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# A CONJECTURE OF BACHMUTH AND MOCHIZUKI ON AUTOMORPHISMS OF SOLUBLE GROUPS

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**1. Introduction.** In [**1**], Bachmuth and Mochizuki conjecture, by analogy with a celebrated result of Tits on linear groups [**8**], that a finitely generated group of automorphisms of a finitely generated soluble group either contains a soluble subgroup of finite index (which may of course be taken to be normal) or contains a non-abelian free subgroup. They point out that their conjecture holds for nilpotent-by-abelian groups and in some other cases. We show here that it breaks down for groups of derived length three. In order to describe more precisely how the conjecture breaks down, we will say that a group $G$ is *perfectly distributed*, if every subgroup of finite index of $G$ contains a non-trivial finitely generated perfect subgroup. Clearly no perfectly distributed group can be soluble-by-finite, and in fact no such group can even have an $SN$-subgroup of finite index. Here $SN$ is the class of all groups having a (generalized) series with abelian factors (see [**6**] or [**7**]).

THEOREM 1. *There exists a finitely-generated soluble group $G$ of derived length three whose automorphism group contains subgroups $\Gamma_0 \lhd \Gamma$ such that*

(a) $\Gamma$ *is finitely generated,*
(b) $\Gamma/\Gamma_0$ *is infinite cyclic,*
(c) $\Gamma_0$ *is perfectly distributed,*
(d) $\Gamma_0$ *is locally finite.*

The proof will show that we can even arrange that $\Gamma_0$ is locally a direct power of any given finite non-abelian simple group. Clearly $\Gamma$ is not soluble-by-finite, nor does it contain a non-abelian free subgroup.

A question which arises naturally is: which groups can be faithfully represented by automorphisms of finitely generated soluble groups? The only restriction I know on such groups is the obvious one that they must be countable, and it would be interesting to know if there are others. In another direction, one may ask for which finitely-generated soluble groups the Bachmuth-Mochizuki Conjecture holds. The group $G$ of Theorem 1 is actually an extension of a locally finite group by an infinite cyclic group, and so the case of torsion-free $G$ seems to merit consideration. But a somewhat more complicated version of the construction for Theorem 1 gives

THEOREM 2. *There exists a finitely-generated soluble group $G$ of derived length*

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*four whose automorphism group contains a torsion-free subgroup* $\Gamma$ *having a normal subgroup* $\Gamma_0$ *such that* (a)–(c) *of Theorem* 1 *hold. Further,*

(d′) *every finitely generated subgroup of* $\Gamma_0$ *is abelian-by-finite.*

**2. The constructions.** A few remarks will clarify the relationship between the problems we are discussing and certain problems of matrix representability. Let $F$ be a finitely-generated soluble group and let $R$ be a ring homomorphic image of the integral group ring $\mathbf{Z}F$. Let $B$ be any subgroup of $M_t(R)^*$, where $M_t(R)$ is the ring of all $t \times t$ matrices over $R$ and stars denote groups of units. Then we can identify $B$ with a group of automorphisms of a free right $R$-module $V$ of rank $t$. We can view $V$ as a $\mathbf{Z}F$-module using the epimorphism from $\mathbf{Z}F$ to $R$ and form the split extension $G = VF$. This is a finitely-generated soluble group, and for each $b \in B$, the map $b\varphi : (v, f) \to (vb, f)$ is an automorphism of it. Clearly $\varphi$ embeds $B$ in Aut $G$. Thus we have

LEMMA 1. *If $F$ is any finitely-generated soluble group and $R$ is any ring image of $\mathbf{Z}F$, then any group of invertible matrices over $R$ can be faithfully represented by automorphisms of some finitely generated soluble group.*

In the light of this, we shall see that Theorems 1 and 2 follow from

THEOREM 3. *Let $k$ be any prime field, and let $p$ be a prime different from the characteristic of $k$. Let $t \geqq 2$ and let $H$ be a subgroup of $M_t(k)^*$ containing a normal finitely-generated subgroup $X$ consisting of diagonal matrices and such that $H/X$ is a finite non-abelian simple group. Then $M_t(k[C_p \wr C_\infty])^*$ contains subgroups $\Gamma_0 \lhd \Gamma$ satisfying* (a)–(c) *of Theorem* 1, *together with*

(d″) $\Gamma_0$ *is locally a finite subdirect power of $H$.*

In more detail, (d″) means that every finite set of elements of $\Gamma_0$ lies in a subgroup of $\Gamma_0$ which is isomorphic to a subdirect product of finitely many copies of $H$. For $1 \leqq n \leqq \infty$, $C_n$ denotes a cyclic group of order $n$.

*Deduction of Theorem* 1. We take $k$ to be any finite prime field, of characteristic $q$ say, and $p$ to be any prime different from $q$. If $H$ is any finite non-abelian simple group, then we can embed $H$ in $M_t(k)^*$ for suitable $t$, taking the normal diagonal subgroup $X$ to be 1. The corresponding group $\Gamma$ produced by Theorem 3 satisfies (a)–(d) of Theorem 1; (d) of course follows from (d″). By Lemma 1, $\Gamma$ is faithfully represented by automorphisms of a finitely-generated soluble group $G$, which is actually a split extension of an elementary abelian $q$-group $A$ by $C_p \wr C_\infty$. Thus $G$ has derived length at most three. Let $D = C_p \wr C_\infty$. Then $D'$ is infinite and so fixes no non-trivial element of the free $kD$-module $A$. Hence $[A, D', D'] \neq 1$, from which it follows that $G'' \neq 1$. Therefore the derived length of $G$ is three exactly.

It is perhaps most natural to try to prove Theorem 2 by applying Lemma 1 with $R = \mathbf{Z}F$ for a suitable finitely-generated soluble group $F$. But we shall see later (Theorem 4) that a group of invertible matrices over such an $R$

cannot be perfectly distributed, which prevents us from obtaining examples like $\Gamma$ in this way. For this reason, we have had to resort to a somewhat more devious approach.

*Deduction of Theorem* 2. We first require a finitely-generated soluble group $T$ whose integral group ring contains an ideal $I$ such that $\mathbf{Z}T/I$ contains a subring (with the same identity) isomorphic to the rational field $\mathbf{Q}$. We may take as $T$ any finitely-generated soluble group containing a free abelian subgroup $Y$ of infinite rank in its centre—such groups are constructed for example in [**2**, Theorem 7] or [**4**]. There is an epimorphism $Y \to \mathbf{Q}^*$, and this can be extended to a ring epimorphism $\mathbf{Z}Y \to \mathbf{Q}$ with kernel $J$, say. Then $I = JT$ is a two-sided ideal of $\mathbf{Z}T$ with $I \cap \mathbf{Z}Y = J$, and so the image of $\mathbf{Z}Y$ in $\mathbf{Z}T/I = S$ is isomorphic to $\mathbf{Q}$. Now let $F = T \times (C_\infty \wr C_\infty)$. Then $\mathbf{Z}F \cong \mathbf{Z}T[C_\infty \wr C_\infty]$, the group ring of $C_\infty \wr C_\infty$ over $\mathbf{Z}T$. Thus, if $p$ is any prime, we have an epimorphism $\mathbf{Z}F \to S[C_p \wr C_\infty] = R$, and this contains a subring (with the same 1) isomorphic to $\mathbf{Q}[C_p \wr C_\infty]$.

Let $L$ be any finite non-abelian simple group and let $E$ be a free group of finite rank containing a normal subgroup $D$ such that $E/D \cong L$. Let $H = E/D'$. Then $H$ is torsion-free [**5**]. Let $X = D/D'$, which is free abelian of finite rank. $X$ has a faithful one-dimensional representation over $\mathbf{Q}$, and inducing this to $H$, we obtain a faithful representation of $H$ over $\mathbf{Q}$ in which $X$ acts diagonally. Thus, for suitable $t \geqq 2$, we may view $H$ as a subgroup of $M_t(\mathbf{Q})^*$, with $X$ a normal diagonal subgroup such that $H/C$ is finite non-abelian simple.

If $\Gamma$ and $\Gamma_0$ are the subgroups of $M_t(\mathbf{Q}[C_p \wr C_\infty])^* \leqq M_t(R)^*$ furnished by Theorem 3, then they satisfy (a)–(c) of Theorem 1, and it follows from (d'') that $\Gamma$ is torsion-free and (d') of Theorem 2 holds, since $H$ is abelian-by-finite by assumption. The group $G$ given by Lemma 1, on which $\Gamma$ operates faithfully, is the split extension $VF$, where $F = T \times (C_\infty \wr C_\infty)$, which is torsion-free, and $V$ is a free $R$-module of rank $t$. Since $R$ contains a subring isomorphic to $\mathbf{Q}$ and containing the identity of $R$, it is additively torsion-free. Hence so is $V$, whence $G$ is torsion-free.

It remains to consider the derived length of $G$. If we take $T$ to be centre-by-metabelian, with $Y \leqq T''$, as we may [**2**, Theorem 7; **4**]), then $G$ clearly has derived length at most four. Let $y$ be an element of $Y$ which is not mapped to 1 under the epimorphism $Y \to \mathbf{Q}^*$ with which we began. Then $y - 1$ is a non-zero element of $S$ lying in a subring isomorphic to $\mathbf{Q}$, and hence $(y - 1)^2 \neq 0$ in $S$ and in $R$. Therefore $V(y - 1)^2 \neq 0$, and $[V, y, y] \neq 1$ in multiplicative notation. Since $y \in T'' \leqq G''$, this tells us that $G''$ is not abelian, and so $G''$ has derived length four exactly. The deduction of Theorem 2 is complete.

Now we must embark on the proof of Theorem 3. Let $k$ be a prime field, and let $C$ be a cyclic group of prime order $p$ different from the characteristic of $k$. Let $A$ be the group algebra $kC$. Since $A$ is commutative and semisimple we can write $A = F \oplus \bar{F}$, where $F$ is a minimal ideal of $A$ generated by an idempotent $e$, and $\bar{F} = A(1-e)$. Let $m \geqq 1$, and for $0 \leqq j \leqq m$, let $c \to c_j$ be an isomorphism of $C$ onto a group $C_j$. Let $R = k[C_0 \times C_1 \times \ldots \times C_m]$, and extend $c \to c_j$ to a $k$-algebra isomorphism of $A$ onto $A_j = kC_j \leqq R$. Then $F_j$ is a field contained in $A_j$ with identity $e_j$; we emphasize that $F_j$ and $R$ have different identity elements.

It will be convenient to think of $k$ as an abstract field, rather than identifying it with any particular subfield of $R$ or $A$. Since $k$ is a prime field, there is a unique ring monomorphism of $k$ into $F_j$, namely $\varphi_j : \lambda \to \lambda e_j (\lambda \in k)$. We also write $\varphi_j$ for the induced monomorphism $M_t(k) \to M_t(F_j)$. Let $H$ be as given, so that $H \leqq M_t(k)^*$, and let

(1) $\quad H_j = \{h\varphi_j + (1_R - e_j)I : h \in H\},$

where $1_R$ is the identity of $R$ and $I$ is the $t \times t$ identity matrix. Since $h\varphi_j \in M_t(F_j)$, $h\varphi_j$ and $(1_R - e_j)I$ annihilate each other, and so

$$(h\varphi_j + (1_R - e_j)I)(h^{-1}\varphi_j + (1_R - e_j)I) = I.$$

Consequently $H_j \leqq M_t(R)^*$, and $H_j \cong H$. Let

(2) $\quad J = \langle H_0, H_1, \ldots, H_m \rangle$

We proceed to establish some facts about $J$.

LEMMA 2. *$J$ is a finite subdirect power of $H$.*

*Proof.* By Maschke's Theorem, $R$ is a commutative semisimple algebra, that is, we can write

(3) $\quad R = \bigoplus_{\lambda \in \Lambda} K_\lambda,$

where $\Lambda$ is a finite set and $K_\lambda$ is a field with identity $f_\lambda$ say. Let $\pi_\lambda$ be the projection of $R$ on $K_\lambda$ associated with (3), and let $\psi_\lambda$ be the unique monomorphism of $k$ into $K_\lambda$. Because of the uniqueness of $\psi_\lambda$, we have

(4) $\quad \psi_\lambda = \varphi_j \pi_\lambda \quad (0 \leqq j \leqq m, \lambda \in \Lambda)$

unless $F_j \pi_\lambda = 0$. The maps induced by $\psi_\lambda$ and $\pi_\lambda$ on the corresponding $t \times t$ matrix rings will be denoted by the same symbol, and then (4) holds for the maps on $t \times t$ matrix rings also. We obtain from (3) corresponding decompositions in the matrix rings, namely

$$M_t(R) = \bigoplus_{\lambda \in \Lambda} M_t(K_\lambda)$$

and

(5) $\quad M_t(R)^* = \operatorname*{Dr}_{\lambda \in \Lambda} G_\lambda,$

where $G_\lambda \cong M_t(K_\lambda)^*$ and Dr denotes direct product of groups. Explicitly,

$$G_\lambda = \{\xi + (1_R - f_\lambda)I ; \xi \in M_t(K_\lambda)^*\}$$

Thus, the projection $\eta_\lambda$ of $M_t(R)^*$ on $G_\lambda$ associated with (5) is given by

(6) $\alpha\eta_\lambda = \alpha\pi_\lambda + (1_R - f_\lambda)I \quad (\alpha \in M_t(R)^*)$

Now let

(7) $M_\lambda = \{h\psi_\lambda + (1_R - f_\lambda)I : h \in H\}.$

Then like $H_j$, $M_\lambda \cong H$ and $M_\lambda \leqq M_t(R)^*$. We wish to describe $H_j\eta_\lambda$. If $F_j\pi_\lambda = 0$, then from (1) we obtain immediately that $H_j\pi_\lambda = \{f_\lambda I\}$, whence (6) gives $H_j\eta_\lambda = \{I\}$. On the other hand, if $F_j\pi_\lambda \neq 0$, then $e_j\pi_\lambda = f_\lambda = 1_R\pi_\lambda$, $(1_R - e_j)\pi_\lambda = 0$, and $H_j\pi_\lambda = \{h\varphi_j\pi_\lambda : h \in H\}$. From (4), (6) and (7), we see that $H_j\eta_\lambda = M_\lambda$ in this case. It follows that $J\eta_\lambda = \langle H_0\eta_\lambda, \ldots, H_m\eta_\lambda\rangle = \{I\}$ or $M_\lambda$, and so $J$ is a subdirect product of those $M_\lambda$ corresponding to indices $\lambda$ such that $J\eta_\lambda \neq \{I\}$. Lemma 2 is established.

By assumption, $H$ contains a normal finitely-generated diagonal subgroup $X$ such that $H/X$ is a finite non-abelian simple group. Let

$$C_\lambda = \{x\psi_\lambda + (1_R - f_\lambda)I : x \in X\}$$

be the subgroup of $M_\lambda$ corresponding to $X$, and

(8) $C = \operatorname*{Dr}_{\lambda\in\Lambda} C_\lambda$

Then by the proof of Lemma 2, $J/J \cap C$ is a subdirect product of certain of the $M_\lambda/C_\lambda$, each of which is isomorphic to the finite non-abelian simple group $H/X$. We deduce

LEMMA 3. *$J/J \cap C$ is a finite direct power of $H/X$. $J \cap C$ is finitely-generated abelian.*

Next we require

LEMMA 4. $H_m \nleqq \langle H_0, \ldots, H_{m-1}\rangle(J \cap C)$.

*Proof.* In $R$, we have $d = (1_R - e_0)(1_R - e_1)\ldots(1_R - e_{m-1})e_m \neq 0$. Hence $d\pi_\lambda \neq 0$ for some $\lambda \in \Lambda$. Then $(1_R - e_j)\pi_\lambda \neq 0$ and so $e_j\pi_\lambda = 0$ $(0 \leqq j \leqq m-1)$. Therefore, as we saw in the proof of Lemma 2, $H_j\eta_\lambda = \{I\}$ $(0 \leqq j \leqq m-1)$. Since $J \cap C$ consists of diagonal matrices, so does $(J \cap C)\eta_\lambda$. Hence

$$\langle H_0, \ldots, H_{m-1}\rangle(J \cap C)\eta_\lambda$$

consists of diagonal matrices. But $e_m\pi_\lambda \neq 0$ and so $H_m\eta_\lambda = M_\lambda \cong H$, which is not even abelian. The claim follows.

Finally, before proving Theorem 3, we note

LEMMA 5. *Let $S$ be any group containing an abelian normal subgroup $U$ such that $S/U$ is perfect. Then $S'$ is perfect.*

*Proof*. We have $S = S'U$, and so, since $U$ is abelian, $[U, S] = [U, S'] \leqq [U, S, S] = [U, S, S'] \leqq S''$. Therefore, passing to $S/[U, S]$, we may assume that $U$ is in the centre of $S$. But then any commutator in $S$ has the form $[su, s'u']$, with $s, s' \in S'$, $u, u' \in U$, and since $u$ and $u'$ are central, this is equal to $[s, s']$, which lies in $S''$. Hence $S' = S''$, as claimed.

*Proof of Theorem* 3. We have to consider $M_t(k[C_p \wr C_\infty])^*$ where $k$ is a prime field of characteristic different from $p$. Let $C$ be the $C_p$ and $\langle x \rangle$ the $C_\infty$, embedded in $C_p \wr C_\infty$ in the usual way. Let $A$, $F$, $e$ be as above, and let $c \rightarrow c_i$ be the isomorphism $c \rightarrow c^{x^i}$ $(c \in C, i \in \mathbf{Z})$ of $C$ onto $C_i = C^{x^i}$. In the notation above, $A$ is now identified with $A_0$. Let $\theta = xI$ and

$$\Gamma = \langle H, \theta \rangle \leqq M_t(k[C \wr \langle x \rangle])^*.$$

Since $H$ is finitely-generated so is $\Gamma$. Thus (a) of Theorem 1 holds. Let

$$\Gamma_0 = \langle H^{\theta^i} : i \in \mathbf{Z} \rangle.$$

Then $\Gamma_0 \lhd \Gamma$, and $\Gamma = \Gamma_0\langle \theta \rangle$. Noting that

$$(\theta^{-1}\xi\theta)_{ij} = x^{-1}\xi_{ij}x \quad (\xi \in M_t(k[C \wr \langle x \rangle])^*)$$

we find that $\Gamma_0 \leqq M_t(\bar{C})^*$, where $C = \langle C^{x^i} : i \in \mathbf{Z} \rangle$ is the base group of $C \wr \langle x \rangle$. Hence $\Gamma_0 \cap \langle \theta \rangle = \{I\}$, and so $\Gamma/\Gamma_0$ is infinite cyclic. This gives (b).

Let $m \geqq 1$, $0 \leqq j \leqq m$. Let $\varphi_0$ be the monomorphism of $k$ into $F_0 = F$. Then $\varphi_j : \lambda \rightarrow (\lambda\varphi_0)^{\theta^j}$ is a monomorphism of $k$ into $F_j$, and must be the unique such. It follows that $H^{\theta^j} = H_j$, where $H_j$ is given by (1). By Lemma 2,

$$(9) \qquad J = \langle H, H^\theta, \ldots, H^{\theta^m} \rangle = \langle H_0, H_1, \ldots, H_m \rangle$$

is a finite subdirect power of $H$. Since any finite subset of $\Gamma_0$ is conjugate under a power of $\theta$ to a subset of some such $J$, (d$''$) of Theorem 3 holds.

It remains to show that $\Gamma_0$ is perfectly distributed. To do this, it suffices to show that each normal subgroup $\Delta$ of finite index in $\Gamma_0$ contains a non-trivial finitely-generated perfect subgroup. Since $|\Gamma_0 : \Delta| < \infty$ we can write $\Gamma_0 = \Delta F$, where $F$ is a subgroup generated by a finite number of the $H^{\theta i}$. Replacing $\Delta$ by a conjugate under a power of $\theta$ if necessary, we may assume that

$$(10) \qquad \Gamma_0 = \Delta\langle H_0, \ldots, H_{m-1} \rangle$$

for some $m \geqq 1$. Let $C$ be defined as in (8) and $J$ be as in (9) above. If $J \cap \Delta = J \cap C$, then we find from (10) that $J = \langle H_0, \ldots, H_{m-1} \rangle (J \cap C)$, contradicting Lemma 4. Therefore $S = J \cap \Delta \nleqq J \cap C$. Since $J \cap \Delta \lhd J$, Lemma 3 shows that $S/J \cap C$ is a non-trivial direct power of the finite non-abelian simple group $H/X$, and so is perfect. Since $J \cap C$ is abelian, Lemma 5 shows that $S'$ is perfect, and clearly $S' \neq 1$. Finally, since $J$ is a finite subdirect power of $H$, which clearly satisfies the maximal condition on subgroups, every subgroup of $J$, and in particular $S'$, is finitely-generated. The proof of Theorem 3 is complete.

**3. Groups of invertible matrices over group rings of generalized soluble groups.** We conclude by drawing attention to a property of groups of invertible matrices over integral group ring of soluble groups, and even of $SI$-groups, where $SI$ is the class of all groups having a normal (in the whole group) series with abelian factors (see [**6**; **7**]). This seems of particular interest because we have been unable to discover any analogous results when the coefficient ring is a field; Theorem 3 at any rate shows that our result becomes false as it stands if $\mathbf{Z}$ is replaced by a field. In fact, we have been unable to discover any group-theoretic restrictions whatever on subgroups of $(kF)^*$, where $k$ is a field and $F$ is an arbitrary finitely generated soluble group.

THEOREM 4. *If $G$ is an $SI$-group and $t \geqq 1$, then $M_t(\mathbf{Z}G)^*$ contains a normal $SI$-subgroup $\Delta$ of finite index.*

In fact, the proof will show that we can arrange that $M_t(\mathbf{Z}G)^*/\Delta$ is a linear group of degree $t$ over any given finite prime field, and that $\Delta$ has an abelian series with terms normal in $M_t(\mathbf{Z}G)^*$.

*Proof.* By refining the given series of $G$ suitably, we see that we may assume that $G$ has a normal series $\{A_\sigma, B_\sigma : \sigma \in \Omega\}$ in which each factor is either torsion-free abelian or elementary abelian. If we have a subgroup of $G$ denoted by a capital Italic letter, we will denote the augmentation ideal of that subgroup by the corresponding small German letter. Also, $XG$ will denote the right ideal of $\mathbf{Z}G$ generated by a non-empty subset $X$ of $\mathbf{Z}G$. We show that

(11) $\{\mathfrak{a}_\sigma G, \mathfrak{b}_\sigma G : \sigma \in \Omega\}$

is a series of two-sided ideals of $\mathfrak{g}$. Here a series of ideals is defined in the same way as a series of subgroups (see [**7**, Part 1, p. 9]), and the only difficulty is to show that if $0 \neq \alpha \in \mathfrak{g}$, then there exists $\tau \in \Omega$ such that $\alpha \in \mathfrak{a}_\tau G$, $\alpha \notin \mathfrak{b}_\tau G$. To see this, let $\Sigma$ be the set of all $\sigma \in \Omega$ such that the sum of the coefficients of $\alpha$ over every coset of $A_\sigma$ is zero. $\Sigma$ is not empty since the support of $\alpha$ must be contained in a suitable $A_\sigma$. For each $\sigma \in \Sigma$ we have a partition $P_\sigma$ of the support $\operatorname{supp}\alpha$ of $\alpha$ determined by the cosets of $A_\sigma$, and we may choose $\sigma \in \Sigma$ such that this partition is as fine as possible. Consider the finite set $X$ of elements $xy^{-1}$, where $x$ and $y$ range over all distinct pairs of elements of $\operatorname{supp}\alpha$ which come from the same coset of $A_\sigma$. If $X = \phi$ then the elements of $\operatorname{supp}\alpha$ all lie in distinct cosets of $A_\sigma$, while the sum of the coefficients of $\alpha$ over each coset of $A_\sigma$ is zero. In other words, $\alpha = 0$. Hence $X \neq \phi$, and there exists $\tau \in \Omega$ such that all members of $X$ belong to $A_\tau$, while not all belong to $B_\tau$. Thus there exist $x_0 \neq y_0$ in $\operatorname{supp}\alpha$ such that $x_0y_0^{-1} \in A_\tau$, $x_0y_0^{-1} \notin B_\tau$. It follows that $\tau \leqq \sigma$ and that $P_\tau = P_\sigma$. Hence $\tau \in \Sigma$, and $\alpha \in \mathfrak{a}_\tau G$. Suppose if possible that $\alpha \in \mathfrak{b}_\tau G$. Since $x_0y_0^{-1} \notin B_\tau$, the partition $Q_\tau$ of $\operatorname{supp}\alpha$ determined by the cosets of $B_\tau$ is a strict refinement of $P_\sigma$. The sum of the coefficients of $\alpha$ over each part of $Q_\tau$ is zero. There exists $\mu \in \Omega$ such that if $Y$ is the set of all $uv^{-1}$, where $u$, $v$ range over all distinct pairs of elements of $\operatorname{supp}\alpha$ which

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lie in the same coset of $B_\tau$ then $Y \leqq A_\mu$, $Y \nleqq B_\mu$. Then $\tau > \mu$. For otherwise $B_\tau \leqq B_\mu$, while there exists an element $u_0 v_0^{-1}$ of $Y$ lying in $B_\tau$ but not in $B_\mu$. Hence $A_\mu \leqq B_\tau$, and $P_\mu = Q_\tau$. Thus $\mu \in \Sigma$, while $P_\mu$ refines $P_\sigma$ properly, a contradiction.

Let $\sigma \in \Omega$ and $\bar{G} = G/B_\sigma$. The natural map $G \to \bar{G}$ induces a ring homomorphism of $\mathbf{Z}G$ onto $\mathbf{Z}\bar{G}$ with kernel $\mathfrak{b}_\sigma G$ and which maps $\mathfrak{a}_\sigma G$ onto $\bar{\mathfrak{a}}_\sigma \bar{G}$. Now $\bar{\mathfrak{a}}_\sigma$ is a residually nilpotent ideal of $\mathbf{Z}\bar{A}_\sigma$, since $\bar{A}_\sigma$ is either torsion-free abelian or elementary abelian ([**3**, Theorem E, Lemma 18]). Since $(\bar{\mathfrak{a}}_\sigma \bar{G})^n = \bar{\mathfrak{a}}_\sigma^n \bar{G} = \bigoplus_{t \in T} \bar{\mathfrak{a}}_\sigma^n s_t$, where $n \geqq 1$ and $T$ is a transversal to $\bar{A}_\sigma$ in $\bar{G}$, we have $\bigcap_{n=1}^\infty (\bar{a}_\sigma \bar{G})^n = 0$. Hence

$$\bigcap_{n=1}^{\infty} ((\mathfrak{a}_\sigma G)^n + \mathfrak{b}_\sigma G) = \mathfrak{b}_\sigma G.$$

It follows that the series (11) may be refined to a series of $\mathfrak{g}$, consisting of two-sided ideals of $\mathbf{Z}G$, with factors of square zero. Let $p$ be any prime. Then since $\mathbf{Z}G/\mathfrak{g} \cong \mathbf{Z}$, and $\bigcap_{n=1}^\infty p^n\mathbf{Z} = 0$, we may extend this series to a series

$$\{\Lambda_\theta, V_\theta : \theta \in \Theta\}$$

of $\mathfrak{g} + p\mathbf{Z}$, with factors of square zero. Then

$$\{M_t(\Lambda_\theta), M_t(V_\theta) : \theta \in \Theta\}$$

is a series of $M_t(\mathfrak{g} + p\mathbf{Z})$, consisting of ideals of $M_t(\mathbf{Z}G)$ and with factors of square zero.

Let $\Gamma = M_t(\mathbf{Z}G)^*$, $\Delta = \Gamma \cap (1 + M_t(\mathfrak{g} + p\mathbf{Z}))$. The augmentation, followed by the natural map $\mathbf{Z} \to \mathbf{Z}/p\mathbf{Z}$, induces a homomorphism of $\Gamma$ into $M_t(\mathbf{Z}_p)^*$ with kernel $\Delta$, and so $\Gamma/\Delta$ is a $\mathbf{Z}_p$-linear group of degree $t$. In particular, $|\Gamma : \Delta| < \infty$. Let $E_\theta = \Delta \cap (1 + M_t(\Lambda_\theta))$, $F_\theta = \Delta \cap (1 + M_t(V_\theta))$. Then

$$\{E_\theta, F_\theta : \theta \in \Theta\} \tag{12}$$

is a series of $\Delta$ consisting of normal subgroups of $\Gamma$. Let $\xi, \eta \in E_\theta$. Then $\xi = 1 + \alpha$, $\eta = 1 + \beta$, with $\alpha, \beta \in M_t(\Lambda_\theta)$. Since

$$(1 + \alpha)(1 - \alpha) = 1 - \alpha^2 \equiv 1 \bmod M_t(V_\theta),$$

we have $\xi^{-1} \equiv 1 - \alpha \bmod M_t(V_\theta)$, and similarly for $\eta$. Therefore $[\xi, \eta] = \xi^{-1}\eta^{-1}\xi\eta \equiv (1 - \alpha)(1 - \beta)(1 + \alpha)(1 + \beta) \equiv 1 \bmod M_t(V_\theta)$, since

$$M_t(\Lambda_\theta)^2 \leqq M_t(V_\theta).$$

Hence $[\xi, \eta] \in F_\theta$, and the factors of (12) are abelian. This proves Theorem 4.

We mention again in conclusion the

PROBLEM. *What can be said about subgroups of $(kF)^*$, where $k$ is a field and $F$ is a finitely-generated soluble group?*

*Added in proof.* A different kind of counterexample to the Bachmuth-Mochizuki conjecture, involving an infinite cyclic extension of the restricted symmetric group on a countable set, has been constructed independently by P. M. Neumann (unpublished).

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