map $\varphi : X \longrightarrow Y$ we will consider the associated (algebra) homomorphism $A : C(Y) \longrightarrow C(X)$ defined by: $Af = f \circ \varphi \quad (f \in C(Y))$. We say that A is *induced* by φ (or φ *induces* A), and we denote the *range* of A by

$$\mathcal{A}_\varphi = \{f \circ \varphi : f \in C(Y)\},$$

which is a subalgebra of $C(X)$; we call $\mathcal{A}_\varphi$ a *composition subalgebra* of $C(X)$. We will consider in the sequel X and Y to be completely regular and Hausdorff topological spaces.

PROPOSITION 1.1. (1) *$A : C(Y) \longrightarrow C(X)$ is a continuous homomorphism if and only if there exists a continuous map $\varphi : X \longrightarrow Y$ inducing A.*

(2) *If Y is a realcompact, every homomorphism between $C(Y)$ and $C(X)$ is automatically continuous (and then it is induced by a continuous map from X to Y).*

Proof. (1) When A is induced by a continuous map, A is clearly continuous.

Conversely, let $A : C(Y) \longrightarrow C(X)$ be a continuous homomorphism. For each $x \in X$ we consider the evaluation homomorphism

$$\delta_x : C(X) \longrightarrow \mathbf{R}$$

defined by:

$$\delta_x(f) = f(x) \quad (f \in C(X)).$$

Then, $\delta_x \circ A : C(Y) \longrightarrow \mathbf{R}$ is a nonzero continuous multiplicative functional on $C(Y)$, and hence we know (see [**18**], 2.20) that $\delta_x \circ A$ is the evaluation at some element of Y; that is, there exists $y \in Y$ such that:

$$(\delta_x \circ A)(f) = f(y) \quad (f \in C(Y)).$$

By defining $y = \varphi(x)$, we obtain a map $\varphi : X \longrightarrow Y$ and it is clear that

$$(Af)(x) = f(\varphi(x)) \quad (f \in C(Y), x \in X);$$

so A is induced by φ. On the other hand, since $f \circ \varphi$ is continuous for each $f \in C(Y)$, and the topology on Y is the initial topology for the family $C(Y)$, we have the continuity of φ.

(2) If Y is realcompact, every multiplicative functional on $C(Y)$ is continuous (see [**15**] 10.5 and 10.6), and the result can be derived as in (1).

Remark 1.2. Notice that if Y is not realcompact, then there are discontinuous multiplicative functionals on $C(Y)$ and hence discontinuous homomorphisms between $C(Y)$ and any $C(X)$ (see [**15**] 10.5 and [**18**], 2.20).

Our purpose now is the study of some properties of induced homomorphisms, in relation with properties of inducing mappings. We start with a result which does not involve the topological structure of algebras (see [**15**], 10.3 for the proof). Recall that a subset $M \subset X$ is said to be *C-embedded* in X if every function in $C(M)$ can be extended to a function in $C(X)$.

PROPOSITION 1.3. *Let $\varphi : X \longrightarrow Y$ be a continuous map and let $A : C(Y) \longrightarrow C(X)$ be the induced homomorphism. Then:*

(1) *A is one-to-one if, and only if, $\varphi(X)$ is a dense subset of Y.*

(2) *A is onto $C(X)$ if, and only if, $\varphi(X)$ is C-embedded in Y and $\varphi : X \longrightarrow \varphi(X)$ is a homeomorphism.*

We are interested in the problem of finding conditions on φ in order to assure that the composition subalgebra $\mathcal{A}_\varphi$ is dense or closed. The question of density is contained in the following result, which is an immediate consequence of the Stone-Weierstrass theorem:

PROPOSITION 1.4. *The composition subalgebra $\mathcal{A}_\varphi$ is dense in $C(X)$ if, and only if, φ is one-to-one.*

The question of closedness of $\mathcal{A}_\varphi$ is more difficult and the remainder of this section will be devoted to it. Recall that $\varphi : X \longrightarrow Z$ is called a *quotient map* if it is onto Z and Z has the final topology (or quotient topology) for φ.

PROPOSITION 1.5. *If $\varphi : X \longrightarrow Z$ is a quotient map, then $\mathcal{A}_\varphi$ is closed in $C(X)$.*

Proof. Let (h_α) be a net in $\mathcal{A}_\varphi$, which is convergent to $h \in C(X)$. For each α, we have that $h_\alpha = f_\alpha \circ \varphi$, where $f_\alpha \in C(Z)$; for each $x \in X$ the net $(f_\alpha(\varphi(x)))$ is convergent to $h(x)$, and therefore $h(x) = h(y)$ whenever $\varphi(x) = \varphi(y)$, so we can define $f : Z \longrightarrow \mathbf{R}$ by:

$$f(\varphi(x)) = h(x) \quad (x \in X);$$

that is, $h = f \circ \varphi$. Since $\varphi : X \longrightarrow Z$ is a quotient map and h is continuous, so is f, and then $h \in \mathcal{A}_\varphi$.

This result allows us to describe the closure of unital subalgebras of $C(X)$ and, in particular, of composition subalgebras, in the following way:

PROPOSITION 1.6. (1) *Let $\mathcal{A}$ be a unital subalgebra of $C(X)$. We consider the equivalence relation on X:*

$$x\mathcal{A}y \Leftrightarrow f(x) = f(y), \forall f \in \mathcal{A};$$

and let $\pi : X \longrightarrow X/\mathcal{A}$ be the quotient map. Then the closure of $\mathcal{A}$ in $C(X)$ is:

$$\overline{\mathcal{A}} = \mathcal{A}_\pi = \{f \in C(X) : f \text{ is constant on each fiber of } \pi\}.$$

(2) *Let $\varphi : X \longrightarrow Y$ be a continuous map. Then, the closure of $\mathcal{A}_\varphi$ in $C(X)$ is*

$$\overline{\mathcal{A}}_\varphi = \{f \in C(X) : f \text{ is constant on each fiber of } \varphi\}.$$

Proof. (1) If we call

$$\mathcal{A}_0 = \{f \in C(X) : f \text{ is constant on each fiber of } \pi\}$$

it is clear that $\mathcal{A}_\pi \subset \mathcal{A}_0$. For each $f \in \mathcal{A}_0$ we define $\hat{f} : X/\mathcal{A} \longrightarrow \mathbf{R}$ by:

$$\hat{f}(\pi(x)) = f(x) \quad (x \in X);$$

then $\hat{f}$ is continuous and therefore $f \in \mathcal{A}_\pi$. This shows that $\mathcal{A}_0 = \mathcal{A}_\pi$. On the other hand, the map $B : \mathcal{A}_\pi \longrightarrow C(X/\mathcal{A})$ defined by $Bf = \hat{f}$ is an algebra homomorphism, and the inverse of B is the homomorphism

$$A : C(X/A) \longrightarrow \mathcal{A}_\pi \hookrightarrow C(X)$$

induced by π. The subalgebra

$$\mathcal{B} = B(\mathcal{A}) \subset C(X/\mathcal{A})$$

contains constant functions and by definition it separates the points of $X/\mathcal{A}$; according to the Stone-Weierstrass theorem, $\mathcal{B}$ is dense in $C(X/\mathcal{A})$ and therefore, since A is continuous, $\mathcal{A}$ is dense in $\mathcal{A}_\pi$. Finally we know by (1.5) that $\mathcal{A}_\pi$ is closed on $C(X)$.

(2) This is a consequence of (1).

COROLLARY 1.7. (1) *$\mathcal{A}$ is a closed unital subalgebra of $C(X)$ if, and only if, there exists a quotient map $\pi : X \longrightarrow Z$ such that $\mathcal{A} = \mathcal{A}_\pi$.*

(2) *If $\mathcal{A}$ is a closed unital subalgebra of $C(X)$ then there exist a completely regular Hausdorff space Y and a continuous map π from X onto Y such that $\mathcal{A} = \mathcal{A}_\pi$.*

Proof. (1) This is a consequence of (1.5) and (1.6).

(2) Let $\mathcal{A}$ be a unital closed subalgebra of $C(X)$. We consider the quotient map $\pi : X \longrightarrow X/\mathcal{A}$ defined in (1.6), and we know that $\mathcal{A} = \mathcal{A}_\pi$. But, as example (1.8) shows, the quotient $X/\mathcal{A}$ could fail to be completely regular; for this reason, we consider on $X/\mathcal{A}$ the initial topology τ for the family $C(X/\mathcal{A})$, instead of the quotient topology; then $Y = (X/\mathcal{A}, \tau)$ is completely regular, $\pi : X \longrightarrow Y$ is continuous and

$$\mathcal{A} = \{f \circ \pi : f \in C(Y)\}.$$

Example 1.8. See [**15**], 3.J). Let

$$x = \mathbf{R}^2 \backslash \left(\left\{ \left(\frac{1}{n}, y \right) : y \neq 0, n \in \mathbf{N} \right\} \cup \{0,0\} \right)$$

with its usual topology; let $Y = \mathbf{R}$ and $\varphi : X \longrightarrow Y$ defined by:

$$\varphi(x,y) = x \quad \text{for each } (x,y) \in X.$$

Let us consider the quotient map $\pi : X \longrightarrow X/\mathcal{A}_\varphi$; then:

1. $X/\mathcal{A}_\varphi$ is not completely regular.
2. $\mathcal{A}_\varphi$ is closed in $C(X)$.

Next we will obtain some conditions for a composition subalgebra to be closed.

LEMMA 1.9. *Let $\varphi : X \longrightarrow Y$ be a continuous map, such that the composition subalgebra $\mathcal{A}_\varphi$ is closed in $C(X)$. Then:*

(1) $\varphi(X)$ is C-embedded in Y.

(2) $X/\mathcal{A}_\varphi$ is completely regular if, and only if, $\varphi : X \longrightarrow \varphi(X)$ is a quotient map.

Proof. Let us consider the canonical factorization of φ:

$$\begin{array}{ccc} X & \xrightarrow{\varphi} & Y \\ {\scriptstyle\pi}\big\downarrow & & \big\uparrow \\ X/A_\varphi & \xrightarrow{\bar{\varphi}} & \varphi(X) \end{array} \qquad \text{(where } \bar{\varphi} \circ \pi = \varphi)$$

Since $\mathcal{A}_\varphi$ is closed in $C(X)$, we know by (1.6) that $\mathcal{A}_\varphi = \mathcal{A}_\pi$.

(1) Now, for each $f \in C(\varphi(X))$ we have that

$$\pi \circ \bar{\varphi} \circ f \in \mathcal{A}_\varphi = \mathcal{A}_\pi$$

and then there exists some $\bar{f} \in C(Y)$ such that

$$\pi \circ \bar{\varphi} \circ f \in \varphi \circ \bar{f};$$

therefore, $\bar{f}$ is an extension of f.

(2) We have that $\varphi : X \longrightarrow \varphi(X)$ is a quotient map if, and only if,

$$\bar{\varphi} : X/\mathcal{A}_\varphi \longrightarrow \varphi(X)$$

is a homeomorphism. So, in this case, $X/\mathcal{A}_\varphi$ must be completely regular. Conversely, if $X/\mathcal{A}_\varphi$ is completely regular, it will be sufficient to prove that for each closed and φ-saturated subset $C \subset X$, $\varphi(C)$ is closed in $\varphi(X)$. And indeed, if $y \in \varphi(X)\backslash\overline{\varphi(C)}$, then $y = \varphi(x)$ for some $x \in X\backslash C$ and therefore $\pi(x) \notin \pi(C)$; since $\pi(C)$ is closed in $X/\mathcal{A}_\varphi$, we can consider $h \in C(X/\mathcal{A}_\varphi)$ such that

$$h(\pi(x)) = 1 \quad \text{and} \quad h(\pi(C)) = 0 :$$

now $h \circ \pi \in \mathcal{A}_\pi = \mathcal{A}_\varphi$, and then there exists $\hat{h} \in C(Y)$ such that $h \circ \pi = \hat{h} \circ \varphi$; therefore,

$$\hat{h}(y) = 1 \quad \text{and} \quad \hat{h}(\varphi(C)) = 0.$$

PROPOSITION 1.10. *Let $\varphi : X \longrightarrow Y$ be a continuous map.*

(1) *The following conditions are equivalent:*

(a) *$\mathcal{A}_\varphi$ is closed in $C(X)$, and $X/\mathcal{A}_\varphi$ is completely regular.*

(b) *$\varphi(X)$ is C-embedded in Y, and $\varphi : X \longrightarrow \varphi(X)$ is a quotient map.*

(c) *$\varphi(X)$ is C-embedded in Y, and for each closed and φ-saturated subset $M \subset X$, $\varphi(M)$ is closed in $\varphi(X)$.*

(2) *If X is Lindelöf and Y is normal, the previous conditions are also equivalent to:*

(d) *For each closed and φ-saturated subset $M \subset X$, $\varphi(M)$ is closed in Y.*

Proof. (1) It is a consequence of (1.9) that (a) implies (b). If (b) holds, we have that

$$\mathcal{A}_\varphi = \{ f \circ \varphi : f \in C(Y) \} = \{ g \circ \varphi : g \in C(\varphi(X)) \}$$

which, according to (1.5), is a closed subalgebra of $C(X)$; now applying (1.9) we obtain (a). Finally, the equivalence between (b) and (c) is straightforward.

(2) If (a) and (b) hold, then $X/\mathcal{A}_\varphi$ is completely regular and Lindelöf, so it is realcompact; therefore, $\varphi(X)$ is realcompact and then, as it is C-embedded in Y, it is closed in Y; hence (d) follows. The converse is clear.

Remark 1.11. Let $\varphi : X \longrightarrow Y$ be a continuous map; $X/\mathcal{A}_\varphi$ is always a Hausdorff space although, as example (1.8) shows, it can fail to be completely regular (even in the case of $\mathcal{A}_\varphi$ being closed in $C(X)$). However, we will see in (1.12) that for some spaces the condition of $X/\mathcal{A}_\varphi$ to be completely regular is always true, and then for these spaces Proposition (1.10) characterizes when $\mathcal{A}_\varphi$ is closed in $C(X)$).

Recall that a topological space T is said to be *hemicompact* if there exists a sequence $(K_n)_{n\in\mathbf{N}}$ of compact subsets of T, such that every compact subset of T is contained in some K_n. And T is said to be a $K_{\mathbf{R}}$-*space* if every real function on T which is continuous on each compact subset of T is continuous on T.

PROPOSITION 1.12. *Let X be a hemicompact $K_{\mathbf{R}}$-space, and let $\pi : X \longrightarrow Z$ be a quotient map such that Z is Hausdorff. Then Z is normal.*

Proof. Let $(K_n)_{n\in\mathbf{N}}$ be an increasing sequence of compact subsets of X, such that every compact subset of X is contained in some K_n. For each $n \in \mathbf{N}$, $\pi(K_n)$ is compact in Z, and hence normal. Now given F and H disjoint nonempty closed subsets of Z, we choose $N \in \mathbf{N}$ such that $\pi(K_N) \cap F$ and $\pi(K_N) \cap H$ are nonempty and, using the normality of $\pi(K_n)$ for $n \geqq N$ we construct by induction a sequence $(f_n)_{n\geqq N}$ of functions such that, for each $n \geqq N$:

1. $f_n : \pi(K_n) \longrightarrow [0,1]$ is continuous.
2. $f_n(\pi(K_n) \cap F) = 0$ and $f_n((K_n) \cap H) = 1$.
3. f_{n+1} is an extension of f_n.

Therefore we can define $f : Z \longrightarrow [0,1]$ such that

$$f|_{\pi(K_n)} = f_n \quad \text{for } n \geqq N.$$

It is clear that $f(F) = 0$ and $f(H) = 1$; and, since

$$(f \circ \pi)|_{K_n} = f_n \circ \pi \quad \text{for each } n \geqq N,$$

we have that f is continuous.

Following [8] we define a map $\varphi : X \longrightarrow Y$ to be *semiproper* if for each compact subset $H \subset Y$, there exists a compact subset $K \subset X$ such that $\varphi(K) = H \cap \varphi(X)$.

Now we examine the relationship between this property and properties of corresponding induced homomorphisms.

PROPOSITION 1.13. *Let $\varphi : X \longrightarrow Y$ be a continuous map, and let $A : C(Y) \longrightarrow C(X)$ be the induced homomorphism, with range $\mathcal{A}_\varphi$. Then $A : C(Y) \longrightarrow \mathcal{A}_\varphi$ is an open map if, and only if, $\varphi(X)$ is closed in Y and $\varphi : X \longrightarrow Y$ is semiproper.*

Proof. $A : C(Y) \longrightarrow \mathcal{A}_\varphi$ is an open map if, and only if, for each nonempty compact subset $H \subset Y$, there exist a nonempty compact subset $K \subset X$ and $\epsilon > 0$ such that, denoting

$$V_H = \{f \in C(Y) : p_H(f) \leqq 1\} \quad \text{and } U_{K,\epsilon} = \{g \in \mathcal{A}_\varphi : p_K(g) \leqq \epsilon\},$$

we have that $A(V_H) \supset U_{K,\epsilon}$.

Suppose this is the case. First we show that $\varphi(X)$ is closed in Y. Indeed, if there exists $y \in \overline{\varphi(X)} \setminus \varphi(X)$ we can consider $H = \{y\}$; for each nonempty compact subset $K \subset X$, there exists $f \in C(Y)$ such that $f(y) = 2$ and

$f(\varphi(K)) = 0$; therefore, $g = f \circ \varphi \in U_{K,\epsilon}$; for every $\epsilon > 0$; but $g \notin A(V_H)$, for, if $f \circ \varphi = \hat{f} \in C(Y)$, then

$$f|_{\overline{\varphi(X)}} = \hat{f}|_{\overline{\varphi(X)}}$$

and consequently $p_H(\hat{f}) \geqq 2$. In order to show that $\varphi : X \longrightarrow Y$ is semiproper, we consider a compact subset $H \subset Y$, and we can suppose that

$$\hat{H} = H \cap \varphi(X) \neq \phi.$$

If the compact subset $K \subset X$ and $\epsilon > 0$ are such that $A(V_{\hat{H}}) \supset U_{K,\epsilon}$, then $\hat{H} = \varphi(K)$; indeed, it is clear that $\hat{H} \supset \varphi(K)$ and, if there exists $y \in \hat{H} \backslash \varphi(K)$, we select $f \in C(Y)$ such that $f(y) = 2$ and $f(\varphi(K)) = 0$; as before, $g = f \circ \varphi \in U_{K,\epsilon}$ but $g \notin A(V_{\hat{H}})$.

Conversely, suppose that $\varphi(X)$ is closed in Y and $\varphi : X \longrightarrow Y$ is semiproper. Let H be a nonempty compact subset of Y; then there is a compact subset $K \subset X$, such that $\varphi(K) = H \cap \varphi(X)$. We consider two cases:

If $\hat{H} = H \cap \varphi(X) = \phi$ then there exists $h \in C(Y)$ such that $h(H) = 0$ and $h(\varphi(X)) = 1$. Hence, for each $f \in C(Y)$ we have that $f \circ \varphi = (fh) \circ \varphi$ and $p_H(fh) = 0$; this shows that $\mathcal{A}_\varphi = A(V_H)$.

If $\hat{H} = H \cap \varphi(X) \neq \phi$ we will see that $A(V_H) \supset U_{K,1/2}$. Indeed, let $f \in C(Y)$ such that $f \circ \varphi \in U_{K,1/2}$ and consider

$$H_0 = \left\{ Y \in H : f(y) \geqq \frac{3}{4} \right\};$$

there exists $h \in C(Y)$ such that $0 \leqq h \leqq 1$, $h(H_0) = 0$ and $h(\varphi(X)) = 1$. Hence,

$$f \circ \varphi = (fh) \circ \varphi \quad \text{and} \quad p_H(fh) \leqq \frac{3}{4} < 1.$$

COROLLARY 1.14. *Let $\varphi : X \longrightarrow Y$ be a continuous map, such that $\overline{\varphi(X)} = Y$ and let $A : C(Y) \longrightarrow C(X)$ be the induced homomorphism, with range $\mathcal{A}_\varphi$. Then, $A : C(Y) \longrightarrow \mathcal{A}_\varphi$ is a topological isomorphism if, and only if, $\varphi(X) = Y$ and $\varphi : X \longrightarrow Y$ is semiproper.*

Proof. This is a consequence of (1.13) (1.1) and (1.3).

Recall that a topological space T is said to be a K*-space* if a subset of T is closed whenever its intersection with every compact subset of T is closed.

For normal K-spaces, we have a relationship between semiproper mappings and closed composition subalgebras (compare with [**8**] and [**5**]).

PROPOSITION 1.15. *Let Y be a normal K-space and let $\varphi : X \longrightarrow Y$ be a semiproper continuous map. Then, the composition subalgebra $\mathcal{A}_\varphi$ is closed in $C(X)$.*

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Proof. It is not difficult to verify that, if Y is a K-space and $\varphi : X \longrightarrow Y$ is a semiproper continuous map, then $\varphi(X)$ is closed in Y and $\varphi : X \longrightarrow \varphi(X)$ is a quotient map. The result follows from (1.10).

The requirement of Y to be a K-space is essential in Proposition (1.15), as the following example shows:

Example 1.16. Let $Y = (l_1, bw)$. We know by **[11]** that Y is normal, but it is not a K-space. Let X be the unit sphere of l_1, as a subspace of Y. We consider the inclusion map $\varphi : X \hookrightarrow Y$ and the corresponding induced homomorphism $A : C(Y) \longrightarrow C(X)$; that is,

$$Af = f|_X \quad \text{for each } f \in C(Y).$$

Then:
1. $\varphi : X \longrightarrow Y$ is semiproper.
2. X is not C-embedded in Y.
3. $\mathcal{A}_\varphi$ is not closed in $C(X)$.

Proof. 1. Let H be a compact subset of Y; then H is w-compact in l_1 and hence, by Schur's Lemma, H is $\|\cdot\|$-compact. Since X is $\|\cdot\|$-closed in l_1, $H \cap X$ is $\|\cdot\|$-compact, and hence bw-compact.

2. Let B be the closed unit ball of l_1 and let $f : B \longrightarrow \mathbf{R}$ be defined by

$$f(x) = \sum_{n=1}^{\infty} 2^{-n} x_n^2, \quad \text{for } x = (x_n)_{n \in \mathbf{N}} \in B.$$

Then f is w-continuous on B: indeed, given $x = (x_n)_{n \in \mathbf{N}} \in B$ and $\epsilon > 0$, we consider $N \in \mathbf{N}$ such that

$$\sum_{n \geqq N} \frac{1}{2^n} < \frac{\epsilon}{4}, \quad \text{and}$$

$$W = \left\{ y = (y_n)_{n \in \mathbf{N}} \in B : |x_1 - y_1| < \frac{\epsilon}{2N}, \dots, |x_N - y_N| < \frac{\epsilon}{2N} \right\},$$

which is a relative w-neighbourhood of x in B; then for every $y = (y_n)_{n \in \mathbf{N}} \in W$, we have that:

$$|f(x) - f(y)| \leqq \sum_{n=1}^{N} 2^{-n} |x_n - y_n| \left(|x_n| + |y_n| \right) + \sum_{n \geqq N} 2^{-n} \left(|x_n^2| + |y_n^2| \right) < \epsilon.$$

Now we consider $h : X \longrightarrow \mathbf{R}$ defined by: $h = 1/f$; it is clear that h is continuous but, since $f(0) = 0$, h does not have any w-continuous extension to $B = \bar{X}^w$; hence, h does not have any bw-continuous extension to Y.

3. This is a consequence of (1.9).

Next we will see that in some cases the fact that the composition subalegbra $\mathcal{A}_\varphi$ is closed implies that the inducing map φ is semiproper. We need the following result, due to Michael (see ([**21**], 12.5) for the proof in the complex case; the proof in the real case is analogous).

PROPOSITION 1.17. *Let X be a Lindelöf space and let $\mathcal{A}$ be a subalgebra of $C(X)$ satisfying:*

1. *If $h \in \mathcal{A}$ and $h(x) \neq 0$ for each $x \in X$ then $1/h \in \mathcal{A}$.*
2. *If $h_1, \dots, h_n \in \mathcal{A}$ and they do not have any common zero, then there exist $g_1, \dots, g_n \in \mathcal{A}$ such that $h_1 g_1 + \cdots + h_n g_n = 1$.*
3. *For each sequence $(h_n)_{n \in \mathbf{N}} \subset \mathcal{A}$, there exists a sequence $(g_n)_{n \in \mathbf{N}} \subset \mathcal{A}$ such that:*

a) *For every $n \in \mathbf{N}$, the zeros of h_n and g_n coincide.*

b) $$0 \leqq g_n \leqq 1.$$

c) $$\sum_{n \in \mathbf{N}} 2^{-n} g_n \in \mathcal{A}.$$

Then, for each nonzero algebra homomorphism $\phi : \mathcal{A} \longrightarrow \mathbf{R}$ there exists $x \in X$ such that $\phi(f) = f(x)$, for every $f \in \mathcal{A}$.

It is not difficult to derive, as a consequence of this result, the following proposition:

PROPOSITION 1.18. *Let X be a Lindelöf space; let $\varphi : X \longrightarrow Y$ be a continuous map such that $\varphi(X)$ is C-embedded in Y. Then, every nonzero algebra homomorphism from $\mathcal{A}_\varphi$ into $\mathbf{R}$ is given by the evaluation at some point of X (in particular, every such homomorphism is continuous).*

PROPOSITION 1.19. *Let X be a Lindelöf space; let $\varphi : X \longrightarrow Y$ be a continuous map such that $\varphi(X)$ is C-embedded in Y, and let us suppose that $\mathcal{A}_\varphi$ is barrelled. Then $\varphi(X)$ is closed in Y, and $\varphi : X \longrightarrow Y$ is semiproper.*

Proof. For each completely regular Hausdorff space Z, we will denote by νz the Hewitt-Nachbin realcompactification of Z (see [**15**], 8).

We consider $\varphi(X) \subset Y \subset \nu Y$ and let

$$M = \overline{\varphi(X)}^{\nu Y}.$$

Since $\varphi(X)$ is C-embedded in Y, $M = \nu(\varphi(X))$. Now we define

$$\hat{\varphi} : X \xrightarrow{\varphi} \varphi(X) \hookrightarrow M,$$

and let $\hat{A}$ be the homomorphism induced by $\hat{\varphi}$, with range $\mathcal{A}_{\hat{\varphi}}$. It is clear that $\mathcal{A}_{\hat{\varphi}} = \mathcal{A}_{\varphi}$ and that

$$\hat{A} : C(M) \longrightarrow \mathcal{A}_{\hat{\varphi}} = \mathcal{A}_{\varphi}$$

is an algebra isomorphism (see (1.3)). $\hat{A}$ is also a topological isomorphism; indeed, for each nonempty compact subset $H \subset M$, we consider

$$V_H = \{f \in C(M) : p_H(f) \leqq 1\} \quad \text{and} \quad U = \hat{A}(V_H),$$

and it will be sufficient to check that U is a barrel in $\mathcal{A}_{\varphi}$; it is clear that U is absorbing, balanced and convex; on the other side, for each $y \in H$ the algebra homomorphism

$$\delta_y \circ \hat{A}^{-1}; \mathcal{A}_{\varphi} \longrightarrow \mathbf{R}$$

is continuous, according (1.18); hence

$$U = \bigcap_{y \in H} \{h \in \mathcal{A}_{\varphi} : |\delta_y \circ \hat{A}^{-1}(h)| \leqq 1\}$$

is closed in $\mathcal{A}_{\varphi}$. Now, since

$$\hat{A} : C(M) \longrightarrow \mathcal{A}_{\hat{\varphi}}$$

is a topological isomorphism, we get from (1.14) that $\hat{\varphi}(X) = M$ and $\hat{\varphi} : X \longrightarrow M$ is semiproper. Then the result follows.

In general it is not easy to know whether a composition subalgebra is barrelled, in order to apply the preceding result. There is, however, an important and simple case:

COROLLARY 1.20. *Let X be a hemicompact $K_{\mathbf{R}}$-space and let $\varphi : X \longrightarrow Y$ be a continuous map such that $\mathcal{A}_{\varphi}$ is closed in $C(X)$. Then, $\varphi : X \longrightarrow Y$ is semiproper.*

Proof. Since X is a hemicompact $K_{\mathbf{R}}$-space, we have that $C(X)$ is a Fréchet space (see [**7**], 21 and 22). The result is then a consequence of (1.9) and (1.19).

Remark 1.21. In Proposition (1.19) X is Lindelöf and hence realcompact; then, according to Nachbin-Shirota theorems (see e.g. [**7**], 8.2) $C(X)$ is bornological and barrelled. Nevertheless, a composition subalgebra $\mathcal{A}_{\varphi}$ of $C(X)$ can be closed in $C(X)$ without being barrelled; in fact, as (1.19) shows, this will always be the case if $\mathcal{A}_{\varphi}$ is closed but φ is not semiproper; (1.8) provides an example of this situation.

2. Homomorphisms between $C_{wbu}(E)$ and $C(X)$. Let E be a real Banach space and let X be a completely regular and Hausdorff topological space. As we have seen in the Introduction, there is a topological algebra isomorphism:

$$C_{wbu}(E) \underset{r}{\overset{e}{\rightleftarrows}} C(E^{**}_{bw^*})$$

where $e(f)$ is the extension of f to E^{**} and $r(f) = f|_E$. This representation allows us to apply the general results of Section 1 to the case of the homomorphism between $C_{wbu}(E)$ and $C(X)$

PROPOSITION 2.1. *Each continuous map* $\varphi : X \longrightarrow E^{**}_{bw^*}$ *induces a continuous homomorphism* $A : C_{wbu}(E) \longrightarrow C(X)$ *by the formula:*

$$Af = e(f) \circ \varphi \quad (f \in C_{wbu}(E)).$$

Conversely, for each homomorphism $A : C_{wbu}(E) \longrightarrow C(X)$ *there exists a continuous map* $\varphi : X \longrightarrow E^{**}_{bw^*}$ *inducing* A *by the above formula; and in particular,* A *is automatically continuous.*

Proof. Since $E^{**}_{bw^*}$ is σ-compact, it is realcompact. The result is then a consequence of (1.1).

PROPOSITION 2.2. *Let* $\varphi : X \longrightarrow E^{**}_{bw^*}$ *be a continuous map and let*

$$A : C_{wbu}(E) \longrightarrow C(X)$$

be the induced homomorphism, with range

$$\mathcal{A}_\varphi = \{e(f) \circ \varphi : f \in C_{wbu}(E)\}.$$

Then:

(1) *A is one-to-one if, and only if, $\varphi(X)$ is dense in $E^{**}_{bw^*}$.*
(2) *$\mathcal{A}_\varphi$ is dense in $C(X)$ if, and only if, φ is one-to-one.*
(3) *$\varphi : X \longrightarrow E^{**}_{bw^*}$ is semiproper if, and only if,*

$$A : C_{wbu}(E) \longrightarrow \mathcal{A}_\varphi$$

is an open map.

(4) *If $\varphi : X \longrightarrow E^{**}_{bw^*}$ is semiproper, then $\mathcal{A}_\varphi$ is closed in $C(X)$.*
(5) *If X is a hemicompact $K_{\mathbf{R}}$-space and $\mathcal{A}_\varphi$ is closed in $C(X)$, then $\varphi : X \longrightarrow E^{**}_{bw^*}$ is semiproper.*

Proof. The results follow from (1.3), (1.4), (1.13), (1.15) and (1.20).

Now we examine some specific situations for the space X:

(I) $X = F^{**}_{bw^*}$; where F is a real Banach space.
In this case we have the homomorphisms between $C_{wbu}(E)$ and $C_{wbu}(F)$, and we obtain the results of ([**20**], 4.6).

(II) $X = (F, \tau_I)$; where F is a real Banach space.
In this case we have the homomorphisms between $C_{wbu}(E)$ and $C_{wb}(F)$, which are induced by the maps of the space:

$$C\left((F,\tau_I), E^{**}_{bw^*}\right) = C\left(F_{bw}, E^{**}_{bw^*}\right) = C_{wb}\left(F, E^{**}_{bw^*}\right).$$

We will obtain a simpler description of inducing maps when the space F does not contain an isomorphic copy of l_1. First we introduce the following classes:

Let T be a topological space; $C_{wk}(F,T)$ denotes the space of all maps $f : F \longrightarrow T$ such that for each weakly compact subset $K \subset F$, $f|_K$ is weakly continuous on K; and $C_{wsc}(F,T)$ denotes the space of all maps $f : F \longrightarrow T$ such that for each weakly convergent sequence $(y_n)_{n\in\mathbf{N}}$ in F, $(f(y_n))_{n\in\mathbf{N}}$ is convergent in T.

With the same proof that ([20], 4.45 and 4.47), we have:

LEMMA 2.3. (a) $C_{wk}(F,T) = C_{wsc}(F,T)$
(b) *If* $F \not\supset l_1$, *then* $C_{wb}(F,T) = C_{wk}(F,T)$.

Therefore, if $F \not\supset l_1$ the homomorphisms between $C_{wbu}(E)$ and $C_{wb}(F) = C_{wk}(F) = C_{wsc}(F)$ are induced by the maps of the space:

$$C_{wb}(F, E^{**}_{bw^*}) = C_{wk}(F, E^{**}_{bw^*}) = C_{wsc}(F, E^{**}_{bw^*}) = C_{wsc}(F, E^{**}_{w^*}).$$

(III) X is a compact Hausdorff topological space.

Let T be a topological space; $C_b(T)$ denotes the space of all bounded continuous real-valued functions on T, with the sup-norm $\|\cdot\|_\infty$. We will use the following "simultaneous extension" theorem due to Arens (see [1], 5.2):

PROPOSITION 2.5. *Let H be a closed subset of a paracompact space T. Let C be a separable closed linear subspace of $C_b(H)$. Then there exists a linear isometry $L : C \longrightarrow C_b(T)$ such that for each $f \in C$, Lf is an extension of f.*

PROPOSITION 2.6. *Let X be a compact Hausdorff space and let $\varphi : X \longrightarrow E^{**}$ be a map.*

(1) *$\varphi : X \longrightarrow E^{**}_{bw^*}$ is continuous if, and only if, $\varphi : X \longrightarrow E^{**}_{w^*}$ is continuous.*

(2) *If $\varphi : X \longrightarrow E^{**}_{bw^*}$ is continuous, then $\varphi : X \longrightarrow E^{**}_{bw^*}$ is semiproper and the composition subalgebra*

$$\mathcal{A}_\varphi = \{e(f) \circ \varphi : f \in C_{wbu}(E)\}$$

is closed in $C(X)$.

(3) *If X is metrizable, and $\varphi : X \longrightarrow E^{**}_{bw^*}$ is one-to-one and continuous, then there exists a continuous linear section*

$$S : C(X) \longrightarrow C_{wbu}(E)$$

of the homomorphism

$$A : C_{wbu}(E) \longrightarrow C(X)$$

induced by φ, and hence $C(X)$ is linearly topologically isomorphic to a complemented subspace of $C_{wbu}(E)$.

Proof. (1) is immediate and (2) is a consequence of (2.2).

(3) If $\varphi : X \longrightarrow E^{**}_{bw^*}$ is continuous and one-to-one, obviously

$$\varphi : X \longrightarrow (\varphi(X), w^*) = K$$

is a homeomorphism. Since $E^{**}_{bw^*}$ is Lindelöf and regular, it is paracompact; if X is metrizable then $C(K)$ is separable and we can apply the preceeding Theorem (2.5), thus obtaining a linear isometry

$$L : C(K) \longrightarrow C_b(E^{**}_{bw^*})$$

such that for each $f \in C(K)$, Lf is a continuous (and bounded) extension of f.

We consider the following conmutative diagram:

$$\begin{tikzcd}
& C(K) \arrow[rr, "T"] \arrow[dl, "L"'] & & C(X) \arrow[dr, dashed, "S"] & \\
C_b(E^{**}_{bw^*}) \arrow[rr, "I"] & & C(E^{**}_{bw^*}) \arrow[ul, "R"'] \arrow[rr, shift left, "r"] & & C_{wbu}(E) \arrow[ll, shift left, "e"] \arrow[ul, "A"]
\end{tikzcd}$$

where $Tf = f \circ \varphi$ and $Rf = f|_K$. Then T is a topological isomorphism and the inclusion I is continuous. The section S is then defined by

$$S = r \circ I \circ L \circ T^{-1}.$$

It is clear that

$$A|_{S(C(X))} : S(C(X)) \longrightarrow C(X)$$

is a linear topological isomorphism and

$$A \circ S : C_{wbu}(E) \longrightarrow S(C(X))$$

is a projection.

3. Interpolation of bounded sequences by C_{wbu}–functions. We will state a "global" formulation of interpolation problems, in order to apply the results obtained before. In particular, for the real Banach space E, we consider the topological algebra isomorphism:

$$\begin{tikzcd}
C_{wbu}(E) \arrow[r, shift left, "e"] & C(E^{**}_{bw^*}) \arrow[l, shift left, "r"]
\end{tikzcd}$$

where $e(f)$ is the extension of f to E^{**} and $r(f) = f|_E$, as we have seen in the introduction.

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Let $(a_n)_{n\in\mathbf{N}}$ be a bounded sequence in E, with pairwise different terms. It is not difficult to check (see [**20**], 4.1.1) that for each $f \in C_{wbu}(E)$, the sequence $(f(a_n))_{n\in\mathbf{N}}$ is bounded; therefore we can consider the "restriction homomorphism":

$$\begin{aligned} R : C_{wbu}(E) &\longrightarrow l_\infty \\ f &\longrightarrow (f(a_n))_{n\in\mathbf{N}} \end{aligned}$$

Our interest is to determine in which cases R is onto l_∞ and in which cases it is onto the subspace c of all convergent sequences.

PROPOSITION 3.1. *The following conditions are equivalent:*

(a) *R is onto l_∞.*

(b) *$M = \{a_n : n \in \mathbf{N}\}$ is weak-discrete and its weak*-closure $\bar{M}^{w^*}$ is weak*-homeomorphic to the Stone-Čech compactification of $\mathbf{N}$, $\beta\mathbf{N}$.*

Proof. Suppose R is onto l_∞. Then M is weak-discrete; for, if there exists $m \in \mathbf{N}$ such that a_m is a weak-accumulation point of M, we can consider $\lambda = (\lambda_n)_{n\in\mathbf{N}} \in l_\infty$ defined by $\lambda_m = 1$ and $\lambda_n = 0$ for $n \neq m$, and there exists no continuous function $f : (M, w) \longrightarrow \mathbf{R}$ such that $f(a_n) = \lambda_n$ for each $n \in \mathbf{N}$. Further, for every bounded function $f : M \longrightarrow \mathbf{R}$, there exists $h \in C_{wbu}(E)$ such that $h(a_n) = f(a_n)$ for each $n \in \mathbf{N}$; hence

$$\hat{f} = e(h)|_{\bar{M}^{w^*}} = (\bar{M}^{w^*}, w^*) \longrightarrow \mathbf{R}$$

is a continuous extension of f. Therefore $(\bar{M}^{w^*}, w^*)$ is homeomorphic to $\beta\mathbf{N}$.

Conversely, suppose (b) holds. Then for every $\lambda = (\lambda_n)_{n\in\mathbf{N}} \in l_\infty$ we consider $f : (M, w) \longrightarrow \mathbf{R}$ defined by: $f(a_n) = \lambda_n$, for each $n \in \mathbf{N}$, and its continuous extension

$$\hat{f} : (\bar{M}^{w^*}, w^*) \longrightarrow \mathbf{R};$$

by the normality of $E^{**}_{bw^*}$ there exists a continuous extension

$$\bar{f} : E^{**}_{bw^*} \longrightarrow \mathbf{R}$$

of $\hat{f}$; therefore,

$$r(\bar{f}) = \bar{f}|_E \in C_{wbu}(E) \quad \text{and} \quad \bar{f}|_E(a_n) = \lambda_n \quad \text{for each } n \in \mathbf{N}.$$

PROPOSITION 3.2. *The range of R is contained in c if, and only if, the sequence $(a_n)_{n\in\mathbf{N}}$ is weak-Cauchy in E.*

Proof. If $(a_n)_{n\in\mathbf{N}}$ is weak-Cauchy in E, then it is weak*-Cauchy in E^{**} and, as $E^{**}_{w^*}$ is sequentially complete (see [**19**], 9.3.1), $(a_n)_{n\in\mathbf{N}}$ is weak*-convergent to some point $a_0 \in E^{**}$. Thus for each $f \in C_{wbu}(E)$ its extension $e(f) \in C(E^{**}_{bw^*})$ satisfies that

$$\big(e(f)(a_n)\big)_{n\in\mathbf{N}} = \big(f(a_n)\big)_{n\in\mathbf{N}}$$

converges to $e(f)(a_0)$.

Conversely, if the range of R is contained in c, in particular we have that for each $\psi \in E^*$ the sequence $(\psi(a_n))_{n\in\mathbf{N}}$ is convergent. Therefore $(a_n)_{n\in\mathbf{N}}$ is weak-Cauchy.

PROPOSITION 3.3. *The following conditions are equivalent:*

(a) *The range of R is the space c.*

(b) *$(a_n)_{n\in\mathbf{N}}$ is weak-Cauchy and, for each $m \in \mathbf{N}$, $(a_n)_{n\in\mathbf{N}}$ is not weak-convergent to a_m.*

(c) *$M = \{a_n : n \in \mathbf{N}\}$ is weak-discrete and its weak*-closure $\bar{M}^{w^*}$ is weak*-homeomorphic to the Alexandroff compactification of $\mathbf{N}, \alpha\mathbf{N}$.*

(d) *The range of R is contained in c, and there exists a continuous linear section $S : c \longrightarrow C_{wbu}(E)$ of R.*

Proof. It is clear that $(a) \Rightarrow (b)$ (see (3.2)) and, since E^{**}_{w*} is sequentially complete, that $(b) \Leftrightarrow (c)$; $(c) \Rightarrow (d)$ is a consequence of (2.6) (here $C(\bar{M}^{w^*}) \approx C(\alpha\mathbf{N}) \approx c$); and finally, $(d) \Rightarrow (a)$ is obvious.

Remark 3.4. Every Banach space E contains sequences satisfying the conditions required for $(a_n)_{n\in\mathbf{N}}$ in (3.3); therefore, the space $C_{wbu}(E)$ contains a complemented subspace isomorphic to c_0.

Let us denote by B_E and B_E^{**} the closed unit ball of E and E^{**}, respectively. We consider the "restriction homomorphism"

$$R : C_{wbu}(E) \longrightarrow l_\infty$$

associated to a sequence $(a_n)_{n\in\mathbf{N}} \subset B_E$ as before, and we also consider the "restriction homomorphism"

$$\bar{R} : C(E^{**}_{bw^*}) \longrightarrow l_\infty$$

associated to a sequence $(a_n^{**})_{n\in\mathbf{N}} \subset B_{E^{**}}$ in the same way. Next we study the existence of sequences in B_E or in $B_{E^{**}}$ for which R or $\bar{R}$ (respectively) are onto l_∞.

PROPOSITION 3.5. *According to the preceding notations, we consider the following conditions:*

(a) *E contains a subspace isomorphic to l_1.*

(b) *There exists a sequence $(a_n)_{n\in\mathbf{N}}$ in B_E for which*

$$R : C_{wbu}(E) \longrightarrow l_\infty$$

is onto.

(c) *There exists a sequence* $(a_n^{**})_{n\in\mathbf{N}}$ *in* $B_{E^{**}}$ *for which*

$$\bar{R} : C(E^{**}_{bw^*}) \longrightarrow l_\infty$$

is onto.

(d) $(B_{E^{**}}, w^*)$ *contains a subspace homeomorphic to* $\beta\mathbf{N}$.

(e) E^* *contains a subspace isomorphic to* $l_1(2^{\aleph_0})$.

Then:

(1) *In general,* $(a) \Leftrightarrow (b) \Rightarrow (c) \Leftrightarrow (d) \Leftrightarrow (e)$.

(2) *If E is separable, the conditions are equivalent.*

(3) *The space* $E = c_0(2^{\aleph_0})$*, which is not separable, shows that in general the conditions are not equivalent.*

Proof. (1) $(a) \Rightarrow (b)$. Let T be an isomorphism from l_1 onto the closed subspace F of E with $\|T\| \leqq 1$; we consider

$$(a_n)_{n\in\mathbf{N}} = (Te_n)_{n\in\mathbf{N}} \subset B_F \subset B_E,$$

where $(e_n)_{n\in\mathbf{N}}$ is the standard basis of l_1. For each $\lambda = (\lambda_n)_{n\in\mathbf{N}} \in l_\infty = l_1^*$ we have that $\psi = \lambda \circ T^{-1} \in F^*$ and, if $\bar{\psi} \in E^*$ is an extension of ψ, it is clear that

$$\bar{\psi} \in C_{wbu}(E) \quad \text{and} \quad \bar{\psi}(a_n) = \lambda_n \quad \text{for each } n \in \mathbf{N}.$$

(b) $\Rightarrow$ (a). If E contains no subspace isomorphic to l_1 and $(a_n)_{n\in\mathbf{N}}$ is a sequence in B_E, from the characterization of Rosenthal ([**22**], Theorem 3) we obtain that there is a weak-Cauchy subsequence $(a_{n_j})_{j\in\mathbf{N}}$ of $(a_n)_{n\in\mathbf{N}}$; hence for each $\lambda = (\lambda_n)_{n\in\mathbf{N}}$ in the range or R, the subsequence $(\lambda_{n_j})_{j\in\mathbf{N}}$ is convergent, and therefore R is not onto l_∞.

(c) $\Rightarrow$ (d). This can be derived as in the proof of (3.1).

(d) $\Rightarrow$ (e). Talagrand showed in ([**24**], II) that for a Banach space X and a cardinal number of the form $\tau = 2^\kappa$, X contains a subspace isomorphic to $l_1(\tau)$ if, and only if, (B_{X^*}, w^*) contains a subspace homeomorphic to $\beta\kappa$ (the Stone-Čech compactification of a discrete space with cardinal κ).

(2) If E is separable then card$(\beta\mathbf{N}) >$ card(E), and from the characterization of Odell-Rosenthal ([**22**], Theroem 3) it follows that (d) $\Rightarrow$ (a).

(3) The space $E = c_0(2^{\aleph_0})$ is not separable and it contains no subspace isomorphic to l_1 (see [**22**], Theorem 3). Nevertheless, $E^* = l_1(2^{\aleph_0})$, so (e) $\not\Rightarrow$ (a).

Now we present some examples of interpolating and non-interpolating sequences for the case $E = l_1$, showing a variety of situations:

Example 3.6. (1) For the standard basis $(e_n)_{n\in\mathbf{N}}$ of l_1, the associated homomorphism R is onto l_∞, and there exists a continuous linear section

$$S : l_\infty \longrightarrow C_{wbu}(l_1)$$

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of R. This follows from the fact that $R|_{l_1^*}$ is the natural isometry from l_1 onto l_∞.

(2) For every bounded inconditional basis $(a_n)_{n\in\mathbf{N}}$ of l_1, we obtain the same result that in (1). For in l_1 every bounded inconditional basis is equivalent to $(e_n)_{n\in\mathbf{N}}$ (see [**23**], II.18-2).

(3) The sequence $(a_n)_{n\in\mathbf{N}} \subset l_1$ defined by: $a_1 = 2e_1$, $a_n = e_{n-1} - e_n$ (for $n \geqq 2$) is a conditional basis of l_1 (see [**23**], II.14.2); in this case the associated homomorphism R is onto l_∞, and there exists also a continuous linear section $S : l_\infty \longrightarrow C_{wbu}(l_1)$ of R.

Proof. Let

$$\xi = (0, 1, 0, 1, \ldots) \in l_\infty = l_1^* \quad \text{and}$$

$$\eta = \frac{1}{2}(1, 1, 1, \ldots) \in l_\infty = l_1^*;$$

we define

$$h_1 = \frac{1}{2}(1 + \xi + \eta) \quad \text{and} \quad h_2 = \frac{1}{2}(1 - \xi - \eta);$$

it is clear that $h_1, h_2 \in C_{wbu}(l_1)$. Now for each $\lambda = (\lambda_n)_{n\in\mathbf{N}} \in l_\infty$ we consider

$$f_1 = \left(\lambda_1/2, 0, -\lambda_3, 0, -\lambda_5, \ldots\right);$$

$$f_2 = (0, -\lambda_2, 0, -\lambda_4, \ldots) \in l_\infty = l_1^*.$$

Then, the function

$$F_\lambda = h_1 f_1 + h_2 f_2 \in C_{wbu}(l_1)$$

satisfies that

$$F_\lambda(a_n) = \lambda_n \quad \text{for each } n \in \mathbf{N}.$$

In addition, the section

$$\begin{aligned} S : l_\infty &\longrightarrow C_{wbu}(l_1) \\ \lambda &\longrightarrow F_\lambda \end{aligned}$$

is linear and, since for each $r > 0$:

$$\sup_{\|x\|_1 \leqq r} |F_\lambda(x)| \leqq \left(1 + r + \frac{r}{2}\right) r \cdot \|\lambda\|_\infty,$$

S is continuous.

(4) Let $b = (b_n)_{n \in \mathbf{N}}$ and $d = (d_n)_{n \in \mathbf{N}}$ be bounded sequences in the Banach space E, such that their respective associated restriction homomorphisms are onto l_∞. The restriction homomorphism associated to the sequence

$$(a_n)_{n \in \mathbf{N}} = (b_n)_{n \in \mathbf{N}} \cup (d_n)_{n \in \mathbf{N}}$$

(the ordering is indifferent) is also onto l_∞ if, and only if, the weak*-closures in E^{**} of $\{b_n : n \in \mathbf{N}\}$ and $\{d_n : n \in \mathbf{N}\}$ are disjoint. This is a consequence of the normality of $E^{**}_{bw^*}$. A particular case of this situation appeared in Example (3) with

$$b = (a_{2n+1})_{n \geqq 0} \quad \text{and} \quad d = (a_{2n})_{n \geqq 1} :$$

there are basic sequences equivalent to the standard basis of l_1 (so the associated homomorphisms are onto l_∞) and the weak*-closures

$$\overline{\{a_{2n+1} : n \geqq 0\}}^{w^*} \quad \text{and} \quad \overline{\{a_{2n} : n \geqq 1\}}^{w^*}$$

are separated in $E^{**}_{bw^*}$ by the extensions of h_1 and h_2.

(5) The sequence $(a_n)_{n \in \mathbf{N}} \subset l_1$ defined by

$$a_{2n-1} = \left(1 + \frac{1}{n}\right) e_n, \quad a_{2n} = e_n \quad (\text{for } n \geqq 1)$$

also satisfies that $(a_{2n-1})_{n \geqq 1}$ and $(a_{2n})_{n \geqq 1}$ are basic sequences equivalent to the standard basis of l_1. But in this case, for every $\xi \in l_\infty = l_1^*$ we have that

$$\left(\xi\left(a_{2n-1} - a_{2n}\right)\right)_{n \in \mathbf{N}}$$

is convergent to 0, and hence the associated homomorphism R is not onto l_∞.

(6) Let $(a_n)_{n \in \mathbf{N}}$ be the sequence in l_1 defined by

$$a_n = e_n + e_{n+1} \quad \text{for each } {}_{n \in \mathbf{N}},$$

and let us consider $M = \{a_n : n \in \mathbf{N}\}$. Then $(a_n)_{n \in \mathbf{N}}$ has no weakly Cauchy subsequence, M is weak-discrete and

$$\text{in}\{\|a_n - a_m\| : n \neq m\} = 2 > 0.$$

However, the homomorphism R associated to $(a_n)_{n \in \mathbf{N}}$ is not onto l_∞.

Proof. For each $m \in \mathbf{N}$ we consider the weak-neighbourhood of 0

$$V_m = \left\{x = (x_n)_{n \in \mathbf{N}} \in l_1 : |x_m| < \frac{1}{2}, |x_{m+1}| < \frac{1}{2}\right\};$$

if $n \neq m$,

$$(a_n + V_n) \cap (a_m + V_m) = \emptyset,$$

so M is weak-discrete.

If $(a_{n_j})_{j\in\mathbf{N}}$ is a subsequence of $(a_n)_{n\in\mathbf{N}}$, there exists $\eta = (\eta_n)_{n\in\mathbf{N}} \in l_\infty = l_1^*$ such that

$$(\eta(a_{n_j}))_{j\in\mathbf{N}} = (\eta_{n_j} - \eta_{n_j+1})_{j\in\mathbf{N}}$$

is not convergent; therefore $(a_{n_j})_{j\in\mathbf{N}}$ is not weak-Cauchy.

In order to prove that R is not onto l_∞, we consider

$$B = \{a_{2n}; n \in \mathbf{N}\} \quad \text{and} \quad C = \{a_{2n+1} : n \in \mathbf{N}\},$$

and we show that

$$\bar{B}^{w^*} \cap \bar{C}^{w^*} \neq \emptyset;$$

indeed, given $\epsilon > 0$ and $\eta_1, \ldots, \eta_N \in l_\infty = l_1^*$, there exist $m, n \in \mathbf{N}$ such that for each $i = 1, \ldots, N$:

$$\left|\eta^i(a_{2n} - a_{2m+1})\right| = \left|\eta^i_{2n} + \eta^i_{2n+1} - \eta^i_{2m} - \eta^i_{2m+1}\right| < \epsilon;$$

hence $0 \in \overline{B - C}^{w^*}$ and then there is a net $(b_\alpha - c_\alpha)_\alpha$ weak*-convergent to 0 with $(b_\alpha)_\alpha \subset B$ and $(c_a)_\alpha \subset C$; since B is bounded, $(b_\alpha)_\alpha$ admits a subnet $(b_\gamma)_\gamma$ weak*-convergent to some point $b \in B$; thus $(c_\gamma)_\gamma$ is also convergent to b and then $b \in \bar{B}^{w^*} \cap \bar{C}^{w^*}$.

(7) In a similar way to Example (6), it is not difficult to check that the countable family $\{e_n - e_m : n < m\}$ in l_1 is weak-discrete and, if

$$P = \{e_{2n} - e_m : 2n < m\} \quad \text{and} \quad I = \{e_{2n+1} - e_m : 2n + 1 < m\},$$

we have that $0 \in \bar{P}^w \cap \bar{I}^w$. Therefore, the homomorphism R associated to this countable family is not onto l_∞.

We have been interested only in interpolation of bounded sequences of scalars. For the case of arbitrary sequences, we just mention the following result, whose proof will be omitted:

PROPOSITION 3.7. *Let $(a_n)_{n\in\mathbf{N}}$ be a sequence in E. We consider the associated "restriction homomorphism"*

$$R : C_{wbu}(E) \longrightarrow \mathbf{R}^{\mathbf{N}}$$

defined by

$$Rf = (f(a_n))_{n\in\mathbf{N}}.$$

Then:

(1) *The range of R is contained in l_∞ if, and only if, $(a_n)_{n\in\mathbf{N}}$ is bounded.*
(2) *R is onto $\mathbf{R}^{\mathbf{N}}$ if, and only if, $\left(\|a_n\|\right)_{n\in\mathbf{N}}$ tends to $+\infty$.*

4. Continuous homomorphisms between $C_{wb}(E)$ and $C_{wb}(F)$. Let E and F be real Banach spaces. In order to apply the general results of Section 1 to the case of homomorphisms between $C_{wb}(E)$ and $C_{wb}(F)$, we consider the topological algebra equalities

$$C_{wb}(E) = C(E, \tau_I) \quad \text{and} \quad C_{wb}(F) = C(F, \tau_I),$$

as we have seen in the introduction.

Remark. According to (1.2), if (E, τ_I) is not realcompact then there are discontinuous homomorphisms between $C_{wb}(E)$ and $C_{wb}(F)$ (here F is arbitrary); an example is $E = C[0, \Omega]$, where Ω denotes the first uncountable ordinal: indeed, since E_{bw} and (E, τ_I) have the same continuous real functions, from the proof of [**10**] it follows that (E, τ_I) is not realcompact.

On the other side, if (E, τ_I) is realcompact, every homomorphism between $C_{wb}(E)$ and $C_{wb}(F)$ is automatically continuous (see 1.1). Next we are going to see that a wide class of spaces are in this situation. Recall that a Banach space is said to be *weakly compactly generated* (W.C.G.) when it has a weakly compact total subset. Then, we have the following result (see [**20**], 4.3.2 and 4.3.3 for the proof):

PROPOSITION 4.1. (1) *If E_w is normal and realcompact (in particular, if E is* W.C.G.*) then $(E, \tau_I) = E_{bw}$ is normal and realcompact.*

(2) *If there exists a sequence $(\xi_n)_{n\in\mathbf{N}} \subset B_{E^*}$ which separates the points of E (in particular, if E is the dual of a separable space), then (E, τ_I) is realcompact.*

As a consequence of (1.1) we have:

PROPOSITION 4.2. *The space of continuous homomorphisms*

$$A : C_{wb}(E) \longrightarrow C_{wb}(F)$$

can be identified with the space $C_{wb}(F, (E, \tau_I))$ of continuous maps

$$\varphi : F_{bw} \longrightarrow (E, \tau_I)$$

(that is, the maps $\varphi : F \longrightarrow E$ such that $f \circ \varphi \in C_{wb}(F)$ for each $f \in C_{wb}(E)$), by the formula:

$$Af = f \circ \varphi, \quad (f \in C_{wb}(E)).$$

In the same way as in (2.4) we obtain:

PROPOSITION 4.3. *If* $F \not\supset l_1$, *the continuous homomorphisms between* $C_{wb}(E)$ *and* $C_{wb}(F) = C_{wk}(F) = C_{wsc}(F)$ *are induced by the maps of the space:*

$$C_{wb}(F, E_{bw}) = C_{wk}(F, E_{bw}) = C_{wsc}(F, E_{bw}) = C_{wsc}(F, E_w).$$

The following result about the range of homomorphisms is immediate:

PROPOSITION 4.4. *Let*

$$\varphi : F_{bw} \longrightarrow (E, \tau_I)$$

[*respectively,* $\varphi : F_{bw} \longrightarrow E^{**}_{bw^*}$] *be a continuous map, and let*

$$A : C_{wb}(E) \longrightarrow C_{wb}(F)$$

[*respectively,* $A : C_{wbu}(E) \longrightarrow C_{wb}(F)$] *be the corresponding induced homomorphism, with range* $\mathcal{A}_\varphi$. *Then:*

(1) $\mathcal{A}_\varphi \subset C_w(F)$ *if, and only if,*

$$\varphi \in C_w(F, (E, \tau_I))$$

[*resp.* $\varphi \in C_w(F, E^{**}_{bw^*})$].

(2) $\mathcal{A}_\varphi \subset C_{wbu}(F)$ *if, and only if,* φ *has an extension*

$$\tilde{\varphi} \in C(F^{**}_{bw^*}, E_w)$$

[*resp.* $\tilde{\varphi} \in C(F^{**}_{bw^*}, E^{**}_{bw^*})$].

Remark 4.5. (1) When F is infinite-dimensional we have (see [**20**], 4.1) that there exists $h \in C_{wb}(F) \backslash C_w(F)$; we consider $v \in E \backslash \{0\}$ and then

$$\varphi = h \otimes v : F \longrightarrow E$$

induces a homomorphism between $C_{wb}(E)$ [respectively, $C_{wbu}(E)$] and $C_{wb}(F)$, whose range is not contained in $C_w(F)$. On the other hand, when F is finite-dimensional it is clear that

$$C_w(F, (E, \tau_I)) = C_{wb}(F, (E, \tau_I))$$

[resp., $C_w(F, E^{**}_{bw^*}) = C_{wb}(F, E^{**}_{bw^*})$].

(2) When F is not refexive we have (see [**20**], 4.1) that there exists $h \in C_{wb}(F) \backslash C_{wbu}(F)$; we consider $v \in E \backslash \{0\}$ and then

$$\varphi = h \otimes v : F \longrightarrow E$$

induces a homomorphism between $C_{wb}(E)$ [respectively, $C_{wbu}(E)$] and $C_{wb}(F)$, whose range is not contained in $C_{wbu}(F)$. When F is reflexive, it is clear that

$$C_{wb}(F,(E,\tau_I)) = C_{wbu}(F,E_w)$$

[resp., $C_{wb}(F,E^{**}_{bw^*}) = C_{wbu}(F,E^{**}_{w^*})$].

PROPOSITION 4.6. *Let $\varphi : F_{bw} \longrightarrow (E,\tau_I)$ be a continuous map, and let*

$$A : C_{wb}(E) \longrightarrow C_{wb}(F)$$

be the corresponding induced homomorphism, with range $\mathcal{A}_\varphi$.

(1) *$\mathcal{A}_\varphi$ is dense in $C_{wb}(F)$ if, and only if, φ is one-to-one.*

(2) *Suppose that E_{bw} is normal and F_{bw} is completely regular, and, in addition, either F_w is Lindelöf or there exists a sequence $(\xi_n)_{n\in\mathbf{N}} \subset B_{E^*}$ separating the points of E. Then, the following conditions are equivalent.*

(a) *$\mathcal{A}_\varphi$ is closed in $C_{wb}(F)$ and $F_{bw}/\mathcal{A}_\varphi$ is completely regular.*

(b) *For each bw-closed and φ-saturated subset C of F, $\varphi(C)$ is bw-closed in E.*

(3) *Suppose that E_{bw} is normal and $E \not\supset l_1$. If*

$$\varphi : F_{bw} \longrightarrow E_{bw}$$

is semiproper, $\mathcal{A}_\varphi$ is closed in $C_{wb}(F)$.

(4) *Suppose that F is reflexive. If $\mathcal{A}_\varphi$ is closed in $C_{wb}(F)$, then*

$$\varphi : F_{bw} \longrightarrow E_{bw}$$

is semiproper.

Proof. (1) This is a consequence of (1.4).

(2) When F_{bw} is Lindelöf (which is equivalent to F_w being Lindelöf), the result follows from (1.10). Now we suppose that there exists a sequence $(\xi_n)_{n\in\mathbf{N}} \subset B_{E^*}$ separating the points of E. We claim that every C-embedded subset of E_{bw} is bw-closed; indeed, if $M \subset E$ is not bw-closed we select $x_0 \in \bar{M}^{bw}\backslash M$ and we consider the function $f : E \longrightarrow \mathbf{R}$ defined by:

$$f(x) = \sum_{n=1}^{\infty} 2^{-n} \frac{|\xi_n(x - x_0)|}{1 + |\xi_n(x - x_0)|};$$

f is w-continuous, and $f(x) = 0$ only for $x = x_0$; hence

$$h = \frac{1}{f} : M \longrightarrow \mathbf{R}$$

is a bw-continuous function on M without bw-continuous extension to $\bar{M}^{bw}$. It is now sufficient to apply (1.10).

(3) From ([20], 4.4.10) and ([7], 2.2.1) we have that: $E \not\supset l_1 \Leftrightarrow C_{wb}(E)$ is complete $\Leftrightarrow (E, \tau_I)$ is a K-space. Therefore the result follows from (1.15).

(4) If F is reflexive, then $F_{bw} = F^{**}_{bw^*}$ is a hemicompact K-space, and the result is a consequence of (1.20).

Remark 4.7. In relation with the preceding proposition (2) and (3), it is interesting to point out that, by ([20], 4.3.1), E_{bw} is normal whenever E_w is normal (in particular when E is W.C.G.). On the other hand, we do not know whether it is possible to remove the requirement of $F_{bw}/\mathcal{A}_\varphi$ being completely regular in (2). When F is reflexive, then $F_{bw}/\mathcal{A}_\varphi$ is always normal (this follows from (1.12): note that F_{bw} is a hemicompact K-space $\Leftrightarrow F_{bw}$ is σ-compact $\Leftrightarrow F$ is reflexive (see [9], 1)); and we will see in (4.8) below that the same is true in some other situations. In these cases, (4.6.(2)) characterizes when $\mathcal{A}_\varphi$ is closed in $C_{wb}(F)$.

PROPOSITION 4.8. *Suppose that E_{bw} is normal and*

$$\varphi : F_{bw} \longrightarrow E_{bw}$$

is a continuous map such that, for each bounded and w-closed subset B of F, $\varphi(B)$ is bw-closed in E. Then $F_{bw}/\mathcal{A}_\varphi$ is normal.

Proof. We have the equivalence relation in F_{bw} (see (1.6)):

$$x\mathcal{A}_\varphi y \Leftrightarrow f(x) = f(y), \quad \forall f \in \mathcal{A}_\varphi \Leftrightarrow \varphi(x) = \varphi(y), \quad (x, y \in F)$$

and we consider the corresponding canonical factorization:

$$\begin{tikzcd} F_{bw} \arrow[rr, "\varphi"] \arrow[dr, "\pi"'] & & E_{bw} \\ & F_{bw}/A_\varphi \arrow[ur, "\bar{\varphi}"'] & \end{tikzcd} \qquad (\text{where } \varphi = \bar{\varphi} \circ \pi).$$

For each $n \in \mathbf{N}$, let B_n be the n-th closed ball of F. Then $\varphi(B_n)$ is closed in E_{bw}, and hence normal. We claim that

$$\bar{\varphi}|_{\pi(B_n)} : \pi(B_n) \longrightarrow \varphi(B_n)$$

is a homeomorphism. Indeed, we have that $\bar{\varphi}$ is continuous; and, on the other side, if C is a closed subset of $\pi(B_n)$, as

$$\pi(B_n) = \bar{\varphi}^{-1}(\varphi(B))$$

is closed in $F_{bw}/\mathcal{A}_\varphi$, we have that C is closed in $F_{bw}/\mathcal{A}_\varphi$; then $\pi^{-1}(C)$ is bw-closed in F and $\pi^{-1}(C) \cap B_n$ is a bounded and w-closed subset of F, so

$$\bar{\varphi}(C) = \varphi(\pi^{-1}(C) \cap B_n)$$

is bw-closed in E. Therefore, $\pi(B_n)$ is normal for each $n \in \mathbf{N}$.
Now the proof can be finished following the same ideas used in (1.12).

The next example will show that the conditions in the preceding Proposition (4.8) do not imply that $\mathcal{A}_\varphi$ is closed:

Example 4.9. We consider the map $\varphi : c_0 \longrightarrow c_0$ defined by:

$$\varphi((x_n)_{n\in\mathbf{N}}) = (\varphi_n(x_n))_{n\in\mathbf{N}}, \quad \text{for every } (x_n)_{n\in\mathbf{N}} \in c_0;$$

where each $\varphi_n : \mathbf{R} \longrightarrow \mathbf{R}$ is defined by

$$\varphi_n(t) = \begin{cases} 1 & ; \text{ for } t \geqq n \\ \frac{1}{2}\left(1 + \frac{2t-1}{2n-1}\right); & \text{for } \frac{1}{2} \leqq t \leqq n \\ t & ; \text{ for } 0 \leqq t \leqq \frac{1}{2} \\ -\varphi_n(-t) & ; \text{ for } t \leqq 0. \end{cases}$$

Then:

1. The range of φ is the closed unit ball Bc_0 of c_0, and

$$\varphi : (c_0, bw) \longrightarrow (c_0, bw)$$

is continuous.

2. For each bounded and w-closed subset B of c_0, $\varphi(B)$ is bw-closed in c_0.
3. $M = \{ne_n : n \in \mathbf{N}\}$ is bw-closed in c_0, but $\varphi(M)$ is not bw-closed in c_0.
4. $\mathcal{A}_\varphi = \{f \circ \varphi : f \in C_{wb}(c_0)\}$ is not closed in $C_{wb}(c_0)$.
5. $(c_0, bw)/\mathcal{A}_\varphi$ is normal.

Proof. 1. It is clear that the range of φ is Bc_0. Now we are going to see that

$$\varphi : (c_0, w) \longrightarrow (Bc_0, w)$$

is continuous (in fact, uniformly continuous):

Given $\xi = (\xi_n)_n \in l_1 - c_0^*$ and given $\epsilon > 0$, we choose $N \in \mathbf{N}$ such that

$$\sum_{n>N} |\xi_n| < \epsilon.$$

Using the uniform continuity of $\varphi_1, \varphi_2, \ldots, \varphi_N$, we can find some $\delta > 0$ such that if $x = (x_n)_{n\in N}$, $y = (y_n)_{n\in\mathbf{N}} \in c_0$ with

$$|x_n - y_n| < \delta \quad \text{for } n = 1, 2, \ldots, N,$$

then

$$|\xi \circ \varphi(x) - \xi \circ \varphi(y)| \leqq \sum_{n=1}^{N} |\xi_n|\,|\varphi_n(x_n) - \varphi_n(y_n)| + \sum_{n>N} 2|\xi_n| < 3\epsilon.$$

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2. Let B be a bounded and bw-closed subset of c_0, and let

$$y \in \overline{\varphi(B)}^{w};$$

since (Bc_0, w) is metrizable, there exists a sequence $(x^m)_{m\in\mathbf{N}} \subset B$ such that

$$(y^m)_{m\in\mathbf{N}} = \big(\varphi(x^m)\big)_{m\in\mathbf{N}}$$

is weakly convergent to y. Let $N \in \mathbf{N}$ be such that $|y_n| < 1/2$ for $n \geqq N$, and $B \subset NBc_0$; for each $n > N$ since

$$\varphi_n : [-n, n] \longrightarrow [-1, 1]$$

is a homeomorphism, we have that $(x_n^m)_{m\in\mathbf{N}}$ is convergent to $\varphi_n^{-1}(y_n) = y_n$; on the other side, there exists a strictly increasing sequence $(m_k)_{k\in\mathbf{N}} \subset \mathbf{N}$ such that for $n = 1, 2, \ldots, N$, $(x_n^{m_k})_{k\in\mathbf{N}}$ is convergent, say to z_n; defining $z_n = y_n$ for $n > N$, we obtain $z = (z_n)_{n\in\mathbf{N}} \in c_0$ such that $(x^{m_k})_{k\in\mathbf{N}}$ is weakly convergent to z and $\varphi(z) = y$; therefore $z \in B$ and $y \in \varphi(B)$.

3. It is clear that M is bw-closed and $\varphi(M) = \{e_n : n \in \mathbf{N}\}$ is not w-closed.

4. We consider the function $h : c_0 \longrightarrow \mathbf{R}$ defined by:

$$h(x)\sum_{n=1}^{\infty} \frac{\min\{n^2, x_n^2\}}{n^2} \quad \text{for } x = (x_n)_{n\in\mathbf{N}} \in c_0.$$

We are going to see that $h \in C_{wb}(c_0)$ (in fact, $h \in C_{wbu}(c_0)$):
Given $R > 0$ and $\epsilon > 0$, we choose $N \in \mathbf{N}$ such that $N \geqq R$ and

$$\sum_{n>N} n^{-2} < \epsilon R^{-2};$$

using the uniform continuity of the functions

$$t \longmapsto \min\{n^2, t^2\} \quad (n = 1, 2, \ldots, N),$$

we can find $\delta > 0$ such that, if

$$x = (x_n)_{n\in\mathbf{N}}, y = (y_n)_{n\in\mathbf{N}} \in c_0 \quad \text{with}$$

$$\|x\|_\infty, \|y\|_\infty \leqq R \quad \text{and} \quad |x_n - y_n| < \delta \quad \text{for } n = 1, 2, \ldots, N,$$

then

$$|h(x) - h(y)| \leqq \sum_{n=1}^{N} \frac{|\min\{n^2, x_n^2\} - \min\{n^2, y_n^2\}|}{n^2} + \sum_{n>N} \frac{2R^2}{n^2} < 3\epsilon.$$

Further, $h(x) = h(y)$ whenever $\varphi(x) = \varphi(y)$; therefore by (1.6) we have that $\varphi \in \overline{\mathcal{A}}_{\varphi}$. Nevertheless, $h \notin \mathcal{A}_{\varphi}$ for, if we suppose that $h = f \circ \varphi$ some $f \in C_{wb}(c_0)$, then

$$1 = h(me_m) = f(\varphi(me_m)) = f(e_m)$$

and $(f(e_m))_{m \in \mathbf{N}}$ is convergent to $f(0) = f(\varphi(0)) = h(0) = 0$, which is a contradiction.

Next we obtain a sufficient condition for a composition subalgebra to be closed, analogous to (4.6.(3)), but without the requirement of $E \not\supset l_1$. We will see in (4.12) that this condition is not necessary.

PROPOSITION 4.10. *Suppose that E_{bw} is normal and*

$$\varphi : F_{bw} \longrightarrow E_{bw}$$

is a continuous map such that for each bounded and w-closed subset $B \subset E$ there exists a w-compact subset $K \subset F$ such that $\varphi(K) = B \cap \varphi(F)$. Then the composition subalgebra

$$\mathcal{A}_{\varphi} = \{f \circ \varphi : f \in C_{wb}(E)\}$$

is closed in $C_{wb}(F)$.

Proof. If M is a τ_l-closed and φ-saturated subset of F, we have that $\varphi(M)$ is bw-closed in E: indeed, for each closed ball B of E there exists a w-compact subset $K \subset F$ such that $\varphi(K) = \varphi(F) \cap B$; hence

$$\varphi(M) \cap B = \varphi(M) \cap \varphi(K) = \varphi(M \cap K)$$

is w-closed in E. Note that in particular $\varphi(F)$ is closed in E_{bw} and therefore it is C-embedded. The result is then a consequence of (1.10).

Remark 4.11. Let X be a completely regular Hausdorff space and let $\varphi : X \longrightarrow E_{bw}$ be a continuous map. Since the inclusion

$$E_{bw} \hookrightarrow E^{**}_{bw^*}$$

is continuous, we have that

$$\tilde{\varphi} : X \longrightarrow E_{bw} \hookrightarrow E^{**}_{bw^*}$$

is also continuous. Let

$$A : C_{bw}(E) \longrightarrow C(X)$$

be the homomorphism induced by φ, with range $\mathcal{A}_\varphi$ and let

$$\tilde{A} : C_{wbu}(E) \longrightarrow C(X)$$

be the homomorphism induced by $\tilde{\varphi}$, with range $\mathcal{A}_{\tilde{\varphi}}$. We have that

$$\tilde{A} = A|_{C_{wbu}(E)} \quad \text{and} \quad \mathcal{A}_{\tilde{\varphi}} \subset \mathcal{A}_\varphi \subset C(X).$$

Note that, according (1.4), $\mathcal{A}_\varphi$ is dense in $C(X) \Leftrightarrow \mathcal{A}_{\tilde{\varphi}}$ is dense in $C(X) \Leftrightarrow \varphi$ is one-to-one. On the other side, since A is continuous and $C_{wbu}(E)$ is dense in $C_{wb}(E)$ (see [**11**], 4.12) we have that $\mathcal{A}_{\tilde{\varphi}}$ is dense in $\mathcal{A}_\varphi$; therefore if $\mathcal{A}_{\tilde{\varphi}}$ is closed in $C(X)$ then $\mathcal{A}_{\tilde{\varphi}} = \mathcal{A}_\varphi$. In (4.12) we have an example for which $\mathcal{A}_\varphi$ is closed in $C(X)$ but $\mathcal{A}_{\tilde{\varphi}}$ is not.

Example 4.12. Let $h : \mathbf{R} \longrightarrow \mathbf{R}$ be the function defined by:

$$h(t) = \begin{cases} 0 & ; \text{ for } t \leqq -1 \\ 1+t & ; \text{ for } -1 \leqq t \leqq 0 \\ 1-t & ; \text{ for } 0 \leqq t \leqq 1 \\ 0 & ; \text{ for } 1 \leqq t \end{cases}$$

We consider $l_1 = l_1(\mathbf{Z})$ and we define the map

$$\varphi : \mathbf{R} \longrightarrow (l_1, bw)$$

by:

$$\varphi(t) = (h(t-n))_{n \in \mathbf{Z}};$$

we also consider

$$\tilde{\varphi} : \mathbf{R} \longrightarrow (l_1, bw) \hookrightarrow (l_1^{**}, bw^*).$$

Then
1. $\varphi : \mathbf{R} \longrightarrow (l_1, bw)$ is bounded, continuous and one-to-one.
2. $\varphi : \mathbf{R} \longrightarrow (l_1 bw)$ does not satisfy the conditions of (4.10), and

$$\tilde{\varphi} : \mathbf{R} \longrightarrow (l_1^{**}, bw^*)$$

is not semiproper.
3. $\mathcal{A}_{\tilde{\varphi}} = \{f \circ \varphi : f \in C_{wbu}(l_1)\}$ is dense, but not closed, in $C(\mathbf{R})$.
4. $\varphi : \mathbf{R} \longrightarrow (l_1, bw)$ is semiproper and

$$\mathcal{A}_\varphi = \{f \circ \varphi : f \in C_{wb}(l_1) = C(\mathbf{R}).$$

Proof. 1. Let $(e_n)_{n\in\mathbf{Z}}$ be the standard basis of l_1. Note that given $t_0 \in \mathbf{R}$ and $n_0 \in \mathbf{Z}$ with $n_0 \leqq t_0 < n_0 + 1$ then for each $t \in (t_0 - 1, t_0 + 1)$ we have that

$$\varphi(t) = \sum_{n=n_0-1}^{n_0+2} h(t-n)e_n.$$

Therefore φ is bounded and continuous. On the other side it is clear that φ is one-to-one.

2. We denoted by B and B^{**} the closed unit balls of l_1 and l_1^{**} respectively. For each $n \in \mathbf{Z}$, since $e_n \in B \subset B^{**}$ we have that

$$n = \varphi^{-1}(e_n) \in \varphi^{-1}(B)$$

and also

$$n \in \tilde{\varphi}^{-1}(B^{**});$$

hence $\varphi^{-1}(B)$ and $\tilde{\varphi}^{-1}(B^{**})$ are not bounded.

3. This follows from (2.2.(2)) and (2.2.(5)).

4. First we are going to see that $\varphi(\mathbf{R})$ is closed in (l_1, bw) and $\varphi : \mathbf{R} \longrightarrow \varphi(\mathbf{R})$ is a homeomorphism. Since for each compact subset $K \subset \mathbf{R}$ we have that

$$\varphi|_K : K \longrightarrow \varphi(K)$$

is a homeomorphism, it is sufficient to show that if $(t_\alpha)_\alpha \in \mathbf{R}$ is a net such that $(\varphi(t_\alpha))_\alpha$ is convergent in (l_1, bw), then there exists some α_0 such that $\{t_\alpha : \alpha \geqq \alpha_0\}$ is bounded. And indeed, if we suppose otherwise we have a net $(t_\alpha)_\alpha \subset \mathbf{R}$ and $x = (x_n)_{n\in\mathbf{Z}} \in l_1$ with $(\varphi(t_\alpha))_\alpha$ being weakly convergent to x, but such that for each $n \in \mathbf{Z}$ and each α there exists $\alpha' \geqq \alpha$ for which

$$h(t_{\alpha'} - n) = 0.$$

Since $(h(t_\alpha - n))_\alpha$ is convergent to x_n for each $n \in \mathbf{Z}$, we have that $x = 0$. But if for each α we choose $m_\alpha \in \mathbf{Z}$ such that

$$|t_\alpha - m_\alpha| \leqq \frac{1}{2}$$

and we define $\xi = (\xi_n)_{n\in\mathbf{Z}} \in l_\infty(\mathbf{Z})$ by:

$$\xi_n = \begin{cases} 0; & \text{if } n \neq m_\alpha \text{ for every } \alpha \\ 1; & \text{if } n = m_\alpha \text{ for some } \alpha \end{cases}$$

we have that

$$\langle x, \varphi(t_\alpha)\rangle \geqq h(t_\alpha - m_\alpha) \geqq \frac{1}{2},$$

for each α; this is a contradiction. Therefore, we have that

$$\varphi : \mathbf{R} \longrightarrow (l_1, bw)$$

is semiproper; and since (l_1, bw) is normal (see 4.1), we have by (1.13) that $\mathcal{A}_\varphi = C(X)$.

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