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On BMO and Hardy regularity estimates for a class of non-local elliptic equations

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Let $\sigma \in (0,\, 2)$, $\chi^{(\sigma)}(y) := \mathbf{1}_{\sigma\in(1,2)} + \mathbf{1}_{\sigma=1}\mathbf{1}_{y\in B(\mathbf{0},\,1)}$, where $\mathbf{0}$ denotes the origin of $\mathbb{R}^n$, and a be a non-negative and bounded measurable function on $\mathbb{R}^n$. In this paper, we obtain the boundedness of the non-local elliptic operator

$$Lu(x) := \int_{\mathbb{R}^n} \left[u(x+y) - u(x) - \chi^{(\sigma)}(y)y\cdot\nabla u(x)\right] a(y)\,\frac{\mathrm{d}y}{|y|^{n+\sigma}}$$

from the Sobolev space based on $\mathrm{BMO}(\mathbb{R}^n)\cap(\bigcup_{p\in(1,\infty)} L^p(\mathbb{R}^n))$ to the space $\mathrm{BMO}(\mathbb{R}^n)$, and from the Sobolev space based on the Hardy space $H^1(\mathbb{R}^n)$ to $H^1(\mathbb{R}^n)$. Moreover, for any $\lambda \in (0,\, \infty)$, we also obtain the unique solvability of the non-local elliptic equation $Lu - \lambda u = f$ in $\mathbb{R}^n$, with $f \in \mathrm{BMO}(\mathbb{R}^n)\cap(\bigcup_{p\in(1,\infty)} L^p(\mathbb{R}^n))$ or $H^1(\mathbb{R}^n)$, in the Sobolev space based on $\mathrm{BMO}(\mathbb{R}^n)$ or $H^1(\mathbb{R}^n)$. The boundedness and unique solvability results given in this paper are further devolvement for the corresponding results in the scale of the Lebesgue space $L^p(\mathbb{R}^n)$ with $p \in (1,\, \infty)$, established by H. Dong and D. Kim [J. Funct. Anal. 262 (2012), 1166–1199], in the endpoint cases of $p = 1$ and $p = \infty$.

Keywords: Non-local elliptic equation; BMO space; Hardy space; Bessel potential space; solvability

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1. Introduction

Let $n \geqslant 1$, $\sigma \in (0,\, 2)$, $\chi^{(\sigma)}(y) := \mathbf{1}_{\sigma\in(1,2)} + \mathbf{1}_{\sigma=1}\mathbf{1}_{y\in B(\mathbf{0},\,1)}$, where $\mathbf{0}$ denotes the origin of $\mathbb{R}^n$, and a be a non-negative and bounded measurable function on $\mathbb{R}^n$. In this paper, we first consider the boundedness of the non-local elliptic operator

$$Lu(x) := \int_{\mathbb{R}^n} \left[u(x+y) - u(x) - \chi^{(\sigma)}(y)y\cdot\nabla u(x)\right] a(y)\,\frac{\mathrm{d}y}{|y|^{n+\sigma}} \tag{1.1}$$

from the Sobolev space based on $\mathrm{BMO}(\mathbb{R}^n)\cap(\bigcup_{p\in(1,\infty)} L^p(\mathbb{R}^n))$ to the BMO (bounded mean oscillation) space $\mathrm{BMO}(\mathbb{R}^n)$, and from the Sobolev space based on the Hardy space $H^1(\mathbb{R}^n)$ to $H^1(\mathbb{R}^n)$. Assume further that $\lambda \in (0,\, \infty)$, $p \in$

$(1, \infty)$, and f belongs to $L^p(\mathbb{R}^n) \cap \mathrm{BMO}(\mathbb{R}^n)$ or the Hardy space $H^1(\mathbb{R}^n)$, we also investigate the unique solvability of the non-local elliptic equation

$$Lu - \lambda u = f \tag{1.2}$$

in the Sobolev space based on $\mathrm{BMO}(\mathbb{R}^n)$ or $H^1(\mathbb{R}^n)$. The results obtained in this paper are further devolvement of the corresponding results in the scale of the Lebesgue space $L^p(\mathbb{R}^n)$ with $p \in (1, \infty)$ established by Dong and Kim [7] in the endpoint cases of $p = 1$ and $p = \infty$.

In particular, when a is a fixed appropriate constant, the corresponding operator L is just the fractional Laplacian $-(-\Delta)^{\sigma/2}$. It is said that the function u is a solution of the equation (1.2), if (1.2) holds true in the sense of almost everywhere.

Denote by $\mathcal{S}(\mathbb{R}^n)$ the classical Schwartz function space, that is, the set of all *infinitely differentiable functions* satisfying that all derivatives decrease rapidly at infinity, and by $\mathcal{S}'(\mathbb{R}^n)$ its *dual space* (namely, the space of all *tempered distributions*).

Recall that, for any given $\alpha \in (0, \infty)$, the *Bessel potential operator* J_α on $\mathcal{S}'(\mathbb{R}^n)$ is defined by, for any $f \in \mathcal{S}'(\mathbb{R}^n)$ and $\xi \in \mathbb{R}^n$,

$$J_\alpha f(\xi) := \mathcal{F}^{-1}\left(\left(1 + |\cdot|^2\right)^{-\alpha/2} \mathcal{F}(f)\right)(\xi)$$

(see, for instance, [14, definition 1.2.4]). Here and hereafter, $\mathcal{F}$ and $\mathcal{F}^{-1}$, respectively, denote the Fourier transform and the inverse Fourier transform. Moreover, for any given $\alpha \in (0, \infty)$, the *Riesz potential operator* I_α on $\mathcal{S}'(\mathbb{R}^n)$ is defined by, for any $f \in \mathcal{S}'(\mathbb{R}^n)$ and $\xi \in \mathbb{R}^n$,

$$I_\alpha f(\xi) := \mathcal{F}^{-1}\left(|\cdot|^{-\alpha} \mathcal{F}(f)\right)(\xi)$$

(see, for instance, [14, definition 1.2.1]). It is worth pointing out that, when $\alpha \in (0, \infty)$, $|\cdot|^{-\alpha}$ has singularity at the origin. Therefore, I_α can only be defined on the space of tempered distributions modulo polynomials. Moreover, for any $\alpha \in (0, \infty)$ and $u \in \mathcal{S}'(\mathbb{R}^n)$, the fractional derivative of u with order α is defined by

$$\partial^\alpha u := -(-\Delta)^{\alpha/2} u = \mathcal{F}^{-1}\left(|\cdot|^\alpha \mathcal{F}(u)\right).$$

Furthermore, for any given $\alpha \in (0, 2)$ and $u \in \mathcal{S}(\mathbb{R}^n)$, the fractional derivative of u with order α has the equivalent definition

$$\begin{aligned} \partial^\alpha u(x) = -(-\Delta)^{\alpha/2} u(x) &= c\,\mathrm{P.V.} \int_{\mathbb{R}^n} [u(x+y) - u(x)] \frac{\mathrm{d}y}{|y|^{n+\alpha}} \\ &= \frac{c}{2} \int_{\mathbb{R}^n} [u(x+y) + u(x-y) - 2u(x)] \frac{\mathrm{d}y}{|y|^{n+\alpha}}, \end{aligned} \tag{1.3}$$

where

$$c := \frac{\alpha(2-\alpha)\Gamma(\frac{n+\alpha}{2})}{\pi^{n-2} 2^{2-\alpha} \Gamma(2 - \frac{\alpha}{2})},$$

Γ is the Gamma function, and P.V. denotes the integral is taken according to the Cauchy principal value sense. It is worth pointing out that (1.3) is well defined for

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any $u \in C_b^2(\mathbb{R}^n)$ (the set of all 2-*times continuously differentiable bounded functions*) (see, for instance, [**13**]).

For any given $\alpha \in (0, \infty)$ and function space X on $\mathbb{R}^n$, the Sobolev spaces based on X, $J_\alpha(X)$ and $I_\alpha(X)$, are defined by the image of X under J_α and I_α, respectively. Furthermore, for any $u \in J_\alpha(X)$ [or $u \in I_\alpha(X)$], the (quasi-)norm of u is given by $\|u\|_{J_\alpha(X)} := \|J_\alpha(u)\|_X$ [or $\|u\|_{I_\alpha(X)} := \|I_\alpha(u)\|_X$]. By this, we find that, for any function $u \in I_\alpha(X)$, the fractional derivative $\partial^\alpha u \in X$.

Moreover, recall that the *Riesz transform* R_j, for any given $j \in \{1, \ldots, n\}$, is defined by, for any $f \in \mathcal{S}(\mathbb{R}^n)$,

$$R_j f(x) = c_n \lim_{\varepsilon \to 0} \int_{|y| \geqslant \varepsilon} \frac{y_j}{|y|^{n+1}} f(x-y)\, \mathrm{d}y,$$

where $c_n := \Gamma(\frac{n+1}{2})\pi^{-\frac{n+1}{2}}$ (see, for instance, [**26**, **27**]). When $n = 1$, the corresponding operator is known as the *Hilbert transform*.

The classical *Hardy space* $H^1(\mathbb{R}^n)$ is defined to be the set of all $f \in L^1(\mathbb{R}^n)$ such that $R_j f \in L^1(\mathbb{R}^n)$ for any $j \in \{1, \ldots, n\}$, with the *norm*

$$\|f\|_{H^1(\mathbb{R}^n)} = \|f\|_{L^1(\mathbb{R}^n)} + \sum_{j=1}^{n} \|R_j f\|_{L^1(\mathbb{R}^n)} \tag{1.4}$$

(see, for instance, [**27**]). Furthermore, denote by $L^1_{\mathrm{loc}}(\mathbb{R}^n)$ the *set of all locally integrable functions* on $\mathbb{R}^n$. Let $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$. It is said that f belongs to the BMO *(bounded mean oscillation) space* $\mathrm{BMO}(\mathbb{R}^n)$, if

$$\|f\|_{\mathrm{BMO}(\mathbb{R}^n)} := \sup_{B \subset \mathbb{R}^n} \frac{1}{|B|} \int_B |f(x) - f_B|\, \mathrm{d}x < \infty,$$

where the supremum is taken over all balls B of $\mathbb{R}^n$ and $f_B := \frac{1}{|B|}\int_B f(y)\,\mathrm{d}y$ (see, for instance, [**14**, **15**, **27**]). Recall that $\|\cdot\|_{\mathrm{BMO}(\mathbb{R}^n)}$ is only a semi-norm and $\mathrm{BMO}(\mathbb{R}^n)$ modulo constants is a Banach space. To make $\mathrm{BMO}(\mathbb{R}^n)$ itself a Banach space, for $f \in \mathrm{BMO}(\mathbb{R}^n)$, we may consider the norm

$$\|f\|_{\mathrm{BMO}+(\mathbb{R}^n)} := \|f\|_{\mathrm{BMO}(\mathbb{R}^n)} + \left| \frac{1}{|B_1(\mathbf{0})|} \int_{B_1(\mathbf{0})} f(x)\, \mathrm{d}x \right|, \tag{1.5}$$

which is useful to consider the pointwise multipliers of $\mathrm{BMO}(\mathbb{R}^n)$, where $B_1(\mathbf{0})$ denotes the ball with the centre $\mathbf{0}$ and the radius 1. It is known that the Hardy space $H^1(\mathbb{R}^n)$ and the BMO space $\mathrm{BMO}(\mathbb{R}^n)$, respectively, are appropriate substitutes of the Lebesgue spaces $L^1(\mathbb{R}^n)$ and $L^\infty(\mathbb{R}^n)$ when studying the boundedness of some linear operators (see, for instance, [**14**, **27**–**29**]). Moreover, it is well known that the space $\mathrm{BMO}(\mathbb{R}^n)$ is the dual space of the Hardy space $H^1(\mathbb{R}^n)$ (see, for instance, [**14**, **27**]).

Non-local equations have aroused extensive research interest in recent years. The non-local equations of the form (1.2) naturally arise in the study of jump Lévy processes; they have extensive applications in many fields, such as, economics, physics and probability theory (see, for instance, [**3**, **5**, **13**, **24**]), and have been extensively studied (see, for instance, [**3**, **4**, **6**–**11**, **16**, **18**–**20**]).

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The study of the boundedness of the non-local elliptic operator L defined as in (1.1) can be founded in many existing literatures. In particular, if the kernel function a satisfies the lower and upper bounds condition, and also satisfies the cancellation condition when $\sigma = 1$, Dong and Kim [**7**, **8**] obtained the boundedness of the operator L from the Sobolev space $J_\sigma(L^p(\mathbb{R}^n))$ with $p \in (1, \infty)$ to $L^p(\mathbb{R}^n)$, and from the Lipschitz space $\Lambda^{\alpha+\sigma}(\mathbb{R}^n)$ to $\Lambda^\alpha(\mathbb{R}^n)$ for any given $\alpha \in (0, \infty)$ (see, for instance, [**26**, **27**] or § 2 below for the definition of the Lipschitz space). Afterwards, for the non-local operator associated with the x-dependent kernel $a(x, \cdot)$ imposed on the Hölder continuity of x, by using the boundedness of the singular integral of convolution type on Lebesgue spaces $L^p(\mathbb{R}^n)$ and the partition of unity argument, Mikulevičius and Pragarauskas [**20**] obtained the boundedness of the operator L from the Sobolev space $J_\sigma(L^p(\mathbb{R}^n))$ to $L^p(\mathbb{R}^n)$ when $p \in (1, \infty)$ is sufficiently large. Recently, Dong *et al.* [**6**] removed the restriction on p and extended the result established by Mikulevičius and Pragarauskas [**20**] to the weighted Lebesgue spaces $L^p_\omega(\mathbb{R}^n)$ for any $p \in (1, \infty)$ and $\omega \in A_p(\mathbb{R}^n)$ (the Muckenhoupt weight class). Furthermore, when the kernel also depends on the temporal variable, the boundedness of parabolic operators with local or non-local time derivatives was also considered in the existing literatures (see, for instance, [**6**, **9**–**11**, **19**, **20**]).

The research on the solvability and regularity of the solutions of non-local equations is even richer. In particular, for the fraction Laplacian problem $(-\Delta)^s u = f$ in $\mathbb{R}^n$, with $s \in (\frac{1}{2}, 1)$ and $f \in L^1(\mathbb{R}^n)$, Karlsen *et al.* [**16**] proved the unique existence by a dual method, and the solution belonging to the local fractional Sobolev space $W^{1-(2-2s)/q,\, q}_{\mathrm{loc}}(\mathbb{R}^n)$ with $q \in (1, \frac{n+2-2s}{n+1-2s})$. For the fractional Laplacian equation with $L^p(\mathbb{R}^n)$-data, the existence and regularity of the solution can be obtained by the classical theory of pseudo-differential operators. However, for the general kernel a, the theory of pseudo-differential operators is no longer effective. In [**7**], by using the boundedness of the non-local operator L as in (1.1) from $J_\sigma(L^p(\mathbb{R}^n))$ to $L^p(\mathbb{R}^n)$ with $p \in (1, \infty)$, Dong and Kim proved that the solution of the non-local elliptic equation (1.2) with $f \in L^p(\mathbb{R}^n)\,(p \in (1, \infty))$ exists and belongs to the Bessel potential space $J_\sigma(L^p(\mathbb{R}^n))$. Moreover, in [**8**], by using the boundedness of the non-local operator L from the Lipschitz space $\Lambda^{\alpha+\sigma}(\mathbb{R}^n)$ to $\Lambda^\alpha(\mathbb{R}^n)$ with any given $\alpha \in (0, \infty)$ and the method of continuity, Dong and Kim established the unique solvability of the equation (1.2) with any given $f \in \Lambda^\alpha(\mathbb{R}^n)$, and also proved that the corresponding solution belongs to $\Lambda^{\alpha+\sigma}(\mathbb{R}^n)$. In the same paper [**8**], the solvability of the equation (1.2), with the kernel being x-dependent, was also established. Furthermore, the solvability of the non-local parabolic equation, the Dirichlet problem of the non-local equation on domains and the semi-linear non-local equations also have been extensively studied in the existing literatures (see, for instance, [**1**, **6**, **11**, **22**–**25**, **31**]).

Throughout this paper, we always assume that the kernel function a satisfies the following assumption.

ASSUMPTION 1.1. Let $\sigma \in (0, 2)$ and a be a non-negative measurable function on $\mathbb{R}^n$.

(i) There are positive constants μ and Λ such that, for any $y \in \mathbb{R}^n$,

$$(2-\sigma)\mu \leqslant a(y) \leqslant (2-\sigma)\Lambda.$$

(ii) If $\sigma = 1$, then, for any $0 < r < R$,

$$\int_{r \leqslant |y| \leqslant R} ya(y) \frac{\mathrm{d}y}{|y|^{n+1}} = 0.$$

Now, we give the main results of this paper.

THEOREM 1.2. *Let $n \geqslant 1$, $\sigma \in (0, 2)$, $p \in (1, \infty)$, and the kernel function a satisfy assumption 1.1. Then the following two assertions hold true.*

(i) *The operator L defined as in (1.1) is a continuous operator from $J_\sigma(L^p(\mathbb{R}^n)) \cap J_\sigma(\mathrm{BMO}(\mathbb{R}^n))$ to $\mathrm{BMO}(\mathbb{R}^n)$, moreover, there exists a positive constant C, depending only on n, σ, μ and Λ, such that, for any $u \in J_\sigma(L^p(\mathbb{R}^n)) \cap J_\sigma(\mathrm{BMO}(\mathbb{R}^n))$,*

$$\|Lu\|_{\mathrm{BMO}(\mathbb{R}^n)} \leqslant C \|\partial^\sigma u\|_{\mathrm{BMO}+(\mathbb{R}^n)}, \tag{1.6}$$

where, for a function $f \in \mathrm{BMO}(\mathbb{R}^n)$, $\|f\|_{\mathrm{BMO}+(\mathbb{R}^n)}$ is defined as in (1.5).

(ii) *The operator L defined as in (1.1) is a continuous operator from $J_\sigma(H^1(\mathbb{R}^n))$ to $H^1(\mathbb{R}^n)$, moreover, there exists a positive constant C, depending only on n, σ, μ and Λ, such that, for any $u \in J_\sigma(H^1(\mathbb{R}^n))$,*

$$\|Lu\|_{H^1(\mathbb{R}^n)} \leqslant C \|\partial^\sigma u\|_{H^1(\mathbb{R}^n)}. \tag{1.7}$$

REMARK 1.3. In theorem 1.2(i), we need a constraint that $u \in J_\sigma(L^p(\mathbb{R}^n))$ for some $p \in (1, \infty)$ to obtain (1.6). This additional condition is due to our proof method (see (3.20) and (3.21) below for the details). Precisely, let $\lambda \in (0, \infty)$ be a constant and $f := -(-\Delta)^{\sigma/2}u - \lambda u$. To guarantee that there exists a unique solution for the equation $Lw - \lambda w = f$ which is important in the proof of theorem 1.2(i), we need to assume that $f \in L^p(\mathbb{R}^n)$ for some $p \in (1, \infty)$. This leads to the constraint that $u \in J_\sigma(L^p(\mathbb{R}^n))$ for some $p \in (1, \infty)$. Meanwhile, because of the lack of the density of $L^p(\mathbb{R}^n) \cap \mathrm{BMO}(\mathbb{R}^n)$ in $\mathrm{BMO}(\mathbb{R}^n)$, we could not replace the condition $u \in J_\sigma(L^p(\mathbb{R}^n)) \cap J_\sigma(\mathrm{BMO}(\mathbb{R}^n))$ with $u \in J_\sigma(\mathrm{BMO}(\mathbb{R}^n))$ by the method used in the proof of theorem 1.2(i).

Next, we show via a counterexample that, for any given $p \in (1, \infty)$, $L^p(\mathbb{R}^n) \cap \mathrm{BMO}(\mathbb{R}^n)$ is not dense in $\mathrm{BMO}(\mathbb{R}^n)$ with respect to $\|\cdot\|_{\mathrm{BMO}(\mathbb{R}^n)}$. Indeed, let $n = 1$ and $f_0(x) := \sin x$ for any $x \in \mathbb{R}$. Then $f_0 \in L^\infty(\mathbb{R}^n)$ and hence $f_0 \in \mathrm{BMO}(\mathbb{R}^n)$. Let $p \in (1, \infty)$. Now, we prove that, for any $g \in L^p(\mathbb{R}^n) \cap \mathrm{BMO}(\mathbb{R}^n)$, $\|f_0 - g\|_{\mathrm{BMO}(\mathbb{R}^n)} \geqslant \frac{2}{\pi}$, which implies that $L^p(\mathbb{R}^n) \cap \mathrm{BMO}(\mathbb{R}^n)$ is not dense in $\mathrm{BMO}(\mathbb{R}^n)$ with respect to $\|\cdot\|_{\mathrm{BMO}(\mathbb{R}^n)}$. For any $k \in \mathbb{N}$, let $I_k := (k\pi, (k+2)\pi)$. Then, for any $k \in \mathbb{N}$,

$$\begin{aligned}
\|f_0 - g\|_{\mathrm{BMO}(\mathbb{R}^n)} &\geqslant \frac{1}{|I_k|} \int_{I_k} |(f_0 - g) - (f_0 - g)_{I_k}| \,\mathrm{d}x \\
&\geqslant \frac{1}{|I_k|} \int_{I_k} |f_0 - (f_0)_{I_k}| \,\mathrm{d}x - \frac{1}{|I_k|} \int_{I_k} |g - (g)_{I_k}| \,\mathrm{d}x \\
&\geqslant \frac{2}{\pi} - \frac{1}{\pi} \int_{k\pi}^{(k+2)\pi} |g| \,\mathrm{d}x.
\end{aligned} \tag{1.8}$$

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Moreover, by $g \in L^p(\mathbb{R})$, we conclude that $\lim_{k\to\infty} \int_{k\pi}^{(k+2)\pi} |g|\,\mathrm{d}x = 0$. Thus, letting $k \to \infty$ in (1.8), we find that, for any $g \in L^p(\mathbb{R}^n) \cap \mathrm{BMO}(\mathbb{R}^n)$, $\|f_0 - g\|_{\mathrm{BMO}(\mathbb{R}^n)} \geqslant \frac{2}{\pi}$.

THEOREM 1.4. *Let $n \geqslant 1$, $\lambda \in (0, \infty)$, $\sigma \in (0, 2)$, $p \in (1, \infty)$, and the kernel function a satisfy assumption 1.1. Then the following two assertions hold true.*

(i) *For any given $f \in \mathrm{BMO}(\mathbb{R}^n) \cap L^p(\mathbb{R}^n)$, the solution u of the equation (1.2) uniquely exists and, moreover, $u \in J_\sigma(\mathrm{BMO}(\mathbb{R}^n))$ and there exists a positive constant C, depending only on n, σ, μ and Λ, such that*

$$\lambda\|u\|_{\mathrm{BMO}(\mathbb{R}^n)} + \|\partial^\sigma u\|_{\mathrm{BMO}(\mathbb{R}^n)} \leqslant C\|f\|_{\mathrm{BMO}+(\mathbb{R}^n)}, \tag{1.9}$$

where, for any $f \in \mathrm{BMO}(\mathbb{R}^n)$, $\|f\|_{\mathrm{BMO}+(\mathbb{R}^n)}$ is defined as in (1.5).

(ii) *For any given $f \in H^1(\mathbb{R}^n)$, the solution u of the equation (1.2) uniquely exists and, moreover, $u \in J_\sigma(H^1(\mathbb{R}^n))$ and there exists a positive constant C, depending only on n, σ, μ and Λ, such that*

$$\lambda\|u\|_{H^1(\mathbb{R}^n)} + \|\partial^\sigma u\|_{H^1(\mathbb{R}^n)} \leqslant C\|f\|_{H^1(\mathbb{R}^n)}. \tag{1.10}$$

REMARK 1.5. (i) Let $\lambda \in (0, \infty)$, $\sigma \in (0, 2)$, $f \in L^\infty(\mathbb{R}^n) \cap L^p(\mathbb{R}^n)$ with some $p \in (1, \infty)$, and u be the solution of the equation (1.2). By the maximum principle, it was proved in [**8**, theorem 1.1] that $\lambda\|u\|_{L^\infty(\mathbb{R}^n)} \leqslant \|f\|_{L^\infty(\mathbb{R}^n)}$. From this and theorem 1.4, it follows that

$$\lambda\|u\|_{L^\infty(\mathbb{R}^n)} + \|\partial^\sigma u\|_{\mathrm{BMO}(\mathbb{R}^n)} \leqslant C\|f\|_{L^\infty(\mathbb{R}^n)},$$

where C is a positive constant depending only on n, σ, μ and Λ.

(ii) When $\lambda = 0$ in theorem 1.4, we could give a priori estimate for the equation $Lu = f$ in $\mathbb{R}^n$. Indeed, if $u \in \mathrm{BMO}(\mathbb{R}^n) \cap L^p(\mathbb{R}^n)$ is a solution of the equation $Lu = f$ with $f \in \mathrm{BMO}(\mathbb{R}^n) \cap L^p(\mathbb{R}^n)$ for some $p \in (1, \infty)$, then, for any $\lambda \in (0, \infty)$, we have $Lu - \lambda u = f - \lambda u$. Since the constant C in (1.9) is independent of λ, u and f, by taking $\lambda \to 0^+$, it follows that (1.9) holds true with $\lambda = 0$. Similarly, if $u \in H^1(\mathbb{R}^n)$ is a solution of the equation $Lu = f$ with $f \in H^1(\mathbb{R}^n)$, we also obtain that (1.10) holds true with $\lambda = 0$.

(iii) The methods used in this paper to show theorems 1.2 and 1.4 are not effective to deal with the general case that the kernel function a depends on both the variables x and y, considered as [**6**, **11**, **20**]. Indeed, in the proofs of theorems 1.2 and 1.4, we use the exchangeability that $(-\Delta)^{\sigma/2}L = L(-\Delta)^{\sigma/2}$ and $R_jL = LR_j$ which plays a key role in the proofs of theorems 1.2 and 1.4, where the operator L is as in (1.1) and R_j with $j \in \{1, \dots, n\}$ denotes the Riesz transform. However, these exchangeable properties may not hold true for the operator L when the kernel function a depends on both the variables x and y.

The remainder of this paper is organized as follows. In § 2, we recall the notions of the Bessel potential space and the Riesz potential space based on $L^p(\mathbb{R}^n)$, $H^1(\mathbb{R}^n)$ or $\mathrm{BMO}(\mathbb{R}^n)$, and the Lipschitz–Zygmund space. Moreover, we also present the boundedness result of the singular integral operator on the Hardy space $H^1(\mathbb{R}^n)$, and some important results established by Dong and Kim [**7**, **8**]. In § 3, we prove theorems 1.2 and 1.4. To prove theorem 1.2(i), the key step is to establish the mean oscillation estimates. This method was originated in [**17**] and used to treat second-order elliptic and parabolic equations with VMO coefficients. Moreover, in [**7**, **8**, **11**], this method was further developed to treat non-local elliptic and parabolic equations. To show theorem 1.2(ii), we use the boundedness of the singular integral operator on the Hardy space $H^1(\mathbb{R}^n)$, which is motivated by [**6**]. Meanwhile, to prove theorem 1.4, we also use the method of mean oscillation estimates. Moreover, a duality argument is also used.

Finally, we make some conventions on notations. Throughout the whole paper, we always denote by C a positive constant which is independent of the main parameters, but it may vary from line to line. The symbol $f \lesssim g$ means that $f \leqslant Cg$. For any $x \in \mathbb{R}^n$ and $r \in (0, \infty)$, let $B_r(x)$ be a ball with centre x and radius r. In particular, when $x = \mathbf{0}$ (the origin of $\mathbb{R}^n$), we let $B_r := B_r(\mathbf{0})$. We denote by $\mathbb{N}$ the set of all positive integers. Moreover, for an open set $\Omega \subset \mathbb{R}^n$, we denote by $C_{\mathrm{c}}^\infty(\Omega)$ the set of all infinitely differentiable functions with compact supports on Ω and by $C_b^\infty(\Omega)$ the set of all infinitely differentiable functions with bound derivatives on Ω. For a multiindex $\gamma := (\gamma_1, \ldots, \gamma_n)$ with each component γ_i being a nonnegative integer, let $|\gamma| = \gamma_1 + \cdots + \gamma_n$ and, for any $|\gamma|$-th differentiable function u, set $D^\gamma u(x) := \frac{\partial^{|\gamma|} u(x)}{\partial x_1^{\gamma_1} \cdots \partial x_n^{\gamma_n}}$.

2. Preliminaries

In this section, we recall the notions of some function spaces, such as, the Bessel potential space and the Riesz potential space based on $L^p(\mathbb{R}^n)$, $H^1(\mathbb{R}^n)$ or $\mathrm{BMO}(\mathbb{R}^n)$, and the Lipschitz–Zygmund space. Moreover, we also present the boundedness result of the singular integral operator on $H^1(\mathbb{R}^n)$, and some important results established in [**7**, **8**].

When X is one of $L^p(\mathbb{R}^n)$, $H^1(\mathbb{R}^n)$ or $\mathrm{BMO}(\mathbb{R}^n)$, we recall the relations of the Sobolev spaces $J_\alpha(X)$ and $I_\alpha(X)$ as follows (see, for instance, [**26**, **28**, **29**]).

PROPOSITION 2.1. *Let $\alpha \in (0, \infty)$. Then the following properties hold true.*

(i) *For any $p \in (1, \infty)$, $J_\alpha(L^p(\mathbb{R}^n)) = L^p(\mathbb{R}^n) \cap I_\alpha(L^p(\mathbb{R}^n))$.*

(ii) *$J_\alpha(H^1(\mathbb{R}^n)) = H^1(\mathbb{R}^n) \cap I_\alpha(H^1(\mathbb{R}^n))$.*

(iii) *$J_\alpha(\mathrm{BMO}(\mathbb{R}^n)) = \mathrm{BMO}(\mathbb{R}^n) \cap I_\alpha(\mathrm{BMO}(\mathbb{R}^n))$.*

Let $u \in L^\infty(\mathbb{R}^n)$. We recall that the harmonic extension of u to $\mathbb{R}^{n+1}_+ := \mathbb{R}^n \times (0, \infty)$ is defined by the convolution

$$U(\cdot, y) := p(\cdot, y) * u(\cdot)$$

for any $y \in (0, \infty)$, where $p(\cdot, y)$ is the classical Poisson kernel on $\mathbb{R}^{n+1}_+$. Let $\alpha \in (0, \infty)$ and ℓ be the smallest integer greater that α. The *Lipschitz–Zygmund space*

$\Lambda^\alpha(\mathbb{R}^n)$ is defined by

$$\Lambda^\alpha(\mathbb{R}^n) := \left\{ u \in L^\infty(\mathbb{R}^n) : \sup_{y\in(0,\infty)} y^{\ell-\alpha} \left\| D_y^\ell U(\cdot, y) \right\|_{L^\infty(\mathbb{R}^n)} < \infty \right\},$$

where D_y^ℓ denotes the ℓ-th derivative with respect to y, which is equipped with the *norm*

$$\|u\|_{\Lambda^\alpha(\mathbb{R}^n)} := \|u\|_{L^\infty(\mathbb{R}^n)} + \sup_{y\in(0,\infty)} y^{\ell-\alpha} \left\| D_y^\ell U(\cdot, y) \right\|_{L^\infty(\mathbb{R}^n)}.$$

Let $\Omega \subset \mathbb{R}^n$ be an open set, $\alpha \in (0, \infty)$ be a non-integer and ℓ the largest integer smaller than α. Denote by $C^\alpha(\Omega)$ the set of all bounded continuous functions on Ω, with satisfying that

$$\|f\|_{C^\alpha(\Omega)} := \sum_{|\gamma|\leqslant \ell} \|D^\gamma f\|_{L^\infty(\Omega)} + \left[D^\ell f\right]_{C^{\alpha-\ell}(\Omega)} < \infty,$$

where $[\cdot]_{C^{\alpha-\ell}(\Omega)}$ denotes the *Hölder semi-norm*, namely, for a function g on Ω,

$$[g]_{C^{\alpha-\ell}(\Omega)} := \sup_{\substack{x,\, y\in\Omega \\ x\neq y}} \left\{ \frac{|g(x) - g(y)|}{|x-y|^{\alpha-\ell}} \right\}.$$

Then we have the following properties of the Lipschitz–Zygmund space (see, for instance, [**26**, chapter V] and [**27**, chapter VI]).

PROPOSITION 2.2. *Let $\alpha \in (0, \infty)$ and $\Lambda^\alpha(\mathbb{R}^n)$ be the Lipschitz–Zygmund space on $\mathbb{R}^n$.*

(i) *For any $0 < \alpha_1 < \alpha_2 < \infty$, $\Lambda^{\alpha_2}(\mathbb{R}^n) \subsetneq \Lambda^{\alpha_1}(\mathbb{R}^n)$.*

(ii) *If α is a non-integer, then $\Lambda^\alpha(\mathbb{R}^n) = C^\alpha(\mathbb{R}^n)$.*

(iii) *If $\alpha \in (0, 2)$, then*

$$\|u\|_{\Lambda^\alpha(\mathbb{R}^n)} = \|u\|_{L^\infty(\mathbb{R}^n)} + \sup_{|h|>0} |h|^{-\alpha} \|u(\cdot+h) + u(\cdot-h) - 2u(\cdot)\|_{L^\infty(\mathbb{R}^n)}.$$

The following is the known result of the boundless of the singular integral operator on $H^1(\mathbb{R}^n)$ (see, for instance, [**27**, chapter III, theorem 3]).

LEMMA 2.3. *Let T be a singular integral operator on $\mathbb{R}^n$. Assume that there exists a kernel function K such that, for any $f \in L^2(\mathbb{R}^n)$ with compact support,*

$$Tf(x) = \int_{\mathbb{R}^n} K(x-y) f(y)\, \mathrm{d}y$$

holds true for any x outside the support of f. Assume further that there exists a positive constant A such that, for any $y \neq 0$,

$$\int_{|x|\geqslant 2|y|} |K(x-y) - K(x)|\, \mathrm{d}x \leqslant A,$$

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and, for any $f \in L^2(\mathbb{R}^n)$,

$$\|Tf\|_{L^2(\mathbb{R}^n)} \leqslant A\|f\|_{L^2(\mathbb{R}^n)}.$$

Then there exists a positive constant C *depending only on the constant* A *such that, for any* $f \in H^1(\mathbb{R}^n)$,

$$\|Tf\|_{L^1(\mathbb{R}^n)} \leqslant C\|f\|_{H^1(\mathbb{R}^n)}.$$

The following conclusions were established in [**7**, **8**].

Lemma 2.4 [**7**, theorem 2.1]. *Let* $p \in (1, \infty)$, $\lambda \in [0, \infty)$, $\sigma \in (0, 2)$, *and the kernel function* a *satisfy assumption* 1.1. *Then the operator* L *defined as in* (1.1) *is a continuous operator from* $J_\sigma(L^p(\mathbb{R}^n))$ *to* $L^p(\mathbb{R}^n)$ *and there exists a positive constant* C, *depending only on* n, p, σ, μ *and* Λ, *such that*

$$\lambda\|u\|_{L^p(\mathbb{R}^n)} + \|Lu\|_{L^p(\mathbb{R}^n)} \leqslant C\left\|-(-\Delta)^{\sigma/2}u - \lambda u\right\|_{L^p(\mathbb{R}^n)}.$$

Moreover, for any $\lambda \in (0, \infty)$ *and* $f \in L^p(\mathbb{R}^n)$, *there exists a unique solution* $u \in J_\sigma(L^p(\mathbb{R}^n))$ *for the equation* (1.2), *and there exists a positive constant* C, *depending only on* n, p, σ, μ *and* Λ, *such that*

$$\lambda\|u\|_{L^p(\mathbb{R}^n)} + \|\partial^\sigma u\|_{L^p(\mathbb{R}^n)} \leqslant C\|f\|_{L^p(\mathbb{R}^n)}.$$

Lemma 2.5 [**8**, theorem 1.3]. *Let* $\alpha \in (0, \infty)$, $\lambda \in (0, \infty)$, $\sigma \in (0, 2)$, *and the kernel function* a *satisfy assumption* 1.1. *Then the operator* $L - \lambda$ *is a continuous operator from* $\Lambda^{\alpha+\sigma}(\mathbb{R}^n)$ *to* $\Lambda^\alpha(\mathbb{R}^n)$, *where* L *is as in (1.1). Moreover, for any* $f \in \Lambda^\alpha(\mathbb{R}^n)$, *there exists a unique solution* $u \in \Lambda^{\alpha+\sigma}(\mathbb{R}^n)$ *for the equation (1.2), and there exists a positive constant* C, *depending only on* n, σ, μ, Λ, λ *and* α, *such that*

$$\|u\|_{\Lambda^{\alpha+\sigma}(\mathbb{R}^n)} \leqslant C\|Lu - \lambda u\|_{\Lambda^\alpha(\mathbb{R}^n)}.$$

3. Proofs of theorems 1.2 and 1.4

In this section, we prove theorems 1.2 and 1.4. Assume that $\sigma \in (0, 2)$. Throughout this paper, we always assume that $\omega(x) := \frac{1}{1+|x|^{n+\sigma}}$ for any $x \in \mathbb{R}^n$ and

$$L^1(\mathbb{R}^n, \omega) := \left\{g \in L^1_{\mathrm{loc}}(\mathbb{R}^n) : \|g\|_{L^1(\mathbb{R}^n,\omega)} := \int_{\mathbb{R}^n} \frac{|g(y)|}{1+|y|^{n+\sigma}}\,\mathrm{d}y < \infty\right\}.$$

Moreover, for an open set $\Omega \subset \mathbb{R}^n$, it is said that a function $f \in C^2_{\mathrm{loc}}(\Omega)$, if, for any $\phi \in C^\infty_{\mathrm{c}}(\Omega)$, $\phi f \in C^2_{\mathrm{c}}(\Omega)$ (the set of all 2-th continuous differentiable functions with compact supports).

We first recall the following property of the space $\mathrm{BMO}(\mathbb{R}^n)$ (see, for instance, [**14**, proposition 3.1.5]).

Proposition 3.1. *Let* $f \in \mathrm{BMO}(\mathbb{R}^n)$. *Then, for any* $\delta \in (0, \infty)$, *there exists a positive constant* C, *depending only on* n *and* δ, *such that, for any* $x_0 \in \mathbb{R}^n$ *and* $R \in (0, \infty)$,

$$R^\delta \int_{\mathbb{R}^n} \frac{|f(x) - (f)_{B_R(x_0)}|}{R^{n+\delta} + |x - x_0|^{n+\delta}}\,\mathrm{d}x \leqslant C\|f\|_{\mathrm{BMO}(\mathbb{R}^n)}.$$

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Now, we need the following lemma 3.2, which was established in [7, corollary 4.3].

Lemma 3.2. *Let* $\lambda \in [0, \infty)$, $\sigma \in (0, 2)$, $f \in L^{\infty}(B_1)$, *and* $u \in C^2_{\text{loc}}(B_1) \cap L^1(\mathbb{R}^n, \omega)$ *be a solution of*

$$Lu - \lambda u = f$$

in B_1, *where the operator* L *is as in (1.1) and the kernel function* a *satisfies assumption 1.1. Then, for any* $\alpha \in (0, \min\{1, \sigma\})$, *there exists a positive constant* C, *depending only* n, σ, μ, Λ, *and* α, *such that*

$$[u]_{C^{\alpha}(B_{1/2})} \leqslant C\left[\|u\|_{L^1(\mathbb{R}^n,\omega)} + \operatorname{osc}_{B_1} f\right],$$

where $\operatorname{osc}_{B_1} f := \sup_{x,\, y \in B_1} |f(x) - f(y)|$.

Moreover, as a corollary of lemma 3.2, we have the following lemma 3.3, which was obtained in [8, proposition 1].

Lemma 3.3. *Let* $\lambda \in [0, \infty)$, $\sigma \in (0, 2)$, $f \in L^{\infty}(B_1)$ *and* $u \in C^2_{\text{loc}}(B_1) \cap L^{\infty}(\mathbb{R}^n)$ *be a solution of*

$$Lu - \lambda u = f \tag{3.1}$$

in B_1, *where* L *is as in lemma 3.2. Then, for any* $\alpha \in (0, \min\{1, \sigma\})$, *there exists a positive constant* C, *depending only* n, σ, μ, Λ *and* α, *such that*

$$[u]_{C^{\alpha}(B_{1/2})} \leqslant C\left[\|u - (u)_{B_1}\|_{L^1(\mathbb{R}^n,\omega)} + \operatorname{osc}_{B_1} f\right], \tag{3.2}$$

where $(u)_{B_1} := \frac{1}{|B_1|}\int_{B_1} u(x)\,\mathrm{d}x$.

Lemma 3.4. *Let* $\lambda \in [0, \infty)$, $\sigma \in (0, 2)$, $f \in C^{\infty}_{\text{loc}}(\mathbb{R}^n) \cap \mathrm{BMO}(\mathbb{R}^n)$ *satisfy that* $f = 0$ *in* B_2, *and* $u \in J_{\sigma}(L^2(\mathbb{R}^n)) \cap C^{\infty}_b(\mathbb{R}^n)$ *be a solution of*

$$Lu - \lambda u = f \tag{3.3}$$

in $\mathbb{R}^n$, *where* L *is as in lemma 3.2. Then, for any* $\alpha \in (0, \min\{1, \sigma\})$, *there exists a positive constant* C, *depending only on* n, σ, μ, Λ *and* α, *such that*

$$[u]_{C^{\alpha}(B_{1/2})} \leqslant C\,\|u - (u)_{B_1}\|_{L^1(\mathbb{R}^n,\omega)} \tag{3.4}$$

and

$$\left[(-\Delta)^{\sigma/2}u\right]_{C^{\alpha}(B_{1/2})} \leqslant C\left[\left\|(-\Delta)^{\sigma/2}u - \left((-\Delta)^{\sigma/2}u\right)_{B_1}\right\|_{L^1(\mathbb{R}^n,\omega)} + \|f\|_{\mathrm{BMO}(\mathbb{R}^n)}\right]. \tag{3.5}$$

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Proof. By lemma 3.3 and the assumption that $f = 0$ in B_2, we find that (3.4) holds true. Now, we show (3.5). Applying $(-\Delta)^{\sigma/2}$ to both sides of (3.3), we conclude that

$$L(-\Delta)^{\sigma/2}u - \lambda(-\Delta)^{\sigma/2}u = (-\Delta)^{\sigma/2}f.$$

For any $x \in B_1$, we have $f(x) = 0$ and, if $y \in B_{1/2}$, then $f(x+y) = 0$. By this, proposition 3.1, and the fact that, for any $x \in B_1$, $(f)_{B_{1/2}(x)} = 0$, we find that, for any $x \in B_1$,

$$\begin{aligned}\left|(-\Delta)^{\sigma/2}f(x)\right| &= c\left|\lim_{\varepsilon\to 0^+}\int_{|y|\geqslant\varepsilon} f(x+y) - f(x)\frac{\mathrm{d}y}{|y|^{n+\sigma}}\right| \\ &= c\left|\int_{|y-x|\geqslant 1/2}\frac{f(y)}{|y-x|^{n+\sigma}}\mathrm{d}y\right| \\ &\lesssim \int_{|y-x|\geqslant 1/2}\frac{\left|f(y)-(f)_{B_{1/2}(x)}\right|}{(1/2)^{n+\sigma}+|y-x|^{n+\sigma}}\mathrm{d}y \lesssim \|f\|_{\mathrm{BMO}(\mathbb{R}^n)},\end{aligned}$$

which, combined with the fact that $\mathrm{osc}_{B_1}(-\Delta)^{\sigma/2}f \leqslant 2\|(-\Delta)^{\sigma/2}f\|_{L^\infty(B_1)}$ and lemma 3.3, further implies that (3.5) holds true. This finishes the proof of lemma 3.4. □

Then, by lemma 3.4 and a scaling and shifting the coordinates argument, we obtain the following lemma.

LEMMA 3.5. *Let $\lambda \in [0, \infty)$, $\sigma \in (0, 2)$, $k \in [2, \infty)$, $f \in C^\infty_{\mathrm{loc}}(\mathbb{R}^n) \cap \mathrm{BMO}(\mathbb{R}^n)$ satisfy that $f = 0$ in $B_{2kr}(x_0)$ for some $x_0 \in \mathbb{R}^n$ and $r \in (0, \infty)$, and $u \in J_\sigma(L^2(\mathbb{R}^n)) \cap C^\infty_b(\mathbb{R}^n)$ be a solution of*

$$Lu - \lambda u = f$$

in $\mathbb{R}^n$, where L is as in lemma 3.2. Then, for any $\alpha \in (0, \min\{1, \sigma\})$, there exists a positive constant C, depending only on n, σ, μ, Λ and α, such that

$$\left(|u-(u)_{B_r(x_0)}|\right)_{B_r(x_0)} \leqslant Ck^{-\alpha}(kr)^\sigma \int_{\mathbb{R}^n}\frac{|u(x)-(u)_{B_{kr}(x_0)}|}{(kr)^{n+\sigma}+|x-x_0|^{n+\sigma}}\,\mathrm{d}x \tag{3.6}$$

and

$$\begin{aligned}&\left(\left|(-\Delta)^{\sigma/2}u - \left((-\Delta)^{\sigma/2}u\right)_{B_r(x_0)}\right|\right)_{B_r(x_0)} \\ &\quad\leqslant C\left[k^{-\alpha}(kr)^\sigma\int_{\mathbb{R}^n}\frac{\left|(-\Delta)^{\sigma/2}u(x)-\left((-\Delta)^{\sigma/2}u\right)_{B_{kr}(x_0)}\right|}{(kr)^{n+\sigma}+|x-x_0|^{n+\sigma}}\,\mathrm{d}x + k^{-\alpha}\|f\|_{\mathrm{BMO}(\mathbb{R}^n)}\right].\end{aligned} \tag{3.7}$$

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Proof. Let $R := kr$, $U(x) := u(Rx + x_0)$, and $F(x) := R^\sigma f(Rx + x_0)$. Then, we conclude that U satisfies the equation

$$L_1 U(x) - R^\sigma \lambda U(x) = F(x)$$

in $\mathbb{R}^n$, where $F(x) = 0$ in B_2 and L_1 is the nonlocal operator with the coefficient $a_1(\cdot) = a(R\cdot)$. Moreover, it is easy to find that a_1 also satisfies assumption 1.1.

Therefore, from lemma 3.4 and a change of variables, it follows that

$$[u]_{C^\alpha(B_{kr/2}(x_0))} = (kr)^{-\alpha}[U]_{C^\alpha(B_{1/2})} \lesssim (kr)^{\sigma-\alpha} \int_{\mathbb{R}^n} \frac{|u(x) - (u)_{B_{kr}(x_0)}|}{(kr)^{n+\sigma} + |x - x_0|^{n+\sigma}}\,\mathrm{d}x \tag{3.8}$$

and

$$\begin{aligned}\left[(-\Delta)^{\sigma/2} u\right]_{C^\alpha B_{kr/2}(x_0)} &= (kr)^{-(\sigma+\alpha)} \left[(-\Delta)^{\sigma/2} U\right]_{C^\alpha(B_{1/2})} \\ &\lesssim (kr)^{\sigma-\alpha} \int_{\mathbb{R}^n} \frac{\left|(-\Delta)^{\sigma/2} u(x) - \left((-\Delta)^{\sigma/2} u\right)_{B_{kr}(x_0)}\right|}{(kr)^{n+\sigma} + |x - x_0|^{n+\sigma}}\,\mathrm{d}x \\ &\quad + (kr)^{-\alpha} \|f\|_{\mathrm{BMO}(\mathbb{R}^n)}.\end{aligned} \tag{3.9}$$

In addition, for any $k \in [2, \infty)$ and any function $g \in C^\alpha(B_{kr/2}(x_0))$, we have

$$\begin{aligned}\left(|g - (g)_{B_r(x_0)}|\right)_{B_r(x_0)} &= \frac{1}{|B_r(x_0)|} \int_{B_r(x_0)} \left| g(y) - \frac{1}{|B_r(x_0)|} \int_{B_r(x_0)} g(x)\,\mathrm{d}x \right| \mathrm{d}y \\ &\leqslant \frac{1}{|B_r(x_0)|} \frac{1}{|B_r(x_0)|} \int_{B_r(x_0)} \int_{B_r(x_0)} |g(x) - g(y)|\,\mathrm{d}x\mathrm{d}y \\ &\lesssim [g]_{C^\alpha(B_{kr/2}(x_0))} r^\alpha,\end{aligned}$$

which, together with (3.8) and (3.9), further implies that (3.6) and (3.7) hold true. This finishes the proof of lemma 3.5. □

LEMMA 3.6. *Let $\sigma \in (0, 2)$, $\lambda \in (0, \infty)$, $k \in [2, \infty)$, $f \in C^\infty_{\mathrm{loc}}(\mathbb{R}^n) \cap \mathrm{BMO}(\mathbb{R}^n)$ and $u \in J_\sigma(L^2(\mathbb{R}^n)) \cap J_\sigma(\mathrm{BMO}(\mathbb{R}^n)) \cap C^\infty_b(\mathbb{R}^n)$ be a solution of*

$$Lu - \lambda u = f$$

in $\mathbb{R}^n$, where L is as in lemma 3.2. Then, for any $\alpha \in (0, \min\{1, \sigma\})$, $x_0 \in \mathbb{R}^n$, and $r \in (0, \infty)$, there exists a positive constant C, depending only n, σ, μ, Λ and α, such that

$$\begin{aligned}&\lambda \left(|u - (u)_{B_r(x_0)}|\right)_{B_r(x_0)} + \left(\left| (-\Delta)^{\sigma/2} u - \left((-\Delta)^{\sigma/2} u\right)_{B_r(x_0)} \right| \right)_{B_r(x_0)} \\ &\quad \leqslant C \left\{ k^{-\alpha} \left[\lambda \|u\|_{\mathrm{BMO}(\mathbb{R}^n)} + \left\| (-\Delta)^{\sigma/2} u \right\|_{\mathrm{BMO}(\mathbb{R}^n)} \right] + k^{n/2} \|f\|_{\mathrm{BMO}(\mathbb{R}^n)} \right\}.\end{aligned}$$

Proof. Let $x_0 \in \mathbb{R}^n$ and $r \in (0, \infty)$. Take $\eta \in C^\infty_{\mathrm{c}}(\mathbb{R}^n)$ such that $\eta \equiv 1$ on $B_{2kr}(x_0)$, $0 \leqslant \eta \leqslant 1$, and $\operatorname{supp}(\eta) \subset B_{4kr}(x_0)$. Then, we have $\eta[f - (f)_{B_{4kr}(x_0)}] \in$

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$C_c^\infty(B_{4kr}(x_0))$. By this, we find that $\eta[f-(f)_{B_{4kr}(x_0)}]\in L^p(\mathbb{R}^n)\cap C^s(\mathbb{R}^n)$ for any $p\in(1,\infty)$ and $s\in(0,1)$. From lemmas 2.4 and 2.5, we deduce that there exists a unique solution $w\in J_\sigma(\cap_{p\in(1,\infty)}L^p(\mathbb{R}^n))\cap\Lambda^{\sigma+s}(\mathbb{R}^n)$ for the equation (1.2) with f replaced by $\eta[f-(f)_{B_{4kr}(x_0)}]$, and, for any $p\in(1,\infty)$, w satisfies that

$$\lambda\|w\|_{L^p(\mathbb{R}^n)}+\left\|(-\Delta)^{\sigma/2}w\right\|_{L^p(\mathbb{R}^n)}\leqslant C\left\|\eta\left[f-(f)_{B_{4kr}(x_0)}\right]\right\|_{L^p(\mathbb{R}^n)}, \tag{3.10}$$

where C is a positive constant independent of λ, η, f and w. Furthermore, by proposition 2.2(iii) and taking $s\in(0,1)$ small enough such that $\sigma+s\in(0,2)$, we conclude that $w\in L^\infty(\mathbb{R}^n)$ and, for any $x\in\mathbb{R}^n$,

$$\begin{aligned}
|\partial^\sigma w(x)|&=\left|\int_{\mathbb{R}^n}\frac{w(x+y)+w(x-y)-2w(x)}{|y|^{n+\sigma}}\,\mathrm{d}y\right|\\
&\leqslant\int_{|y|\leqslant1}\frac{|w(x+y)+w(x-y)-2w(x)|}{|y|^{n+\sigma}}\,\mathrm{d}y\\
&\quad+\int_{|y|>1}\frac{|w(x+y)+w(x-y)-2w(x)|}{|y|^{n+\sigma}}\,\mathrm{d}y\\
&\lesssim\|w\|_{\Lambda^{\sigma+s}(\mathbb{R}^n)}\int_{|y|\leqslant1}\frac{1}{|y|^{n-s}}\,\mathrm{d}y+\|w\|_{L^\infty(\mathbb{R}^n)}\int_{|y|>1}\frac{1}{|y|^{n+\sigma}}\,\mathrm{d}y\\
&\lesssim\|w\|_{\Lambda^{\sigma+s}(\mathbb{R}^n)}.
\end{aligned}$$

Thus, $w\in J_\sigma(\mathrm{BMO}(\mathbb{R}^n))$. In addition, from the classical theory of the Fourier transform (see, for instance, [**2**, remark 2.2]), it follows that $w\in C_b^\infty(\mathbb{R}^n)$.

Let $v:=u-w$. Then, we have $v\in J_\sigma(\mathrm{BMO}(\mathbb{R}^n))\cap J_\sigma(L^2(\mathbb{R}^n))\cap C_b^\infty(\mathbb{R}^n)$ and

$$Lv-\lambda v=(1-\eta)\left[f-(f)_{B_{4kr}(x_0)}\right]+(f)_{B_{4kr}(x_0)}. \tag{3.11}$$

By the fact that $(1-\eta)[f-(f)_{B_{4kr}(x_0)}]+(f)_{B_{4kr}(x_0)}$ is a constant in $B_{2kr}(x_0)$, similarly to the proof of lemma 3.5, we find that

$$\left(\left|v-(v)_{B_r(x_0)}\right|\right)_{B_r(x_0)}\lesssim k^{-\alpha}(kr)^\sigma\int_{\mathbb{R}^n}\frac{|v(x)-(v)_{B_{kr}(x_0)}|}{(kr)^{n+\sigma}+|x-x_0|^{n+\sigma}}\,\mathrm{d}x. \tag{3.12}$$

Applying $(-\Delta)^{\sigma/2}$ to both sides of (3.11), we conclude that

$$L(-\Delta)^{\sigma/2}u-\lambda(-\Delta)^{\sigma/2}u=(-\Delta)^{\sigma/2}\left((1-\eta)\left[f-(f)_{B_{4kr}(x_0)}\right]\right).$$

For any $x\in B_{kr}(x_0)$, we have $(1-\eta)(x)=0$ and, if $y\in B_{kr/2}(x)$, then $(1-\eta)(x+y)=0$. By this, proposition 3.1, and the fact that, for any $x\in B_{kr}(x_0)$ and

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$y \notin B_{kr/2}(x)$, $|y-x| \gtrsim |y-x_0|$, we find that, for any $x \in B_{kr}(x_0)$,

$$
\begin{aligned}
&\left|(-\Delta)^{\sigma/2}\left((1-\eta)\left[f-(f)_{B_{4kr}(x_0)}\right]\right)(x)\right| \\
&\quad = c\left|\lim_{\varepsilon\to 0^+}\int_{|y|\geqslant\varepsilon}(1-\eta)\left[f-(f)_{B_{4kr}(x_0)}\right](x+y)\frac{\mathrm{d}y}{|y|^{n+\sigma}}\right| \\
&\quad \lesssim \int_{|y-x|\geqslant kr/2}\frac{\left|f(y)-(f)_{B_{4kr}(x_0)}\right|}{|y-x|^{n+\sigma}}\mathrm{d}y \\
&\quad \lesssim \int_{|y-x|\geqslant kr/2}\frac{\left|f(y)-(f)_{B_{4kr}(x_0)}\right|}{(4kr)^{n+\sigma}+|y-x|^{n+\sigma}}\mathrm{d}y \\
&\quad \lesssim \int_{|y-x|\geqslant kr/2}\frac{\left|f(y)-(f)_{B_{4kr}(x_0)}\right|}{(4kr)^{n+\sigma}+|y-x_0|^{n+\sigma}}\mathrm{d}y \\
&\quad \lesssim (kr)^{-\sigma}\|f\|_{\mathrm{BMO}(\mathbb{R}^n)}.
\end{aligned}
$$

This, together with lemma 3.3 and the scaling and shifting the coordinates argument as in lemma 3.5, implies that

$$
\begin{aligned}
&\left(\left|(-\Delta)^{\sigma/2}v-\left((-\Delta)^{\sigma/2}v\right)_{B_r(x_0)}\right|\right)_{B_r(x_0)} \\
&\quad \lesssim k^{-\alpha}(kr)^{\sigma}\int_{\mathbb{R}^n}\frac{\left|(-\Delta)^{\sigma/2}v(x)-\left((-\Delta)^{\sigma/2}v\right)_{B_{kr}(x_0)}\right|}{(kr)^{n+\sigma}+|x-x_0|^{n+\sigma}}\,\mathrm{d}x+k^{-\alpha}\|f\|_{\mathrm{BMO}(\mathbb{R}^n)}.
\end{aligned}
\tag{3.13}
$$

From (3.13), we deduce that

$$
\begin{aligned}
&\left(\left|(-\Delta)^{\sigma/2}u-\left((-\Delta)^{\sigma/2}u\right)_{B_r(x_0)}\right|\right)_{B_r(x_0)} \\
&\quad \leqslant \left(\left|(-\Delta)^{\sigma/2}v-\left((-\Delta)^{\sigma/2}v\right)_{B_r(x_0)}\right|\right)_{B_r(x_0)}+2\left(\left|(-\Delta)^{\sigma/2}w\right|\right)_{B_r(x_0)} \\
&\quad \lesssim k^{-\alpha}(kr)^{\sigma}\int_{\mathbb{R}^n}\frac{\left|(-\Delta)^{\sigma/2}v(x)-\left((-\Delta)^{\sigma/2}v\right)_{B_{kr}(x_0)}\right|}{(kr)^{n+\sigma}+|x-x_0|^{n+\sigma}}\,\mathrm{d}x \\
&\qquad +\left(\left|(-\Delta)^{\sigma/2}w\right|\right)_{B_r(x_0)}+k^{-\alpha}\|f\|_{\mathrm{BMO}(\mathbb{R}^n)} \\
&\quad \lesssim k^{-\alpha}(kr)^{\sigma}\int_{\mathbb{R}^n}\frac{\left|(-\Delta)^{\sigma/2}u(x)-\left((-\Delta)^{\sigma/2}u\right)_{B_{kr}(x_0)}\right|}{(kr)^{n+\sigma}+|x-x_0|^{n+\sigma}}\,\mathrm{d}x \\
&\qquad +k^{-\alpha}(kr)^{\sigma}\int_{\mathbb{R}^n}\frac{\left|(-\Delta)^{\sigma/2}w(x)-\left((-\Delta)^{\sigma/2}w\right)_{B_{kr}(x_0)}\right|}{(kr)^{n+\sigma}+|x-x_0|^{n+\sigma}}\,\mathrm{d}x \\
&\qquad +\left(\left|(-\Delta)^{\sigma/2}w\right|\right)_{B_r(x_0)}+k^{-\alpha}\|f\|_{\mathrm{BMO}(\mathbb{R}^n)}.
\end{aligned}
\tag{3.14}
$$

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Moreover, by (3.10) and the equivalent characterization of $\|f\|_{\mathrm{BMO}(\mathbb{R}^n)}$ (see, for instance, [**14**, corollary 3.1.9]), we conclude that, for any $p \in (1, \infty)$ and $R \in [r, \infty)$,

$$
\begin{aligned}
\left(\left|(-\Delta)^{\sigma/2} w\right|\right)_{B_R(x_0)} &\leqslant \left(\frac{1}{|B_R(x_0)|}\int_{B_R(x_0)} \left|(-\Delta)^{\sigma/2} w\right|^p \,\mathrm{d}x\right)^{1/p} \\
&\leqslant \frac{1}{|B_R(x_0)|^{1/p}} \left(\int_{\mathbb{R}^n} \left|\eta\left[f - (f)_{B_{4kr}(x_0)}\right]\right|^p \,\mathrm{d}x\right)^{1/p} \\
&\lesssim \left(\frac{kr}{R}\right)^{n/p} \|f\|_{\mathrm{BMO}(\mathbb{R}^n)}.
\end{aligned} \tag{3.15}
$$

Furthermore, take $p \in (1, \infty)$ small enough such that $n/p + \sigma > 1$. Then, from (3.15), it follows that

$$
\begin{aligned}
&(kr)^\sigma \int_{\mathbb{R}^n} \frac{\left|(-\Delta)^{\sigma/2} w(x) - \left((-\Delta)^{\sigma/2} w\right)_{B_{kr}(x_0)}\right|}{(kr)^{n+\sigma} + |x - x_0|^{n+\sigma}} \,\mathrm{d}x \\
&\leqslant (kr)^\sigma \int_{\mathbb{R}^n} \frac{\left|(-\Delta)^{\sigma/2} w(x)\right|}{(kr)^{n+\sigma} + |x - x_0|^{n+\sigma}} \,\mathrm{d}x + (kr)^\sigma \int_{\mathbb{R}^n} \frac{\left|\left((-\Delta)^{\sigma/2} w\right)_{B_{kr}(x_0)}\right|}{(kr)^{n+\sigma} + |x - x_0|^{n+\sigma}} \,\mathrm{d}x \\
&\lesssim (kr)^\sigma \sum_{j=0}^{\infty} \int_{jkr \leqslant |x-x_0| < (j+1)kr} \frac{\left|(-\Delta)^{\sigma/2} w(x)\right|}{(kr)^{n+\sigma} + |x - x_0|^{n+\sigma}} \,\mathrm{d}x + \|f\|_{\mathrm{BMO}(\mathbb{R}^n)} \\
&\lesssim (kr)^\sigma \sum_{j=0}^{\infty} \frac{1}{(j^{n+\sigma} + 1)(kr)^{n+\sigma}} \int_{|x-x_0| < (j+1)kr} \left|(-\Delta)^{\sigma/2} w(x)\right| \,\mathrm{d}x \\
&\quad + \|f\|_{\mathrm{BMO}(\mathbb{R}^n)} \\
&\lesssim \sum_{j=0}^{\infty} \frac{(j+1)^{n-n/p}}{j^{n+\sigma} + 1} \|f\|_{\mathrm{BMO}(\mathbb{R}^n)} + \|f\|_{\mathrm{BMO}(\mathbb{R}^n)} \lesssim \|f\|_{\mathrm{BMO}(\mathbb{R}^n)},
\end{aligned} \tag{3.16}
$$

which, together with (3.14), (3.15) and proposition 3.1, further implies that

$$
\begin{aligned}
&\left(\left|(-\Delta)^{\sigma/2} u - \left((-\Delta)^{\sigma/2} u\right)_{B_r(x_0)}\right|\right)_{B_r(x_0)} \\
&\lesssim k^{-\alpha} (kr)^\sigma \int_{\mathbb{R}^n} \frac{\left|(-\Delta)^{\sigma/2} u(x) - \left((-\Delta)^{\sigma/2} u\right)_{B_{kr}(x_0)}\right|}{(kr)^{n+\sigma} + |x - x_0|^{n+\sigma}} \,\mathrm{d}x \\
&+ k^{n/p} \|f\|_{\mathrm{BMO}(\mathbb{R}^n)} + k^{-\alpha} \|f\|_{\mathrm{BMO}(\mathbb{R}^n)} \\
&\lesssim k^{-\alpha} \left\|(-\Delta)^{\sigma/2} u\right\|_{\mathrm{BMO}(\mathbb{R}^n)} + k^{n/2} \|f\|_{\mathrm{BMO}(\mathbb{R}^n)} + k^{-\alpha} \|f\|_{\mathrm{BMO}(\mathbb{R}^n)} \\
&\lesssim k^{-\alpha} \left\|(-\Delta)^{\sigma/2} u\right\|_{\mathrm{BMO}(\mathbb{R}^n)} + k^{n/2} \|f\|_{\mathrm{BMO}(\mathbb{R}^n)}.
\end{aligned} \tag{3.17}
$$

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Similarly, by (3.10), (3.12) and proposition 3.1, we find that

$$\begin{aligned}
&\lambda \left(\left|u-(u)_{B_r(x_0)}\right|\right)_{B_r(x_0)}\\
&\quad \leqslant \lambda \left(\left|v-(v)_{B_r(x_0)}\right|\right)_{B_r(x_0)} + 2\lambda(|w|)_{B_r(x_0)}\\
&\quad \lesssim \lambda k^{-\alpha}(kr)^{\sigma}\int_{\mathbb{R}^n}\frac{|v(x)-(v)_{B_{kr}(x_0)}|}{(kr)^{n+\sigma}+|x-x_0|^{n+\sigma}}\,\mathrm{d}x + k^{n/2}\|f\|_{\mathrm{BMO}(\mathbb{R}^n)}\\
&\quad \lesssim \lambda k^{-\alpha}(kr)^{\sigma}\int_{\mathbb{R}^n}\frac{|u(x)-(u)_{B_{kr}(x_0)}|}{(kr)^{n+\sigma}+|x-x_0|^{n+\sigma}}\,\mathrm{d}x\\
&\qquad + \lambda k^{-\alpha}(kr)^{\sigma}\int_{\mathbb{R}^n}\frac{|w(x)-(w)_{B_{kr}(x_0)}|}{(kr)^{n+\sigma}+|x-x_0|^{n+\sigma}}\,\mathrm{d}x + k^{n/2}\|f\|_{\mathrm{BMO}(\mathbb{R}^n)}\\
&\quad \lesssim \lambda k^{-\alpha}\,\|u\|_{\mathrm{BMO}(\mathbb{R}^n)} + k^{n/2}\|f\|_{\mathrm{BMO}(\mathbb{R}^n)} + k^{-\alpha}\|f\|_{\mathrm{BMO}(\mathbb{R}^n)}\\
&\quad \lesssim \lambda k^{-\alpha}\,\|u\|_{\mathrm{BMO}(\mathbb{R}^n)} + k^{n/2}\|f\|_{\mathrm{BMO}(\mathbb{R}^n)},
\end{aligned}$$

which, combined with (3.17), further implies that lemma 3.6 holds true. This finishes the proof of lemma 3.6. □

Let ϕ be a non-negative, real-valued function in $C_{\mathrm{c}}^{\infty}(\mathbb{R}^n)$ with the property that $\int_{\mathbb{R}^n}\phi(x)\,\mathrm{d}x=1$ and $\mathrm{supp}\,(\phi)\subset B_1$. For any $\varepsilon\in(0,\,\infty)$, let $\phi_\varepsilon(\cdot):=\frac{1}{\varepsilon^n}\phi(\frac{\cdot}{\varepsilon})$. Let $u\in L^p(\mathbb{R}^n)\cap\mathrm{BMO}(\mathbb{R}^n)$ for some $p\in(1,\,\infty)$. The mollification u_ε of u is defined by, for any $x\in\mathbb{R}^n$,

$$u_\varepsilon(x):=\phi_\varepsilon * u(x)=\int_{\mathbb{R}^n}\phi_\varepsilon(x-y)u(y)\,\mathrm{d}y.$$

Then, we have the following well-known properties of u_ε (see, for instance, [**32**, theorem 1.6.1]).

Lemma 3.7. *Let $p\in(1,\,\infty)$, $u\in L^p(\mathbb{R}^n)\cap L^\infty(\mathbb{R}^n)$ and u_ε be the mollification of u. Then the following properties hold true.*

(i) *For any $\varepsilon\in(0,\,\infty)$, $u_\varepsilon\in C^\infty(\mathbb{R}^n)$.*

(ii) *For any $\varepsilon\in(0,\,\infty)$, $u_\varepsilon\in L^p(\mathbb{R}^n)$ and $\lim_{\varepsilon\to0}\|u-u_\varepsilon\|_{L^p(\mathbb{R}^n)}=0$.*

(iii) *For any $\varepsilon\in(0,\,\infty)$, $\|u_\varepsilon\|_{L^\infty(\mathbb{R}^n)}\leqslant\|u\|_{L^\infty(\mathbb{R}^n)}$.*

In addition, when $u\in\mathrm{BMO}(\mathbb{R}^n)$, we have the following property of u_ε.

Lemma 3.8. *Let $u\in\mathrm{BMO}(\mathbb{R}^n)$ and u_ε be the mollification of u. Then, for any $\varepsilon\in(0,\,\infty)$, $u_\varepsilon\in\mathrm{BMO}(\mathbb{R}^n)$ and*

$$\|u_\varepsilon\|_{\mathrm{BMO}(\mathbb{R}^n)}\leqslant C\|u\|_{\mathrm{BMO}(\mathbb{R}^n)},$$

where C is a positive constant independent of ε and u.

Proof. Let $\varepsilon\in(0,\,\infty)$ and $B_r(x_0)\subset\mathbb{R}^n$ be a ball. By the equivalent characterization of $\|u_\varepsilon\|_{\mathrm{BMO}(\mathbb{R}^n)}$ (see, for instance, [**14**, proposition 3.1.2(4)]), to show lemma

3.8, we only need to prove that, for any $B_r(x_0) \subset \mathbb{R}^n$, there exists a constant c such that

$$\frac{1}{|B_r(x_0)|} \int_{B_r(x_0)} |u_\varepsilon(x) - c| \,\mathrm{d}x \leqslant C\|u\|_{\mathrm{BMO}(\mathbb{R}^n)}. \tag{3.18}$$

We first assume that $r \leqslant \varepsilon$. In this case, let $c := (u)_{B_{3\varepsilon}(x_0)}$. Then, by the fact that, for any $x \in B_r(x_0)$ with $r \leqslant \varepsilon$ and $y \in B_\varepsilon(x)$, $y \in B_{3\varepsilon}(x_0)$, we have

$$\begin{aligned}
&\frac{1}{|B_r(x_0)|} \int_{B_r(x_0)} |u_\varepsilon(x) - c| \,\mathrm{d}x \\
&\quad \leqslant \frac{1}{|B_r(x_0)|} \int_{B_r(x_0)} \frac{1}{\varepsilon^n} \int_{B_\varepsilon(x)} \phi\left(\frac{x-y}{\varepsilon}\right) |u(y) - c| \,\mathrm{d}y\mathrm{d}x \\
&\quad \leqslant \frac{1}{|B_r(x_0)|} \int_{B_r(x_0)} \frac{1}{\varepsilon^n} \int_{B_{3\varepsilon}(x_0)} |u(y) - (u)_{B_{3\varepsilon}(x_0)}| \,\mathrm{d}y\mathrm{d}x \\
&\quad \lesssim \|u\|_{\mathrm{BMO}(\mathbb{R}^n)}.
\end{aligned} \tag{3.19}$$

Now, we assume that $r \geqslant \varepsilon$. In this case, let $c := (u)_{B_{2r}(x_0)}$. Then, from the fact that, for any $y \in B_\varepsilon$ and $x \in B_r(x_0 - y)$ with $r \geqslant \varepsilon$, $x \in B_{2r}(x_0)$, it follows that

$$\begin{aligned}
&\frac{1}{|B_r(x_0)|} \int_{B_r(x_0)} |u_\varepsilon(x) - c| \,\mathrm{d}x \\
&\quad \leqslant \frac{1}{|B_r(x_0)|} \int_{B_r(x_0)} \frac{1}{\varepsilon^n} \int_{B_\varepsilon} \phi\left(\frac{y}{\varepsilon}\right) |u(x-y) - c| \,\mathrm{d}y\mathrm{d}x \\
&\quad \leqslant \frac{1}{\varepsilon^n} \int_{B_\varepsilon} \phi\left(\frac{y}{\varepsilon}\right) \frac{1}{|B_r(x_0)|} \int_{B_r(x_0)} |u(x-y) - c| \,\mathrm{d}x\mathrm{d}y \\
&\quad \leqslant \frac{1}{\varepsilon^n} \int_{B_\varepsilon} \phi\left(\frac{y}{\varepsilon}\right) \frac{1}{|B_r(x_0)|} \int_{B_r(x_0-y)} |u(x) - c| \,\mathrm{d}x\mathrm{d}y \\
&\quad \leqslant \frac{1}{\varepsilon^n} \int_{B_\varepsilon} \phi\left(\frac{y}{\varepsilon}\right) \frac{1}{|B_r(x_0)|} \int_{B_{2r}(x_0)} |u(x) - c| \,\mathrm{d}x\mathrm{d}y \\
&\quad \lesssim \|u\|_{\mathrm{BMO}(\mathbb{R}^n)},
\end{aligned}$$

which, together with (3.19), further implies that (3.18) holds true. This finishes the proof of lemma 3.8. □

To prove theorems 1.2 and 1.4, we also need the following convergence lemma on the space $\mathrm{BMO}(\mathbb{R}^n)$.

LEMMA 3.9. *Let $p \in (1, \infty)$, $\{f_k\}_{k\in\mathbb{N}} \subset \mathrm{BMO}(\mathbb{R}^n) \cap L^p(\mathbb{R}^n)$ be a sequence of functions and $f \in L^p(\mathbb{R}^n)$. Assume that $\lim_{k\to\infty} \|f - f_k\|_{L^p(\mathbb{R}^n)} = 0$ and $\lim_{k\to\infty} f_k = f$ in the sense of almost everywhere. Then, $f \in \mathrm{BMO}(\mathbb{R}^n)$ and*

$$\|f\|_{\mathrm{BMO}(\mathbb{R}^n)} \leqslant \varliminf_{k\to\infty} \|f_k\|_{\mathrm{BMO}(\mathbb{R}^n)}.$$

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Proof. Let $B \subset \mathbb{R}^n$ be a ball. Then, by the Hölder inequality, we conclude that, for any $k \in \mathbb{N}$,

$$\begin{aligned} \left| |f-(f)_B| - |f_k-(f_k)_B| \right| &\leqslant |f-f_k| + \frac{1}{|B|}\int_B |f-f_k|\,\mathrm{d}x \\ &\leqslant |f-f_k| + \left(\frac{1}{|B|}\int_B |f-f_k|^p\,\mathrm{d}x\right)^{1/p}. \end{aligned}$$

Furthermore, from the assumptions that $\lim_{k\to\infty}\|f-f_k\|_{L^p(\mathbb{R}^n)} = 0$ and $\lim_{k\to\infty} f_k = f$ in the sense of almost everywhere, we deduce that

$$\lim_{k\to\infty} |f_k-(f_k)_B| = |f-(f)_B|,$$

which, together with the Fatou lemma, further implies that

$$\frac{1}{|B|}\int_B |f-(f)_B|\,\mathrm{d}x \leqslant \varliminf_{k\to\infty} \frac{1}{|B|}\int_B |f_k-(f_k)_B|\,\mathrm{d}x \leqslant \varliminf_{k\to\infty} \|f_k\|_{\mathrm{BMO}(\mathbb{R}^n)}.$$

Since the ball B is arbitrary, it follows that

$$\|f\|_{\mathrm{BMO}(\mathbb{R}^n)} \leqslant \varliminf_{k\to\infty} \|f_k\|_{\mathrm{BMO}(\mathbb{R}^n)}.$$

This finishes the proof of lemma 3.9. □

Now, we prove theorem 1.2 by using lemmas 2.3, 2.4, 2.5, 3.3, 3.6, 3.7 and 3.9.

Proof of theorem 1.2. We first show (i). Let $\lambda \in (0, \infty)$, $u \in J_\sigma(L^p(\mathbb{R}^n)) \cap J_\sigma(\mathrm{BMO}(\mathbb{R}^n))$ and $f = -(-\Delta)^{\sigma/2}u - \lambda u$. Then, we have $f \in L^p(\mathbb{R}^n) \cap \mathrm{BMO}(\mathbb{R}^n)$. Let f_ε be the mollification of f. Then, by lemmas 3.7 and 3.8, we find that, for any $\varepsilon \in (0, \infty)$, $f_\varepsilon \in L^p(\mathbb{R}^n) \cap \mathrm{BMO}(\mathbb{R}^n)$. From lemma 2.4, it follows that there exists a $u_\varepsilon \in J_\sigma(L^p(\mathbb{R}^n))$ such that

$$-(-\Delta)^{\sigma/2}u_\varepsilon - \lambda u_\varepsilon = f_\varepsilon, \tag{3.20}$$

moreover, there exists a positive constant C, independent of f, f_ε, u, u_ε and λ, such that

$$\|Lu_\varepsilon - Lu\|_{L^p(\mathbb{R}^n)} \leqslant C\|f_\varepsilon - f\|_{L^p(\mathbb{R}^n)}. \tag{3.21}$$

Let $\{\chi_j\}_{j\in\mathbb{N}}$ be a sequence of smooth functions satisfying that $\chi_j = 1$ on the ball B_j, $\operatorname{supp}(\chi_j) \subset B_{j+1}$, and $0 \leqslant \chi_j \leqslant 1$, where, for any $j \in \mathbb{N}$, $B_j := B(\mathbf{0}, j)$. Then, we have $\chi_j f_\varepsilon \in C_{\mathrm{c}}^\infty(\mathbb{R}^n)$ for any $j \in \mathbb{N}$ and

$$\lim_{j\to\infty} \|\chi_j f_\varepsilon - f_\varepsilon\|_{L^p(\mathbb{R}^n)} = 0. \tag{3.22}$$

Moreover, from lemma 2.4 and the theory of Fourier transform (see, for instance, [**2**, remark 2.2]), we deduce that there exists a unique $u_{\varepsilon,j} \in C_b^\infty(\mathbb{R}^n) \cap J_\sigma(L^2(\mathbb{R}^n))$

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such that

$$- (-\Delta)^{\sigma/2} u_{\varepsilon,j} - \lambda u_{\varepsilon,j} = \chi_j f_\varepsilon, \tag{3.23}$$

meanwhile, there exists a positive constant C, independent of f_ε, u_ε, $u_{\varepsilon,j}$, χ_j and λ, such that

$$\|Lu_{\varepsilon,j} - Lu_\varepsilon\|_{L^p(\mathbb{R}^n)} \leqslant C \|\chi_j f_\varepsilon - f_\varepsilon\|_{L^p(\mathbb{R}^n)} . \tag{3.24}$$

Take $\eta \in C_c^\infty(\mathbb{R}^n)$ such that $\eta \equiv 1$ on B_2, $\operatorname{supp}(\eta) \subset B_4$ and $0 \leqslant \eta \leqslant 1$. Then, we have $\eta\chi_j f_\varepsilon \subset C_c^\infty(\mathbb{R}^n)$. By using lemma 2.4 and the theory of Fourier transform again, we find that there exists a unique $w_{\varepsilon,j} \in J_\sigma(\cap_{q\in(1,\infty)} L^q(\mathbb{R}^n)) \cap C_b^\infty(\mathbb{R}^n)$ such that

$$-(-\Delta)^{\sigma/2} w_{\varepsilon,j} - \lambda w_{\varepsilon,j} = \eta\chi_j f_\varepsilon$$

and, for any $q \in (1, \infty)$,

$$\|Lw_{\varepsilon,j}\|_{L^q(\mathbb{R}^n)} \leqslant C \|\eta\chi_j f_\varepsilon\|_{L^q(\mathbb{R}^n)} , \tag{3.25}$$

where C is a positive constant independent of $w_{\varepsilon,j}$, η, χ_j, f_ε and λ.

Let $v_{\varepsilon,j} := u_{\varepsilon,j} - w_{\varepsilon,j} \in J_\sigma(L^2(\mathbb{R}^n)) \cap C_b^\infty(\mathbb{R}^n)$. Then

$$- (-\Delta)^{\sigma/2} v_{\varepsilon,j} - \lambda v_{\varepsilon,j} = (1-\eta)\chi_j f_\varepsilon. \tag{3.26}$$

By applying L to both sides of (3.26), we conclude that

$$-(-\Delta)^{\sigma/2} Lv_{\varepsilon,j} - \lambda Lv_{\varepsilon,j} = L\left[(1-\eta)\chi_j f_\varepsilon\right].$$

From the fact that $v_{\varepsilon,j} \in C_b^\infty(\mathbb{R}^n) \subset \Lambda^s(\mathbb{R}^n)$ for any $s \in (0, \infty)$ and lemma 2.5, we deduce that $Lv_{\varepsilon,j} \in \Lambda^s(\mathbb{R}^n)$ for any $s \in (0, \infty)$. Then, by proposition 2.2, we find that $Lv_{\varepsilon,j} \in L^\infty(\mathbb{R}^n) \cap C^2(\mathbb{R}^n)$, which, together with lemma 3.3, further implies that there exists $\alpha \in (0, \min\{1, \sigma\})$ such that

$$[Lv_{\varepsilon,j}]_{C^\alpha(B_{1/2})} \leqslant C \left\{ \|Lv_{\varepsilon,j} - (Lv_{\varepsilon,j})_{B_1}\|_{L^1(\mathbb{R}^n,\omega)} + \|L\left[(1-\eta)\chi_j f_\varepsilon\right]\|_{L^\infty(B_1)} \right\}, \tag{3.27}$$

where C is a positive constant independent of $v_{\varepsilon,j}$, η, χ_j, f_ε and λ. For any $x \in B_1$, we have that $(1-\eta)\chi_j f_\varepsilon(x) = 0$, and if $y \in B_{1/2}$, then $(1-\eta)\chi_j f_\varepsilon(x+y) = 0$. Meanwhile, by $f \in \mathrm{BMO}(\mathbb{R}^n)$ and lemma 3.8, we find that, for any $\varepsilon \in (0, \infty)$, $f_\varepsilon \in \mathrm{BMO}(\mathbb{R}^n)$, which, combined with the characterization of pointwise multipliers for functions of bounded mean oscillation (see, for instance, [**21**, theorem 1]), implies that, for any $j \in \mathbb{N}$ and $\varepsilon \in (0, \infty)$, $(1-\eta)\chi_j f_\varepsilon \in \mathrm{BMO}(\mathbb{R}^n)$. Moreover, from [**21**, lemmas 3.1 and 3.3] and the proof of [**21**, theorem 1] (see [**21**, pp. 215-216]), we

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deduce that

$$\|(1-\eta)\chi_j f_\varepsilon\|_{\mathrm{BMO}(\mathbb{R}^n)} \lesssim \|f_\varepsilon\|_{\mathrm{BMO+}(\mathbb{R}^n)},$$

which, together with the fact that, for any $x \in B_1$, $((1-\eta)\chi_j f_\varepsilon)_{B_{1/2}(x)} = 0$, proposition 3.1, and lemma 3.8, further implies that, for any $x \in B_1$,

$$\begin{aligned}
|L[(1-\eta)\chi_j f_\varepsilon(x)]| &\leqslant C \int_{|y|\geqslant\frac{1}{2}} \frac{|(1-\eta)\chi_j f_\varepsilon(x+y)|}{|y|^{n+\sigma}}\,\mathrm{d}y \\
&= C \int_{|y-x|\geqslant\frac{1}{2}} \frac{|(1-\eta)\chi_j f_\varepsilon(y) - ((1-\eta)\chi_j f_\varepsilon)_{B_{1/2}(x)}|}{|y-x|^{n+\sigma}}\,\mathrm{d}y \\
&\leqslant C\,\|(1-\eta)\chi_j f_\varepsilon\|_{\mathrm{BMO}(\mathbb{R}^n)} \leqslant C\,\|f_\varepsilon\|_{\mathrm{BMO+}(\mathbb{R}^n)} \\
&\leqslant C\left[\|f\|_{\mathrm{BMO}(\mathbb{R}^n)} + |(f_\varepsilon)_{B_1}|\right],
\end{aligned}$$

where C is a positive constant independent of η, χ_j, f_ε and λ. By this and (3.27), we conclude that

$$[Lv_{\varepsilon,j}]_{C^\alpha(B_{1/2})} \leqslant C\left[\|Lv_{\varepsilon,j} - (Lv_{\varepsilon,j})_{B_1}\|_{L^1(\mathbb{R}^n,\omega)} + \|f\|_{\mathrm{BMO}(\mathbb{R}^n)} + |(f_\varepsilon)_{B_1}|\right]. \tag{3.28}$$

Then, similarly to the proofs of lemmas 3.5 and 3.6, by (3.25), (3.28), and a scaling and shifting the coordinates argument, we conclude that, for any $k \in [2, \infty)$,

$$\begin{aligned}
&\left(\left|Lu_{\varepsilon,j} - (Lu_{\varepsilon,j})_{B_r(x_0)}\right|\right)_{B_r(x_0)} \\
&\quad\leqslant C\left\{k^{-\alpha}\|Lu_{\varepsilon,j}\|_{\mathrm{BMO}(\mathbb{R}^n)} + k^{n/2}\left[\|f\|_{\mathrm{BMO}(\mathbb{R}^n)} + |(f_\varepsilon)_{B_1}|\right]\right\},
\end{aligned}$$

where C is a positive constant independent of x_0, r, k, $u_{\varepsilon,j}$, f and λ. Since x_0 and r are arbitrary, it follows that, by taking a sufficient large k such that $Ck^{-\alpha} \leqslant \frac{1}{2}$, we have

$$\|Lu_{\varepsilon,j}\|_{\mathrm{BMO}(\mathbb{R}^n)} \leqslant C\left[\|f\|_{\mathrm{BMO}(\mathbb{R}^n)} + |(f_\varepsilon)_{B_1}|\right], \tag{3.29}$$

where C is a positive constant independent of f, $u_{\varepsilon,j}$ and λ.

Furthermore, by (3.22) and (3.24), we find that there exists a subsequence of $\{Lu_{\varepsilon,j}\}_{j\in\mathbb{N}}$, still denoted by $\{Lu_{\varepsilon,j}\}_{j\in\mathbb{N}}$, such that

$$\lim_{j\to\infty} Lu_{\varepsilon,j} = Lu_\varepsilon$$

in the sense of almost everywhere, which, together with lemma 3.9 and (3.29), further implies that

$$\|Lu_\varepsilon\|_{\mathrm{BMO}(\mathbb{R}^n)} \leqslant C\left[\|f\|_{\mathrm{BMO}(\mathbb{R}^n)} + |(f_\varepsilon)_{B_1}|\right]. \tag{3.30}$$

Similarly, from (3.30), (3.21) and lemmas 3.7 and 3.9, we deduce that

$$\begin{aligned}
\|Lu\|_{\mathrm{BMO}(\mathbb{R}^n)} &\leqslant C\left[\|f\|_{\mathrm{BMO}(\mathbb{R}^n)} + |(f)_{B_1}|\right] \\
&= C\left[\left\|-(-\Delta)^{\sigma/2}u - \lambda u\right\|_{\mathrm{BMO}(\mathbb{R}^n)} + \left|\left(-(-\Delta)^{\sigma/2}u - \lambda u\right)_{B_1}\right|\right].
\end{aligned} \tag{3.31}$$

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Since the constant C in (3.31) is independent of λ, by taking $\lambda \to 0^+$, we obtain (i). This finishes the proof of (i).

Next, we prove (ii) by borrowing some ideas from [6] (see also [20]). We first assume that $\sigma \in (0,\,1)$. From the proof of [6, proposition 4.1], it follows that

$$\begin{aligned} Lu &= \int_{\mathbb{R}^n} (u(x+y)-u(x))a(y)\frac{\mathrm{d}y}{|y|^{n+\sigma}} \\ &= \lim_{\varepsilon\to 0^+} C_0\,\mathrm{P.V.}\int_{\mathbb{R}^n}\left(\int_{\mathbb{R}^n} k^\sigma(z,y)a_\varepsilon(y)\frac{\mathrm{d}y}{|y|^{n+\sigma}}\right)\partial^\sigma u(x-z)\,\mathrm{d}z, \end{aligned}$$

where $C_0 := \frac{\Gamma((n-\alpha)/2)}{2^\alpha \pi^{n/2}\Gamma(\alpha/2)}$, $\varepsilon\in(0,\,1)$, $a_\varepsilon(y) := a(y)\mathbf{1}_{\varepsilon\leqslant 1\leqslant\frac{1}{\varepsilon}}$, and $k^\sigma(z,\,y) := |z+y|^{-n+\sigma} - |z|^{-n+\sigma}$.

Let

$$k_\varepsilon(z) := \int_{\mathbb{R}^n} k^\sigma(z,y)a_\varepsilon(y)\frac{\mathrm{d}y}{|y|^{n+\sigma}}.$$

Then, we have

$$Lu(x) = \lim_{\varepsilon\to 0^+}\int_{\mathbb{R}^n} k_\varepsilon(z)\partial^\sigma u(x-z)\,\mathrm{d}z =: \lim_{\varepsilon\to 0^+} C_0 T^\varepsilon \partial^\sigma u(x),$$

where T^ε denotes the singular integral operator associated with the kernel k_ε. By [6, lemmas 4.4 and 4.5], we conclude that the assumptions in lemma 2.3 are satisfied. Thus, from lemma 2.3 and the Fatou lemma, we deduce that

$$\|Lu\|_{L^1(\mathbb{R}^n)} \lesssim \varliminf_{\varepsilon\to 0^+}\|T^\varepsilon\partial^\sigma u\|_{L^1(\mathbb{R}^n)} \lesssim \|\partial^\sigma u\|_{H^1(\mathbb{R}^n)}. \tag{3.32}$$

For the case $\sigma = 1$ and $\sigma\in(1,\,2)$, (1.7) also holds true. Indeed, if $\sigma\in(1,\,2)$, $L_\varepsilon u$ can be written as

$$\sum_{i=1}^n \int_{\mathbb{R}^n}\left[D_i u(x+y) - D_i u(x)\right](a_\varepsilon)_i(y)\frac{\mathrm{d}y}{|y|^{n+\sigma-1}} = \sum_{i=1}^n L^{(a_\varepsilon)_i}(D_i u)(x),$$

where

$$(a_\varepsilon)_i(y) := \frac{y_i}{|y|}\int_0^1 a_\varepsilon\left(\frac{y}{s}\right)s^{-1+\sigma}\,ds,$$

and $L^{(a_\varepsilon)_i}$ denotes the non-local elliptic operator defined by

$$L^{(a_\varepsilon)_i}u(x) := \int_{\mathbb{R}^n}[u(x+y)-u(x)](a_\varepsilon)_i(y)\frac{\mathrm{d}y}{|y|^{n+\sigma-1}}.$$

Then, by (3.32) and the boundedness of the Riesz transform on $H^1(\mathbb{R}^n)$ (see, for instance, [27, chapter III, theorem 4]), we conclude that

$$\begin{aligned} \|Lu\|_{L^1(\mathbb{R}^n)} &\lesssim \varliminf_{\varepsilon\to 0}\sum_{i=1}^n\left\|L^{(a_\varepsilon)_i}(D_i(u))\right\|_{L^1(\mathbb{R}^n)} \\ &\lesssim \sum_{i=1}^n\left\|\partial^{\sigma-1}D_i u\right\|_{H^1(\mathbb{R}^n)} \lesssim \|\partial^\sigma u\|_{H^1(\mathbb{R}^n)}. \end{aligned} \tag{3.33}$$

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Thus, (1.7) holds true in the case of $\sigma \in (1,\, 2)$.

If $\sigma = 1$, via using assumption 1.1(ii) and an argument used in [6, p. 18], we find that

$$\begin{aligned} Lu(x) &= \lim_{\varepsilon\to 0^+} L_\varepsilon u(x) \\ &:= \lim_{\varepsilon\to 0^+} C_0\,\text{P.V.}\int_{\mathbb{R}^n} \left(\partial^{1/2}u(x-z) - \partial^{1/2}u(x)\right) m_\varepsilon(z)\frac{\mathrm{d}z}{|z|^{n+\sigma-1/2}}, \end{aligned}$$

where, for any $\varepsilon \in (0,\, \infty)$ and $z \in \mathbb{R}^n$ with $z \neq \mathbf{0}$,

$$\begin{aligned} m_\varepsilon(z) := &\int_{|y|\leqslant\frac{1}{2}} \left[\frac{1}{|z/|z|+y|^{n-1/2}} - 1 - \left(-n+\frac{1}{2}\right)\left(\frac{z}{|z|}, y\right)\right] a_\varepsilon(|z|y)\,\frac{\mathrm{d}y}{|y|^{n+\sigma}} \\ &+ \int_{|y|>\frac{1}{2}} \left(\frac{1}{|z/|z|+y|^{n-1/2}} - 1\right) a_\varepsilon\left(|z|y\right)\frac{\mathrm{d}y}{|y|^{n+\sigma}} \end{aligned}$$

satisfies that there exists a positive constant C, depending only on n, such that $|m_\varepsilon(z)| \leqslant C$. Since $\sigma - \frac{1}{2} \in (\frac{1}{4},\, \frac{3}{4})$ and $|m_\varepsilon(z)| \lesssim 1$ for any $\varepsilon \in (0,\, \infty)$ and $z \in \mathbb{R}^n$ with $z \neq \mathbf{0}$, similar to the proof of (3.32), it follows that

$$\|Lu\|_{L^1(\mathbb{R}^n)} \leqslant \varliminf_{\varepsilon\to 0^+} \|L_\varepsilon u\|_{L^1(\mathbb{R}^n)} \lesssim \left\|\partial^{1/2}\partial^{1/2}u\right\|_{H^1(\mathbb{R}^n)} \lesssim \left\|\partial^1 u\right\|_{H^1(\mathbb{R}^n)}.$$

This, together with (3.32) and (3.33), implies that

$$\|Lu\|_{L^1(\mathbb{R}^n)} \lesssim \|\partial^\sigma u\|_{H^1(\mathbb{R}^n)} \tag{3.34}$$

holds true for any $\sigma \in (0,\, 2)$.

Furthermore, it is known that $J_\sigma(L^2(\mathbb{R}^n)) \cap I_\sigma(H^1(\mathbb{R}^n))$ is dense in $I_\sigma(H^1(\mathbb{R}^n))$ (see, for instance, [30, chapter 5]). Therefore, for any $u \in J_\sigma(H^1(\mathbb{R}^n))$, there exists a Cauchy sequence $\{u_k\}_{k\in\mathbb{N}} \subset J_\sigma(L^2(\mathbb{R}^n)) \cap I_\sigma(H^1(\mathbb{R}^n))$ such that u_k converges to u in $I_\sigma(H^1(\mathbb{R}^n))$. By lemma 2.4 and (3.34), we find that, for any $k \in \mathbb{N}$, $Lu_k \in L^2(\mathbb{R}^n) \cap L^1(\mathbb{R}^n)$ and

$$\|Lu_k\|_{L^1(\mathbb{R}^n)} \lesssim \|\partial^\sigma u_k\|_{H^1(\mathbb{R}^n)}. \tag{3.35}$$

Moreover, from the boundedness of the Riesz transform R_j on $H^1(\mathbb{R}^n)$ and (3.34), we deduce that, for any $k \in \mathbb{N}$,

$$\begin{aligned} \|R_j L u_k\|_{L^1(\mathbb{R}^n)} &= \|L R_j u_k\|_{L^1(\mathbb{R}^n)} \lesssim \|\partial^\sigma R_j u_k\|_{H^1(\mathbb{R}^n)} \\ &= \|R_j \partial^\sigma u_k\|_{H^1(\mathbb{R}^n)} \lesssim \|\partial^\sigma u_k\|_{H^1(\mathbb{R}^n)}, \end{aligned}$$

which, combined with (3.35), further implies that, for any $k \in \mathbb{N}$,

$$\|Lu_k\|_{H^1(\mathbb{R}^n)} \lesssim \|\partial^\sigma u_k\|_{H^1(\mathbb{R}^n)}.$$

By this estimate and the density of $J_\sigma(L^2(\mathbb{R}^n)) \cap I_\sigma(H^1(\mathbb{R}^n))$ in $I_\sigma(H^1(\mathbb{R}^n))$, we conclude that (1.7) holds true. Therefore, this finishes the proof of (ii) and hence of theorem 1.2. $\square$

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Next, we prove theorem 1.4 by using lemmas 2.4, 2.5, 3.6, 3.7 and 3.9, and theorem 1.2.

Proof of theorem 1.4. We first show (i). Let $p \in (1, \infty)$, $f \in L^p(\mathbb{R}^n) \cap \mathrm{BMO}(\mathbb{R}^n)$ and f_ε be the mollification of f. Then, for any $\varepsilon \in (0, \infty)$, $f_\varepsilon \in L^p(\mathbb{R}^n) \cap \mathrm{BMO}(\mathbb{R}^n) \cap C^\infty(\mathbb{R}^n)$. From lemma 2.4, it follows that there exist solutions $u, u_\varepsilon \in J_\sigma(L^p(\mathbb{R}^n))$ for the equation (1.1) with respect to f and f_ε, respectively, with satisfying that

$$\lambda\|u\|_{L^p(\mathbb{R}^n)} + \|\partial^\sigma u\|_{L^p(\mathbb{R}^n)} \leqslant C\|f\|_{L^p(\mathbb{R}^n)},$$
$$\lambda\|u_\varepsilon\|_{L^p(\mathbb{R}^n)} + \|\partial^\sigma u_\varepsilon\|_{L^p(\mathbb{R}^n)} \leqslant C\|f_\varepsilon\|_{L^p(\mathbb{R}^n)},$$

and

$$\lambda\|u - u_\varepsilon\|_{L^p(\mathbb{R}^n)} + \|\partial^\sigma u - \partial^\sigma u_\varepsilon\|_{L^p(\mathbb{R}^n)} \leqslant C\|f - f_\varepsilon\|_{L^p(\mathbb{R}^n)}, \tag{3.36}$$

where C is a positive constant independent of u, f, u_ε, f_ε and λ.

Let $\{\eta_j\}_{j\in\mathbb{N}}$ be a sequence of smooth functions satisfying that $\eta_j = 1$ on the ball B_j, $\operatorname{supp}(\eta_j) \subset B_{j+1}$ and $0 \leqslant \eta_j \leqslant 1$, where, for any $j \in \mathbb{N}$, $B_j := B(\mathbf{0}, j)$. For any $j \in \mathbb{N}$, we have $\eta_j f_\varepsilon \in C_{\mathrm{c}}^\infty(\mathbb{R}^n) \cap L^2(\mathbb{R}^n) \cap L^p(\mathbb{R}^n) \cap \mathrm{BMO}(\mathbb{R}^n)$ and

$$\lim_{j\to\infty} \|\eta_j f_\varepsilon - f_\varepsilon\|_{L^p(\mathbb{R}^n)} = 0. \tag{3.37}$$

By lemma 2.4, we find that there exists a unique solution $u_{\varepsilon,j} \in J_\sigma(L^2(\mathbb{R}^n)) \cap J_\sigma(L^p(\mathbb{R}^n))$ for the equation (1.1) with f replaced by $\eta_j f_\varepsilon$, moreover, there exists a positive constant C, independent of u_ε, f_ε, $u_{\varepsilon,j}$, η_j and λ, such that

$$\lambda \|u_{\varepsilon,j} - u_\varepsilon\|_{L^p(\mathbb{R}^n)} + \|\partial^\sigma u_{\varepsilon,j} - \partial^\sigma u_\varepsilon\|_{L^p(\mathbb{R}^n)} \leqslant C \|\eta_j f_\varepsilon - f_\varepsilon\|_{L^p(\mathbb{R}^n)}. \tag{3.38}$$

Since $\eta_j f_\varepsilon \in C_{\mathrm{c}}^\infty(\mathbb{R}^n)$, it follows that $\eta_j f_\varepsilon \in C^s(\mathbb{R}^n)$ for any $s \in (0, 1)$. Then, by lemma 2.5, we conclude that $u_{\varepsilon,j} \in \Lambda^{s+\sigma}(\mathbb{R}^n)$, which, together with proposition 2.2(iii), further implies that $u_{\varepsilon,j}$ and $\partial^\sigma u_{\varepsilon,j}$ belong to $L^\infty(\mathbb{R}^n)$. Thus, $u_{\varepsilon,j} \in J_\sigma(\mathrm{BMO})(\mathbb{R}^n)$. From the fact that $\eta_j f_\varepsilon \in C_{\mathrm{c}}^\infty(\mathbb{R}^n)$ and the theory of Fourier transform, we deduce that $u_{\varepsilon,j} \in C_b^\infty(\mathbb{R}^n)$. Then, by lemma 3.6, we find that

$$\begin{aligned}
&\lambda \left(\left|u_{\varepsilon,j} - (u_{\varepsilon,j})_{B_r(x_0)}\right|\right)_{B_r(x_0)} + \left(\left|\partial^\sigma u_{\varepsilon,j} - (\partial^\sigma u_{\varepsilon,j})_{B_r(x_0)}\right|\right)_{B_r(x_0)} \\
&\quad \leqslant C \left\{k^{-\alpha} \left[\lambda \|u_{\varepsilon,j}\|_{\mathrm{BMO}(\mathbb{R}^n)} + \|\partial^\sigma u_{\varepsilon,j}\|_{\mathrm{BMO}(\mathbb{R}^n)}\right] + k^{n/2} \|\eta_j f_\varepsilon\|_{\mathrm{BMO}(\mathbb{R}^n)}\right\},
\end{aligned}$$

where C is a positive constant independent of $u_{\varepsilon,j}$, f_ε, η_j, x_0, r, k and λ. Since $x_0 \in \mathbb{R}^n$ and $r \in (0, \infty)$ are arbitrary, it follows that

$$\begin{aligned}
&\lambda \|u_{\varepsilon,j}\|_{\mathrm{BMO}(\mathbb{R}^n)} + \|\partial^\sigma u_{\varepsilon,j}\|_{\mathrm{BMO}(\mathbb{R}^n)} \\
&\quad \leqslant C \left\{k^{-\alpha} \left[\lambda \|u_{\varepsilon,j}\|_{\mathrm{BMO}(\mathbb{R}^n)} + \|\partial^\sigma u_{\varepsilon,j}\|_{\mathrm{BMO}(\mathbb{R}^n)}\right] + k^{n/2} \|\eta_j f_\varepsilon\|_{\mathrm{BMO}(\mathbb{R}^n)}\right\}.
\end{aligned}$$

Via taking a sufficient large k such that $Ck^{-\alpha} \leqslant \frac{1}{2}$, we then obtain that

$$\lambda \|u_{\varepsilon,j}\|_{\mathrm{BMO}(\mathbb{R}^n)} + \|\partial^\sigma u_{\varepsilon,j}\|_{\mathrm{BMO}(\mathbb{R}^n)} \leqslant C \|\eta_j f_\varepsilon\|_{\mathrm{BMO}(\mathbb{R}^n)}.$$

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By the characterization of pointwise multipliers for functions of bounded mean oscillation (see, for instance, [21]) and lemmas 3.8, we conclude that

$$\lambda \|u_{\varepsilon,j}\|_{\mathrm{BMO}(\mathbb{R}^n)} + \|\partial^\sigma u_{\varepsilon,j}\|_{\mathrm{BMO}(\mathbb{R}^n)} \leqslant C\left[\|f\|_{\mathrm{BMO}(\mathbb{R}^n)} + |(f_\varepsilon)_{B_1}|\right]. \tag{3.39}$$

Moreover, from (3.37) and (3.38), we deduce that there exists a subsequence of $\{u_{\varepsilon,j}\}_{j\in\mathbb{N}}$, still denoted by $\{u_{\varepsilon,j}\}_{j\in\mathbb{N}}$, such that

$$\lim_{j\to\infty} u_{\varepsilon,j} = u_\varepsilon$$

and

$$\lim_{j\to\infty} \partial^\sigma u_{\varepsilon,j} = \partial^\sigma u_\varepsilon$$

in the sense of almost everywhere, which, combined with (3.39) and lemma 3.9, further implies that

$$\lambda \|u_{\varepsilon}\|_{\mathrm{BMO}(\mathbb{R}^n)} + \|\partial^\sigma u_{\varepsilon}\|_{\mathrm{BMO}(\mathbb{R}^n)} \leqslant C\left[\|f\|_{\mathrm{BMO}(\mathbb{R}^n)} + |(f_\varepsilon)_{B_1}|\right]. \tag{3.40}$$

Similarly, by (3.36), lemma 3.7(ii), (3.40) and lemma 3.9, we find that (1.9) holds true. This finishes the proof of (i).

Next, we prove (ii). We first assume that $f \in H^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$. Let L^* be the non-local operator associated with the kernel $a(-\cdot)$. Then, we observe that $a(-\cdot)$ also satisfies assumption 1.1. For any $g \in L^\infty(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$, by (i) and lemma 2.4, we conclude that there exists a unique $u \in J_\sigma(\mathrm{BMO}(\mathbb{R}^n)) \cap J_\sigma(L^2(\mathbb{R}^n))$ such that

$$L^* u - \lambda u = g,$$

moreover, there exists a positive constant C, independent of u, g and λ, such that

$$\lambda\|u\|_{\mathrm{BMO}(\mathbb{R}^n)} + \|\partial^\sigma u\|_{\mathrm{BMO}(\mathbb{R}^n)} \leqslant C\left[\|g\|_{\mathrm{BMO}(\mathbb{R}^n)} + |(g)_{B_1}|\right] \leqslant C\|g\|_{L^\infty(\mathbb{R}^n)}. \tag{3.41}$$

Furthermore, from lemma 2.4, it follows that there exists a unique $v \in J_\sigma(L^2(\mathbb{R}^n))$ such that

$$Lv - \lambda v = f. \tag{3.42}$$

Then, we find that

$$\int_{\mathbb{R}^n} vg\,\mathrm{d}x = \int_{\mathbb{R}^n} v(L^*u - \lambda u)\,\mathrm{d}x = \int_{\mathbb{R}^n} (Lv - \lambda v)u\,\mathrm{d}x = \int_{\mathbb{R}^n} fu\,\mathrm{d}x,$$

which, together with (3.41) and the characterization of the norm of $L^1(\mathbb{R}^n)$ (see, for instance, [12, theorem 6.14]) and the fact that $\mathrm{BMO}(\mathbb{R}^n)$ is the dual space of $H^1(\mathbb{R}^n)$ (see, for instance, [14, theorem 3.2.2] and [27, p. 142, theorem 1]), further

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implies that

$$\begin{aligned}\lambda\|v\|_{L^1(\mathbb{R}^n)} &\leqslant \sup_{\substack{\|g\|_{L^\infty(\mathbb{R}^n)}\leqslant 1\\ g\in L^\infty(\mathbb{R}^n)\cap L^2(\mathbb{R}^n)}} \left|\int_{\mathbb{R}^n}\lambda vg\,\mathrm{d}x\right| = \sup_{\substack{\|g\|_{L^\infty(\mathbb{R}^n)}\leqslant 1\\ g\in L^\infty(\mathbb{R}^n)\cap L^2(\mathbb{R}^n)}} \left|\int_{\mathbb{R}^n}\lambda fu\,\mathrm{d}x\right| \\ &\leqslant \sup_{\substack{\|g\|_{L^\infty(\mathbb{R}^n)}\leqslant 1\\ g\in L^\infty(\mathbb{R}^n)\cap L^2(\mathbb{R}^n)}} \lambda\|u\|_{\mathrm{BMO}(\mathbb{R}^n)}\|f\|_{H^1(\mathbb{R}^n)} \lesssim \|f\|_{H^1(\mathbb{R}^n)}. \end{aligned} \tag{3.43}$$

Similarly, for $(-\Delta)^{\sigma/2}v$, we have

$$\begin{aligned}\int_{\mathbb{R}^n}(-\Delta)^{\sigma/2}vg\,\mathrm{d}x &= \int_{\mathbb{R}^n}(-\Delta)^{\sigma/2}v(L^*u-\lambda u)\,\mathrm{d}x \\ &= \int_{\mathbb{R}^n}(Lv-\lambda v)(-\Delta)^{\sigma/2}u\,\mathrm{d}x = \int_{\mathbb{R}^n}f(-\Delta)^{\sigma/2}u\,\mathrm{d}x,\end{aligned}$$

which, combined with (3.41) and the characterization of the norm of $L^1(\mathbb{R}^n)$, implies that

$$\begin{aligned}\left\|(-\Delta)^{\sigma/2}v\right\|_{L^1(\mathbb{R}^n)} &\leqslant \sup_{\substack{\|g\|_{L^\infty(\mathbb{R}^n)}\leqslant 1\\ g\in L^\infty(\mathbb{R}^n)\cap L^2(\mathbb{R}^n)}} \left|\int_{\mathbb{R}^n}(-\Delta)^{\sigma/2}vg\,\mathrm{d}x\right| \\ &= \sup_{\substack{\|g\|_{L^\infty(\mathbb{R}^n)}\leqslant 1\\ g\in L^\infty(\mathbb{R}^n)\cap L^2(\mathbb{R}^n)}} \left|\int_{\mathbb{R}^n}f(-\Delta)^{\sigma/2}u\,\mathrm{d}x\right| \\ &\leqslant \sup_{\substack{\|g\|_{L^\infty(\mathbb{R}^n)}\leqslant 1\\ g\in L^\infty(\mathbb{R}^n)\cap L^2(\mathbb{R}^n)}} \left\|(-\Delta)^{\sigma/2}u\right\|_{\mathrm{BMO}(\mathbb{R}^n)}\|f\|_{H^1(\mathbb{R}^n)} \\ &\lesssim \|f\|_{H^1(\mathbb{R}^n)}.\end{aligned} \tag{3.44}$$

By applying the Riesz transform R_j to the two sides of (3.42), we obtain that

$$LR_jv-\lambda R_jv = R_jf.$$

Since the Riesz transform R_j is bounded on both $L^2(\mathbb{R}^n)$ and $H^1(\mathbb{R}^n)$, it follows that $R_jf\in L^2(\mathbb{R}^n)\cap H^1(\mathbb{R}^n)$. Then, by using an argument similar to that used in (3.43) and (3.44), we conclude that

$$\begin{aligned}\lambda\|R_jv\|_{L^1(\mathbb{R}^n)} + \left\|R_j(-\Delta)^{\sigma/2}v\right\|_{L^1(\mathbb{R}^n)} &= \lambda\|R_jv\|_{L^1(\mathbb{R}^n)} + \left\|(-\Delta)^{\sigma/2}R_jv\right\|_{L^1(\mathbb{R}^n)} \\ &\lesssim \|R_jf\|_{H^1(\mathbb{R}^n)} \lesssim \|f\|_{H^1(\mathbb{R}^n)},\end{aligned}$$

which, together with (3.43), (3.44) and (1.4), further implies that

$$\lambda\|v\|_{H^1(\mathbb{R}^n)} + \left\|(-\Delta)^{\sigma/2}v\right\|_{H^1(\mathbb{R}^n)} \lesssim \|f\|_{H^1(\mathbb{R}^n)}. \tag{3.45}$$

Finally, for any $f\in H^1(\mathbb{R}^n)$, it is known that there exists a Cauchy sequence $\{f_k\}_{k\in\mathbb{N}}\subset L^2(\mathbb{R}^n)\cap H^1(\mathbb{R}^n)$ such that f_k converges to f in $H^1(\mathbb{R}^n)$ (see, for

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instance, [**14**, proposition 2.1.7] and [**27**]). Then, from (3.45) and lemma 2.4, we deduce that, for f_k and f_m with $k,\ m \in \mathbb{N}$, there exist $v_k,\ v_m \in J_\sigma(L^2(\mathbb{R}^n)) \cap J_\sigma(H^1(\mathbb{R}^n))$ such that

$$Lv_k - \lambda v_k = f_k$$

and

$$Lv_m - \lambda v_m = f_m,$$

moreover, we have

$$\lambda \|v_k\|_{H^1(\mathbb{R}^n)} + \left\|(-\Delta)^{\sigma/2} v_k\right\|_{H^1(\mathbb{R}^n)} \lesssim \|f_k\|_{H^1(\mathbb{R}^n)}$$

and

$$\lambda \|v_k - v_m\|_{H^1(\mathbb{R}^n)} + \left\|(-\Delta)^{\sigma/2} v_k - (-\Delta)^{\sigma/2} v_m\right\|_{H^1(\mathbb{R}^n)} \lesssim \|f_k - f_m\|_{H^1(\mathbb{R}^n)}.$$

Therefore, $\{v_k\}_{k\in\mathbb{N}}$ is a Cauchy sequence in $J_\sigma(H^1(\mathbb{R}^n))$, and there exists a $v \in J_\sigma(H^1(\mathbb{R}^n))$ such that v_k converges to v in $J_\sigma(H^1(\mathbb{R}^n))$. Then, by theorem 1.2(ii), we conclude that

$$\|Lv_k - Lv\|_{L^1(\mathbb{R}^n)} \lesssim \|\partial^\sigma v_k - \partial^\sigma v\|_{H^1(\mathbb{R}^n)},$$

and Lv_k converges to Lv in $L^1(\mathbb{R}^n)$. Furthermore, v is a solution of $Lv - \lambda v = f$ and

$$\lambda \|v\|_{H^1(\mathbb{R}^n)} + \left\|(-\Delta)^{\sigma/2} v\right\|_{H^1(\mathbb{R}^n)} \lesssim \|f\|_{H^1(\mathbb{R}^n)}.$$

Meanwhile, the uniqueness follows from the above estimate. This finishes the proof of theorem 1.4. □

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