Canad. Math. Bull. Vol. **26** (2), 1983

A COVERING PROBLEM FOR IDEMPOTENT LATIN SQUARES

BY

KATHERINE HEINRICH*

ABSTRACT. Let $A=(a_{ij})$ be an idempotent latin square of order n, $n\geq 3$, in which $a_{ii}=i$, $1\leq i\leq n$. A set $S\subseteq N=\{1,2,\ldots,n\}$ is a cover of A if $(N\times N)\backslash\{(i,i):i\notin S\}=\{(i,j):i\in S, j\in N\}\cup\{(j,i):i\in S, j\in N\}\cup\{(i,j):a_{ij}\in S\}$. A cover S is minimum for A if $|S|\leq|T|$ for every cover T of A and we write $c(A)=|S|$. We denote by $c(n)$ the maximum value of $c(A)$ over all idempotent latin squares A of order n and in this paper show that $(7n/10)-3.8\leq c(n)<n-n^{1/3}+1$ for all $n\geq 15$. The problem of determining $c(n)$ was first raised by J. Schönheim.

1. **Introduction.** Both W. W. Rouse Ball [1] and Maurice Kraitchik [4] have raised the question of determining the minimum number of chess pieces of one type which can be placed on an $n\times n$ chessboard so as to attach all cells. For example, in [1] it is given that for an 8×8 board, the number for kings is 9; for queens 5; for bishops 8; for knights 12; and for rooks 8. Clearly, on an $n\times n$ board the minimum number of rooks required is n.

In this paper we are interested in the somewhat related problem posed by J. Schönheim [5]. Suppose we have an $n\times n$ chessboard, $n\geq 3$, each cell of which is coloured from a set of n colours so that the n cells of each row, column and of the main left-right diagonal are coloured in distinct colours. A superrook placed on a cell of the board attacks all cells of the same row, column and colour as that on which it stands. The problem is to determine the minimum number of superrooks to be placed on the main left-right diagonal so as to attack all n^2-n off-diagonal cells in *any* colouring of the board. For 3×3 boards this number is 1; for 4×4 boards it is 2; and for 5×5 boards it is 3.

We can think of the board as an idempotent latin square of order n which we denote by $A=(a_{ij})$ where $a_{ii}=i$, $1\leq i\leq n$. (For all definitions pertaining to latin squares the reader is referred to J. Dénes and A. D. Keedwell [3].) The superrooks are then to be placed on cells a_{ii}, $1\leq i\leq n$. A set $S\subseteq\{1,2,\ldots,n\}=N$ will be called a *cover* of A if $(N\times N)\backslash\{(i,i):i\notin S\}=\{(i,j):i\in S, j\in N\}\cup\{(j,i):i\in S, j\in N\}\cup\{(i,j):a_{ij}\in S\}$. The set S then corresponds to placing superrooks on cells a_{ii}, $i\in S$, which attack every off-main diagonal cell. If S is a

Received by the editors May 26, 1981 and, in revised form, April 19, 1982.

AMS Subject Classification Number-05B15.

* This research was supported in part by Natural Sciences and Engineering Research Council of Canada through grant U0165.

minimum cover of A (that is, there is no $T \subseteq N$, $|T|<|S|$ so that T is a cover of A) and $|S|=k$, we say A has *covering number* k and denote this by $c(A)=k$. Thus $c(n)$ is the maximum value of $c(A)$ over all idempotent latin squares A of order n. Our problem is to determine $c(n)$.

We shall show in Section 2 that no latin square of order n can have covering number less that $\lfloor n/2 \rfloor$ and that for every order $n \geq 6$ there is a latin square A of order n for which $c(A)=\lfloor n/2 \rfloor$. In Section 3 we shall show that $(7n/10)-3.8 \leq c(n) < n-n^{1/3}+1$.

2. **The minimum is attained**

THEOREM 1. *No latin square of order n has covering number less than* $\lfloor n/2 \rfloor$.

Proof. Let A be an idempotent latin square of order n with $c(A)=k$. We can suppose (on permutations of rows, of columns and of elements) that $S=\{1,2,\ldots,k\}$ covers A. Let B be the subarray defined by rows and columns $k+1, k+2, \ldots, n$. The $(n-k)(n-k-1)$ off diagonal cells of B can contain only the elements of S, each of which can occur at most $n-k$ times. Thus we need $k(n-k) \geq (n-k)(n-k-1)$ and hence $k \geq \lfloor n/2 \rfloor$. □

THEOREM 2. *There exists a latin square A of order n for which* $c(A)=\lfloor n/2 \rfloor$, *provided* $n \neq 5$.

Proof. If $n=2m+1$, $m \neq 2$, let A be a latin square of order $2m+1$, with a subsquare B of order m (see [3, Theorem 1.5.1] for the construction of A). We can suppose that the elements of B are $m+2, m+3, \ldots, 2m+1$ and that B has a transversal. (Such a subsquare B is easily constructed when $m=6$ and for $m \neq 2, 6$ follows from the existence of a pair of orthogonal latin squares of order m [2].) The rows and columns of A can then be permuted so that A is idempotent. Then $S=\{m+2, m+3, \ldots, 2m+1\}$ is a cover of A. When $m=2$ it is not difficult to see that for every latin square A of order 5, $c(A)=3$.

If $n=2m$, $m \neq 2$, let B be an idempotent latin square of order m with a transversal disjoint from the main diagonal. (Again, B is easily constructed when $m=6$ and otherwise follows from [2].) Let B' be a copy of B in which elements $1, 2, \ldots, m$ are replaced with $m+1, m+2, \ldots, 2m$. Now, A' is a latin square of order $2m$ with a copy of B in the upper left and lower right, and of B' in the upper right and lower left. In $A'=(a'_{ij})$ the cells $a'_{ii}=a'_{m+i,m+i}=i$ and $a'_{i,m+i}=a'_{m+i,i}=m+i$, $1 \leq i \leq m$, form subsquares of order 2. In each of these subsquares exchange elements i and $m+i$; so obtaining the latin square A in which $a_{ii}=a_{m+i,m+i}=m+i$, and $a_{i,m+i}=a_{m+i,i}=i$, $1 \leq i \leq m$. We can now permute the first m rows and columns of A so that, using the second transversal of B, A becomes idempotent. The set $S=\{1, 2, \ldots, m\}$ is now a cover of A. Also, for every idempotent latin square A of order 4, $c(A)=2$. □

3. **Bounds on** $c(n)$. We now study upper and lower bounds on $c(n)$.

THEOREM 3 (*Lower bound*). *For all* $n\geq 15$, $c(n)\geq(7n/10)-3.8$.

Proof. We shall construct a latin square of order n with cover S, $|S|\geq(7n/10)-3.8$. We consider first the case $n=10m+5$, $m\geq 1$.

Let $A=(a_{ij})$ be the idempotent latin square of order 5 on the residues $\{1,2,3,4,5\}$ modulo 5 with $a_{ij}=2j-i$. Let A_{rs}, $1\leq r,s\leq 2m+1$ be the latin square obtained by adding $5(r+s-2)$ to each element of A and reducing modulo $10m+5$ to the residues $\{1,2,\ldots,10m+5\}$. Consider the idempotent latin square B of order $10m+5$ formed by the blocks A_{rs}.

We want to show that if S is any cover of B then $|S|\geq 7m+3$. Let S be a cover of B. It is only possible to cover the off-main diagonal cells of A_{rr} if S contains at least three elements from $\{1+5(r-1),2+5(r-1),3+5(r-1),4+5(r-1),5r\}$ as $c(A_{rr})=3$. However, this may not be sufficient to cover B. Suppose that from one of the A_{rr}, and we can assume this is A_{11}, we have exactly 3 elements in S. (If no such A_{rr} exists, $|S|\geq 4(2m+1)$ and we are done.) In B the other copies of A_{11} are $A_{r,2-r}$, $2\leq r\leq 2m+1$. Consider the m order 10 subarrays in B consisting of A_{rr}, $A_{r,2-r}$, $A_{2-r,r}$ and $A_{2-r,2-r}$, $2\leq r\leq m+1$. We shall show that at least seven elements from A_{rr} and $A_{2-r,2-r}$ must be contained in S. From this it will follow immediately that $|S|\geq 7m+3$.

Suppose that exactly three elements from A_{rr} are in S; that $x,y\notin S$, $x\neq y$ and $x,y\in\{1,2,3,4,5\}$; and that $u'\equiv u \pmod 5$, $v'\equiv v \pmod 5$, $u\neq v$, $u',v'\in\{1+5(i-1),2+5(i-1),\ldots,5i\}$ and $u',v'\notin S$.

For any z in A, $a_{2k-z,k}=z$ for $1\leq k\leq 5$. Thus in $A_{r,2-r}$ $a_{u,x+u}$, $a_{v,(x+v)/2}$, $a_{u,(y+u)/2}$ and $a_{v,(y+v)/2}$ are not covered by the elements in S from A_{11} and A_{rr}. If four elements from $A_{2-r,2-r}$ are in S it is possible to cover these cells and we are finished. Otherwise we must have either $(x+u)/2\equiv(y+v)/2 \pmod 5$ or $x+v/2\equiv(y+u)/2 \pmod 5$. Both equivalences cannot hold as A has no order two subsquare and so we assume $(x+u)/2\equiv(y+v)/2 \pmod 5$. Applying a similar argument to the cells of $A_{2-r,r}$ not covered by elements in S from A_{11} and A_{rr}, we see that if we are not to have four elements from $A_{2-r,2-r}$ in S we must have either $2u-x\equiv 2v-y \pmod 5$ or $2v-x\equiv 2u-y \pmod 5$. However, neither of these is consistent with $(x+u)/2\equiv(y+v)/2 \pmod 5$ and so we cannot have only three elements from $A_{2-r,2-r}$ in S.

We must now take care of all other values of n. In B there are two symmetrically placed sets of $2m+1$ blocks, $T_1=\{A_{r,r+1}:1\leq r\leq 2m+1\}$ and $T_2=\{A_{r,r-1}:1\leq r\leq 2m+1\}$. Since A has five disjoint transversals, then B has (using T_1 and T_2) ten disjoint transversals. Select p, $1\leq p\leq 9$, $p\neq 2$, of these and use them to prolongate (see [3, p. 39]) p times from B inserting an idempotent latin square $D=(d_{ij})$ of order p with $d_{ii}=10m+5+i$, $1\leq i\leq p$. The result is an idempotent latin square E of order $10m+5+p$. By an argument

https://doi.org/10.4153/CMB-1983-023-7 Published online by Cambridge University Press

similar to that given above and as T_1 and T_2 are symmetrically placed then (allowing $c(1)=0$) $c(E)\geq 7(m-1)+9+c(p)\geq 7m+2+\lfloor p/2\rfloor$, $1\leq p\leq 9$, $p\neq 2$.

This leaves only the case when $p=2$. Again, begin with the idempotent latin square B of order $10m+5$. Using the transversal T_1 described above prolongate once and insert the element $10m+6$ in cell $(10m+6, 10m+6)$. This yields an idempotent latin square $D=(d_{ij})$ of order $10m+6$ with $2m+1$ subsquares of order 6 each containing $d_{10m+6,10m+6}$. Select one of these subsquares and replace it with a latin square of order 6 in which $d_{10m+6,10m+6}=10m+6$, and there is a transversal which avoids this cell. Now, prolongate once from D using T_1 and this new subsquare of order 6. Insert $10m+7$ in cell $(10m+7, 10m+7)$. This results in an idempotent latin square E of order $10m+7$ and an argument similar to that given earlier shows that $c(E)\geq 7(m-1)+9=7m+2$.

Combining these results we have $c(n)\geq(7n/10)-3.8$. □

THEOREM 4 (*Upper bound*). *For all n*, $c(n)<n-n^{1/3}+1$.

Proof. Let A be an idempotent latin square of order n with $c(A)\geq n-k$; that is, if S covers A then $|S|\geq n-k$. Then every order $k+1$ subarray of A positioned symmetrically about the main diagonal must have at least one element from its set of main diagonal elements in an off diagonal position as otherwise there is an obvious cover of A with $n-(k+1)$ elements.

Any element a_{ij}, $i\neq j$, lies in both a main diagonal and off diagonal position in $\binom{n-3}{k-2}$ of these subarrays as the subarray must contain a_{ii}, a_{jj} and a_{ij}. There are $n(n-1)$ off diagonal elements in A and $\binom{n}{k+1}$ such order $k+1$ subarrays. If $n(n-1)\binom{n-3}{k-2}<\binom{n}{k+1}$ there is some order $k+1$ subarray without the necessary property and we have a contradiction. Thus $n(n-1)\binom{n-3}{k-2}\geq\binom{n}{k+1}$ implying $k^3-k+2\geq n$.

We have shown that if $n>k^3-k+2$, then $c(n)<n-k$. Consequently, for $k>2$, if $k^3\leq n<(k+1)^3$, $c(n)<n-k<n-n^{1/3}+1$ when $n\geq 27$.

It is easy to see that $c(n)\leq n-2$ for all n and thus $c(n)<n-n^{1/3}+1$ for all n. □

REFERENCES

1. W. W. Rouse Ball and H. S. M. Coxeter, *Mathematical Recreations and Essays*. University of Toronto Press, 1974.

2. R. C. Bose, S. S. Shrikhande and E. T. Parker, *Further results on the construction of mutually orthogonal latin squares and the falsity of Euler's conjecture*. Canad. J. Math. **12** (1960), 189–203.

3. J. Dénes and A. D. Keedwell, *Latin Squares and their Applications.* Akadémiai Kiadó, Budapest, 1974.

4. Maurice Kraitchik, *Mathematical Recreations.* Dover, 1953.

5. J. Schönheim, Problem 32 in *Combinatorial Structures and their Applications.* Proceedings of the Calgary Conference, Gordon and Breach, 1969, p. 505.

Department of Mathematics
Simon Fraser University
Burnaby B.C. VSA IS6