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# A Schreier theorem for free topological groups

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M.I. Graev has shown that subgroups of free topological groups need not be free. Brown and Hardy, however, have proved that any open subgroup of the free topological group on a $k_\omega$-space is again a free topological group: indeed, this is true for any closed subgroup for which a Schreier transversal can be chosen continuously. This note provides a proof of this result more direct than that of Brown and Hardy. An example is also given to show that the condition stated in the theorem is not a necessary condition for freeness of a subgroup. Finally, a sharpened version of a particular case of the theorem is obtained, and is applied to the preceding example.

The well-known Nielsen-Schreier theorem [4] states that every subgroup of a free group is free. Examples have been given, however, to show that the analogous statement for subgroups of a free topological group is false ([3] and [1]). Nevertheless, Brown and Hardy have proved in [2] (see also [5]) that a closed subgroup of the free topological group on a $k_\omega$-space will again be free, provided that a Schreier transversal for the subgroup can be chosen continuously. (Two immediate corollaries deserve mention. Firstly, any open subgroup satisfies the condition and is therefore free.

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Secondly, if a topological group $G$ is a $k_\omega$-space, then the canonical quotient from the free topological group on $G$ onto $G$ has a free topological group as its kernel.)

Brown and Hardy proved their result by means of the theory of topological groupoids ([2], [5]; see also [6]) and their techniques also yielded an open subgroup version of the Kurosh subgroup theorem for free products of topological groups. Our main aim in this note is to provide a short proof of their topological Nielsen-Schreier theorem, without recourse to the theory of topological groupoids. We then discuss an example of a free topological group with a subgroup which is free, but for which a Schreier transversal cannot be chosen continuously. A sharpened version of the theorem in a special case is then proved, and applied to the preceding example.

We now list a few facts, and establish the notation to be used later. We assume the reader to be familiar with the concept of the free topological group on a pointed space, as defined by Graev [3]. (See also [9] and [7].)

Recall that a Hausdorff space $X$ is a $k_\omega$-space if it is the union of an increasing sequence $\{X_n\}$ of compact sets, with the property that a set $A \subseteq X$ is closed if and only if $A \cap X_n$ is compact for each $n$. (We refer to $\cup X_n$ as the decomposition of $X$.) Any compact set in such a space lies in some $X_n$. For this and further information, see [7] and [8].

If $Y$ is a subset of a group $G$, we denote by $\mathrm{gp}(Y)$ the subgroup of $G$ generated by $Y$, and by $\mathrm{gp}_n(Y)$ those elements of $\mathrm{gp}(Y)$ which can be written as words of lengths less than or equal to $n$ with respect to $Y$. In particular, if $F(X)$ is the free topological group on a pointed space $X$, we shall write $F_n(X)$ for $\mathrm{gp}_n(X)$; and if $X$ is a $k_\omega$-space with decomposition $\cup X_n$, we shall write $F_n(X_n)$ for $\mathrm{gp}_n(X_n)$.

With this notation, Theorem 1 of [7] shows that if $X$ is a $k_\omega$-space as above, then $F(X)$ is also a $k_\omega$-space, with decomposition $\cup F_n(X_n)$.

Theorem 3 of [7] gives conditions under which a subgroup of a free topological group is again free. We begin with a slightly sharpened version of this result.

**THEOREM 1.** *Let* $X$ *be a* $k_\omega$*-space with decomposition* $\cup X_n$ *, and let* $Y$ *be a subspace of* $F(X)$ *such that* $Y\setminus\{e\}$ *freely generates* $\mathrm{gp}(Y)$ *. Then the following are equivalent:*

(1) $Y$ *is the union of an increasing sequence of compact sets* $\{Y_n\}$ *, such that for each* $n$ *there exists an* $m$ *for which* $\mathrm{gp}(Y) \cap F_n(X_n) \subseteq \mathrm{gp}_m(Y_m)$ *;*

(2) *there is a decomposition* $\cup Y_n$ *of* $Y$ *as a* $k_\omega$*-space, such that for each* $n$ *there exists an* $m$ *for which* $\mathrm{gp}(Y) \cap F_n(X_n) \subseteq \mathrm{gp}_m(Y_m)$ *;*

(3) $\mathrm{gp}(Y)$ *is* $F(Y)$ *, and* $\mathrm{gp}(Y)$ *and* $Y$ *are closed in* $F(X)$ *.*

Proof. That (2) implies (3) is Theorem 3 of [7], and that (2) implies (1) is trivial.

Suppose (1) holds: we shall show that the given union $Y = \cup Y_n$ is a decomposition of $Y$ as a $k_\omega$-space (*cf.* Theorem 4 of [7]). Now

$$\begin{aligned} Y \cap F_n(X_n) &= Y \cap \mathrm{gp}(Y) \cap F_n(X_n) \\ &\subseteq Y \cap \mathrm{gp}_m(Y_m) \quad (\text{for some } m) \\ &= Y_m . \end{aligned}$$

Therefore $Y \cap F_n(X_n) \subseteq Y_m \cap F_n(X_n)$ , and since $Y_m \subseteq Y$ , we have $Y \cap F_n(X_n) = Y_m \cap F_n(X_n)$ . But both $Y_m$ and $F_n(X_n)$ are compact, and so $Y \cap F_n(X_n)$ is compact. Since this holds for each $n$ , and $F(X) = \cup F_n(X_n)$ is a $k_\omega$-space, $Y$ is closed in $F(X)$ . Hence $Y$ is a $k_\omega$-space with decomposition $\cup\left(Y \cap F_n(X_n)\right)$ . To show that $\cup Y_n$ is also such a decomposition, we need only find for each $n$ an $m$ for which $Y \cap F_n(X_n) \subseteq Y_m$ (see §2 of [7]); and this we have already done. Thus (2) is proved.

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Suppose now that (3) holds. If we set $Y_n = Y \cap F_n(X_n)$ , the fact that $Y$ is closed implies that $\cup Y_n$ is a decomposition of $Y$ as a $k_\omega$-space. By Theorem 1 of [7], $\mathrm{gp}(Y) = F(Y)$ is then a $k_\omega$-space, with decomposition $\cup \mathrm{gp}_m(Y_m)$ . But $\mathrm{gp}(Y)$ is closed in $F(X)$ , and so $\mathrm{gp}(Y) \cap F_n(X_n)$ is compact for each $n$ , and there is an $m$ such that $\mathrm{gp}(Y) \cap F_n(X_n) \subseteq \mathrm{gp}_m(Y_m)$ . Therefore (2) holds, completing the proof.

We now prove the topological Nielsen-Schreier theorem of Brown and Hardy.

THEOREM 2 [2]. *Let $G = F(X)$ be the free topological group on a $k_\omega$-space $X = \cup X_n$ , and let $H$ be a subgroup of $G$ . Suppose that the projection $p$ from $G$ onto $G/H$ (the space of right cosets of $H$ ) has a continuous section $s : G/H \to G$ such that $s(G/H)$ is a Schreier transversal for $H$ in $G$ . Then $H$ is closed and is a free topological group.*

Proof. If we set $B = \{s(Hg)xs(Hgx)^{-1} : g \in G, x \in X\}$ , the usual proof of the Nielsen-Schreier theorem (see [4]) shows that $B\backslash\{e\}$ is algebraically a free basis for $H$ .

Define $\phi : G \times X \to G$ by $\phi : (g, x) \mapsto s(Hg)xs(Hgx)^{-1}$ , for $g \in G$ $x \in X$ . Clearly $B = \phi(G \times X)$ . Now each of the following functions is continuous on $G \times X$ :

$$(g, x) \mapsto g \mapsto p(g) = Hg \mapsto s(Hg) ,$$

$$(g, x) \mapsto x ,$$

$$(g, x) \mapsto gx \mapsto p(gx) = Hgx \mapsto s(Hgx) ;$$

and thus $\phi$ is itself continuous, by the continuity of multiplication and inversion in $G$ . Defining $B_n$ to be $\phi(F_n(X_n) \times X_n)$ we see that each $B_n$ is compact, that $B_1 \subseteq B_2 \subseteq \ldots$ , and that $B = \bigcup_{n=1}^{\infty} B_n$ . According to the previous theorem, to show that $H$ is $F(B)$ and is closed in $G$ we need only find for each $n$ an $m$ for which $H \cap F_n(X_n) \subseteq \mathrm{gp}_m(B_m)$ .

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To this end, let $h \in H \cap F_n(X_n)$ . Then $h$ can be written in reduced form $x_1^{\varepsilon_1}x_2^{\varepsilon_2} \ldots x_k^{\varepsilon_k}$ (that is, $\varepsilon_i = \varepsilon_{i+1}$ whenever $x_i = x_{i+1}$ ) where $k \le n$ and $x_i \in X_n$ for $i = 1, 2, \ldots, k$ . Since $h \in H$ , the proof of the Nielsen-Schreier theorem shows that we can also write
$h = c_1c_2 \ldots c_k$ , where

$$c_i = s\left(Hx_1^{\varepsilon_1}x_2^{\varepsilon_2} \ldots x_{i-1}^{\varepsilon_{i-1}}\right)x_i^{\varepsilon_i}s\left(Hx_1^{\varepsilon_1}x_2^{\varepsilon_2} \ldots x_{i-1}^{\varepsilon_{i-1}}x_i^{\varepsilon_i}\right)^{-1} ,$$

for $i = 1, 2, \ldots, k$ . Now it is clear that for each $i \le k$ , $x_1^{\varepsilon_1}x_2^{\varepsilon_2} \ldots x_{i-1}^{\varepsilon_{i-1}}$ and $x_1^{\varepsilon_1}x_2^{\varepsilon_2} \ldots x_i^{\varepsilon_i} \in F_n(X_n)$ , and $x_i \in X_n$ . Noting that if $\varepsilon_i = 1$ then $c_i = \phi\left(x_1^{\varepsilon_1}x_2^{\varepsilon_2} \ldots x_{i-1}^{\varepsilon_{i-1}}, x_i\right)$ , we see that $c_i \in \phi\left(F_n(X_n) \times X_n\right) = B_n$ . Similarly, if $\varepsilon_i = -1$ , then $c_i^{-1} = \phi\left(x_1^{\varepsilon_1}x_2^{\varepsilon_2} \ldots x_i^{\varepsilon_i}, x_i\right) \in \phi\left(F_n(X_n) \times X_n\right) = B_n$ . Hence $h = c_1c_2 \ldots c_k \in \mathrm{gp}_n(B_n)$ , since $k \le n$ ; that is, $H \cap F_n(X_n) \subseteq \mathrm{gp}_n(B_n)$ for all $n$ . This proves the theorem.

The following example shows that the condition given in Theorem 2 for a subgroup of $F(X)$ to be free is not a necessary condition. Let $X$ be a compact Hausdorff space, and $Y$ a closed subset of $X$ containing $e$ , the base-point of $X$ . By Theorem 1, the subgroup $H = \mathrm{gp}(Y)$ of $F(X)$ is $F(Y)$ . Suppose that there exists a continuous section $s$ of the projection $p$ as in Theorem 2. Let $x \in X \backslash Y$ and suppose that $s(Hx) = x_1^{\varepsilon_1}x_2^{\varepsilon_2} \ldots x_n^{\varepsilon_n}$ in reduced form. Then $x_1^{\varepsilon_1}x_2^{\varepsilon_2} \ldots x_n^{\varepsilon_n}x^{-1} \in H$ . If $x_n^{\varepsilon_n} \ne x$ , then $x_1^{\varepsilon_1}x_2^{\varepsilon_2} \ldots x_n^{\varepsilon_n}x^{-1}$ is reduced (as written) with respect to $X$ , and therefore with respect to $Y$ , and so $x_1, x_2, \ldots, x_n$ , $x \in Y$ , contradicting $x \in X \backslash Y$ . We must therefore have $x_n^{\varepsilon_n} = x$ . But this implies that $x_1^{\varepsilon_1}x_1^{\varepsilon_2} \ldots x_{n-1}^{\varepsilon_{n-1}} \in H$ , and since we already know that

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$x_1^{\varepsilon_1} x_2^{\varepsilon_2} \ldots x_{n-1}^{\varepsilon_{n-1}}$ is in the Schreier transversal for $H$ (because it is an initial segment of $x_1^{\varepsilon_1} x_2^{\varepsilon_2} \ldots x_n^{\varepsilon_n}$ ), we have $x_1^{\varepsilon_1} x_2^{\varepsilon_2} \ldots x_{n-1}^{\varepsilon_{n-1}} = e$ ; that is, $s(Hx) = x_n^{\varepsilon_n} = x$ , for $x \in X \backslash Y$ . For $x \in Y$ , $x \in H$ also, and thus $s(Hx) = e$ . Since $X$ is compact, and $s$ and $p$ are continuous, we find that $s(p(X)) = X \backslash Y \cup \{e\}$ is compact. Taking $X = [0, 1]$ , $Y = [0, \frac{1}{2}]$ , and $e = 0$ , for example, we have $X \backslash Y \cup \{e\} = \{0\} \cup (\frac{1}{2}, 1]$ , which is certainly not compact.

The above paragraph describes a situation in which Theorem 2 does not apply. Under some circumstances, however, a result somewhat stronger can be obtained from the proof of Theorem 2.

**COROLLARY.** *Let* $X = \cup X_n$ *be a* $k_\omega$*-space and let* $H$ *be a subgroup of* $G = F(X)$ . *Suppose that there is a compact set* $Y$ *in* $H$ *such that* $Y \backslash \{e\}$ *is algebraically a free basis for* $H$ , *and that* $Y \backslash \{e\}$ *can be obtained as a free basis from some Schreier transversal for* $H$ *in* $G$ . *Then* $H$ *is* $F(Y)$ *and is closed in* $G$ .

*Proof.* By Theorem 1 we have only to show that for each $n$ there is an $m$ such that $H \cap F_n(X_n) \subseteq \mathrm{gp}_m(Y)$ . This statement is proved exactly as the corresponding statement was proved in Theorem 2, except that the functions $s$ and $\phi$ need no longer be continuous.

If we now return to our example above, the Corollary shows easily that $H = F(Y)$ . A Schreier transversal for $H$ in $F(X)$ certainly exists (it may be constructed in the usual inductive way [4]). If $r(g)$ denotes the representative of an element $g \in F(X)$ , the free basis defined by the Schreier transversal contains the elements $r(e)yr(y)^{-1}$ for each $y \in Y$ . But since $y \in H$ , $r(e) = r(y) = e$ , and so the free basis contains $Y$ : because $Y$ is already a free basis however, the new basis must be precisely $Y$ . Thus the hypotheses of the Corollary are satisfied, and $H = F(Y)$ .

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