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ARTICLE

Fair Infinite Lotteries, Qualitative Probability, and Regularity

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Abstract

A number of philosophers have thought that fair lotteries over countably infinite sets of outcomes are conceptually incoherent by virtue of violating countable additivity. In this article, I show that a qualitative analogue of this argument generalizes to an argument against the conceptual coherence of a much wider class of fair infinite lotteries—including continuous uniform distributions. I argue that this result suggests that fair lotteries over countably infinite sets of outcomes are no more conceptually problematic than continuous uniform distributions. Along the way, I provide a novel argument for a weak qualitative, epistemic version of regularity.

1. Introduction

Is a fair lottery over a countably infinite set of outcomes conceptually coherent? Philosophers of probability have been divided on the question since at least as far back as when de Finetti (1972) answered it in the affirmative. The standard argument *against* the conceptual coherence of such lotteries is that any probability function that might be thought to represent such a lottery violates *countable additivity*. Countable additivity is the following constraint on a given probability function P:

Countable additivity. Let $\{A_1, A_2, \ldots\}$ be a countably infinite set of mutually incompatible propositions. Then,

$$P(A_1 \cup A_2 \cup \ldots) = P(A_1) + P(A_2) + \ldots$$

It is straightforward to show that any probability function that treats each member of a countably infinite set of outcomes as equiprobable violates countable additivity. However, many philosophers have thought that violations of countable additivity lead to intuitively undesirable consequences. For example, any probability function that violates countable additivity is *nonconglomerable*: the probability of some proposition, conditional on every member of a particular partition, is bounded between two values, yet the unconditional probability of that proposition lies outside those values (see Kadane, Schervish, and Seidenfeld 1986). Moreover, if one's subjective probability function is nonconglomerable, then one cannot employ decision-theoretic dominance reasoning (see Seidenfeld and Schervish 1983). It has also been argued that if one's subjective probability function violates countable additivity, then one will be subject to a Dutch Book (see Williamson 1999). As a result, many philosophers have regarded fair lotteries over countably infinite sets of outcomes as conceptually incoherent.[1]

My focus in this article is not on the conceptual coherence of fair lotteries over countably infinite sets of outcomes but rather on the conceptual coherence of fair infinite lotteries in general, including fair lotteries over uncountably infinite sets of outcomes. I am especially concerned with what probability theorists call *continuous uniform distributions*. A continuous uniform distribution is a distribution over a given interval of real numbers such that (1) each real number is equiprobable and (2) the probability of any subinterval is proportional to the length of that interval. It is standard in probability theory to represent a continuous uniform distribution by a probability function whose probability density function is simply a constant value. This sort of probabilistic representation finds widespread application in such fields as statistical theory and statistical mechanics. Moreover, unlike probability functions that purportedly represent fair lotteries over countably infinite sets of outcomes, the probability functions that are standardly taken to represent continuous uniform distributions *do* satisfy countable additivity. Accordingly, there has been little dispute concerning the conceptual coherence (and theoretical fruitfulness) of continuous uniform distributions.

In this article, I raise some concerns about the conceptual coherence of continuous uniform distributions. I do so by formulating a qualitative analogue of the standard argument against the conceptual coherence of fair lotteries over countably infinite sets of outcomes. In particular, I develop an argument that any representation of a continuous uniform distribution that is formulated purely in terms of *qualitative* probability—that is, the binary relation of one proposition being at least as probable as another—violates the qualitative analogue of countable additivity known as *monotone continuity*. This argument relies on assuming that any qualitative probability relation satisfies the standard constraints of *nonnegativity*, *additivity*, and *transitivity* as well as a weak qualitative version of *regularity*. The latter principle states that no impossible proposition is at least as possible as some possible proposition. I argue that, *if* one accepts countable additivity and thereby rejects the conceptual coherence of fair lotteries over countably infinite sets of outcomes, *then* one should also accept monotone continuity and thereby reject the conceptual coherence of continuous uniform distributions.

Assuming the Axiom of Choice and that qualitative probability is defined over a σ-algebra, I then generalize this argument to show that any qualitative probability representation of *any* fair infinite lottery that satisfies the aforementioned constraints violates monotone continuity. Thus monotone continuity is violated by all such qualitative probability representations of fair lotteries over countably infinite sets of outcomes, fair lotteries over continuum-sized sets of outcomes, fair lotteries over $\aleph_{17}$-sized sets of outcomes, and so on. In short, the standard argument for rejecting the conceptual coherence of a fair lottery over a countably infinite set of

[1] See Bartha (2004) for further discussion of how countable additivity relates to the conceptual coherence of fair lotteries over countably infinite sets of outcomes.

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outcomes generalizes to an argument for rejecting the conceptual coherence of *any* fair infinite lottery.

The preceding arguments crucially rely on assuming a weak qualitative version of regularity. Because regularity is a controversial principle—and because virtually all extant arguments for regularity have only concerned a *quantitative* version of the principle—I provide a novel argument for a weak *qualitative* version of the principle on which qualitative probability is interpreted as rational comparative confidence. According to the principle I defend, one should not be at least as confident in some epistemically impossible proposition as in some epistemically possible proposition.

The plan for the article is as follows. In section 2, I describe the standard numerical probabilistic representation of continuous uniform distributions in more detail. In section 3, I introduce the formalism of qualitative probability and explain how the constraint of monotone continuity is a qualitative analogue of countable additivity. In section 4, I show that any qualitative probability representation of the continuous uniform distribution that satisfies nonnegativity, additivity, transitivity, and what I call *weak regularity* must violate monotone continuity. In section 5, I prove a generalization of this result for arbitrary fair infinite lotteries. In section 6, I argue for the aforementioned weak qualitative, epistemic version of regularity. I close in section 7 with some remarks about the import of the foregoing discussion.

2. The continuous uniform distribution: its standard numerical representation

The continuous uniform distribution over a given interval of real numbers $[a, b]$—or, more generally, over a given set of outcomes that can be parameterized as such—is the probability distribution that satisfies the following constraints:

(U1). Each real number in $[a, b]$ is equiprobable. (More precisely, each singleton included in $[a, b]$ is equiprobable.)

(U2). The probability of any subinterval of $[a, b]$ is proportional to the length of that subinterval. Equivalently, given subintervals $[a_1, b_1)$ and $[a_2, b_2)$, $[a_1, b_1)$ is $(b_1 - a_1)/(b_2 - a_2)$ times as probable as $[a_2, b_2)$, provided $a_2 \neq b_2$.

The foregoing description may be regarded as a pretheoretic characterization of the continuous uniform distribution over $[a, b]$. To understand it, one need only an intuitive grasp of the notions of *equiprobability* and *probability ratio* (as in one proposition being *twice* as probable as another proposition). Nonetheless, this distribution may be mathematically represented in multiple ways. I now describe its standard representation in terms of real-valued numerical probability. In section 4, I describe how it can be represented in terms of qualitative probability.

The standard numerical representation of the continuous uniform distribution involves a real-valued numerical probability function P whose probability density function f is as follows:

$$f(x) = \begin{cases} \frac{1}{b-a} & \text{if } a \leq x \leq b, \\ 0 & \text{otherwise.} \end{cases}$$

For simplicity, I focus on the case in which $a = 0$ and $b = 1$ in what follows, and I call this case "the" continuous uniform distribution. In this case, for any $c, d \in [0, 1]$,

$P(\{c\}) = P(\{d\}) = 0$. So, P satisfies (U1). Additionally, the probability of a given subinterval $[a_1, b_1) \subseteq [0, 1]$ is simply given by $P([a_1, b_1)) = b_1 - a_1$. So, given subinterval $[a_2, b_2) \subseteq [0, 1]$, $\frac{P([a_1,b_1))}{P([a_2,b_2))} = \frac{b_1-a_1}{b_2-a_2}$, provided $a_2 \neq b_2$. As a result, P satisfies (U2) as well. Hence it is appropriate to call P a "representation" of the continuous uniform distribution.

It is customary in probability theory to define P not only for subintervals of $[0, 1]$ but for various other subsets as well (namely, the "measurable" subsets). The standard way to do so is to have P assign every Borel set in $[0, 1]$ its Lebesgue measure (see Billingsley 1995, chap. 3). The measure-theoretic details of this procedure do not matter for our purposes, but it yields a number of probability assignments that one would intuitively expect. For example, this procedure yields that $P([0, \frac{1}{2}) \cup [\frac{3}{4}, 1)) = \frac{1}{2} + \frac{1}{4} = \frac{3}{4}$.

It is well known that, when P is extended in the preceding manner, it satisfies countable additivity. In this respect, P is unlike any probability function that might be taken to represent a fair lottery over a countably infinite set of outcomes. Accordingly, there has been little worry among defenders of countable additivity that the probability distribution that P represents—that is, the continuous uniform distribution—is conceptually coherent. Later, once we consider a qualitative representation of the continuous uniform distribution, I will argue that this complacency has been misplaced.

3. Qualitative probability

In what follows, I understand qualitative probability to be the binary relation of one proposition being *at least as probable* as another proposition. I employ the following notation:

- $A \succeq B$: A is at least as probable as B.
- $A \approx B$: A and B are *equiprobable*.
- $A \succ B$: A is strictly *more* probable than B.

It is standard to define $\approx$ and $\succ$ in terms of $\succeq$, as follows:

- $A \approx B$ if and only if $A \succeq B$ and $B \succeq A$.
- $A \succ B$ if and only if $A \succeq B$ and $B \not\succeq A$.

In what follows, I assume the preceding definition of $\approx$ but not that of $\succ$, as the latter is more controversial.[2]

The central object of formal theories about qualitative probability is a *qualitative probability structure* $\langle \Omega, \mathcal{F}, \succeq \rangle$, which consists of a nonempty set Ω of possible outcomes, an algebra $\mathcal{F}$ on Ω, and a binary relation $\succeq$ on $\mathcal{F}$. In what follows, I generally focus on just a given qualitative probability relation, though it is important to bear in mind that every such relation is also associated with some Ω and some $\mathcal{F}$. As is standard, I also treat propositions as being members of $\mathcal{F}$.

[2] See Konek (2019, note 4) for an argument against the latter definition when qualitative probability is interpreted as an agent's comparative confidence relation.

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A number of axiomatizations of qualitative probability have been developed in the literature (for a survey, see Krantz et al. 1971, chap. 5). I do not assume any particular axiomatization in what follows, but I do assume that any qualitative probability relation $\succeq$ satisfies at least the following constraints for any propositions A, B, C:

Nonnegativity. $A \succeq \emptyset$.
Additivity. Suppose $(A \cap C) = (B \cap C) = \emptyset$. Then, $A \succeq B$ if and only if $(A \cup C) \succeq (B \cup C)$.
Transitivity. If $A \succeq B$ and $B \succeq C$, then $A \succeq C$.
Weak regularity. If $A \neq \emptyset$, then $\emptyset \not\succeq A$.

Note that, if we assume the preceding definition of $\succ$, then weak regularity is a consequence of the following:

Regularity. If $A \neq \emptyset$, then $A \succ \emptyset$.

That said, I do not assume regularity in what follows. I also do not assume that $\succeq$ satisfies the following:

Totality. $A \succeq B$ or $B \succeq A$.

Nonnegativity, additivity, and transitivity are entailed by all major axiomatizations of qualitative probability. Weak regularity, regularity, and totality are entailed by some axiomatizations, while others are silent on them.[3] Because weak regularity is the most controversial constraint I will assume, I will say much more about it later.

Historically, the main interest in studying qualitative probability has been to prove various "representation theorems" (for a survey of such theorems, see Fishburn 1986). These are theorems that state conditions under which a given qualitative probability relation can be "represented by" some numerical probability function. More precisely, these theorems state conditions under which, for a given qualitative probability relation $\succeq$, there exists some numerical probability function P such that $A \succeq B$ if and only if $P(A) \geq P(B)$.[4] Although I am not primarily concerned with representation theorems in this article, one particular class of representation theorems is worth noting in the present context. These are representation theorems that state conditions under which a given qualitative probability relation can be represented by a *countably additive* probability function. Most notably, Villegas (1964) shows that a necessary condition for a qualitative probability relation $\succeq$ to be representable by a countably additive probability function is that $\succeq$ satisfies the following constraint:

Monotone continuity. Consider propositions $A_1, A_2, \ldots$, and B. Suppose

(i) $A_1 \subseteq A_2 \subseteq \ldots,$
(ii) $A = \cup_i A_i$,

[3] For example, regularity and totality are stated as axioms of qualitative probability by de Finetti (1937), but only the former is a consequence of Koopman's (1940) axiomatization.

[4] Strictly speaking, not all representation theorems have exactly this character, though the strongest such theorems do.

(iii) $B \succeq A_i$ for all i.
Then, $B \succeq A$.[5]

Additional representation theorems have revealed further connections between monotone continuity and countable additivity (see Chuaqui and Malitz 1983; Chateauneuf and Jaffray 1984; Schwarze 1989). Such theorems generally assume that qualitative probability satisfies nonnegativity, additivity, transitivity, *and* weak regularity.[6] This fact suggests that monotone continuity is, in some sense, a qualitative analogue of countable additivity when these constraints are assumed.

Indeed, there is more to say about the connection between monotone continuity and countable additivity than merely that they both appear in certain representation theorems. Consider the following constraint on a given numerical probability function P:

Continuity. Consider propositions $A_1, A_2, \ldots$ Suppose

(i) $A_1 \subseteq A_2 \subseteq \ldots$
(ii) $A = \cup_i A_i$.
Then, $P(A) = \lim_{i \to \infty} P(A_i)$.

Continuity is sometimes included in the axiomatization of numerical probability,[7] and indeed, countable additivity entails continuity. Moreover, note that continuity entails a numerical version of monotone continuity. In particular, suppose that P satisfies continuity and that $P(B) \geq P(A_i)$ for all i, for suitable A_i and B. Then, passing to the limit, it follows that $P(B) \geq P(A)$—which is just a numerical version of the consequent of monotone continuity. Thus, if P is a countably additive probability function, then any qualitative probability relation that P represents satisfies monotone continuity.

The foregoing considerations suggest an argument for a conditional conclusion: if we should accept countable additivity as a constraint on numerical probability, then we should accept monotone continuity as a constraint on qualitative probability. This conclusion follows if we assume that any countably additive probability function indeed represents some qualitative probability relation. However, even if we reject this assumption, it is hard to see why we should accept countable additivity yet reject monotone continuity.[8] As we just saw, continuity entails a structurally analogous numerical version of monotone continuity. So, it seems that monotone continuity at least partly captures the intuitive motivation for accepting continuity. Moreover, because countable additivity entails continuity, it seems that monotone continuity at least partly captures the intuitive motivation for accepting countable additivity as well. These considerations are by no means decisive, but prima facie,

[5] Here I follow Fishburn's (1986) formulation of monotone continuity, which does not require $\mathcal{F}$ to be a σ-algebra. Also, following most authors, I take $A_1, A_2, \ldots$ to be a countably infinite sequence here.

[6] Indeed, I am not aware of any such theorem that does not assume weak regularity.

[7] For example, Kolmogorov's (1950) theory includes an equivalent axiom.

[8] Note that the countably additive probability function P from section 2 violates this assumption if weak regularity is assumed. For example, $P(\{0\}) = P(\{0.5\}) = P(\{0, 0.5\}) = 0$, so $P(\{0\}) \geq P(\{0, 0.5\})$. If P represents some qualitative probability relation $\succeq$, then $\{0\} \succeq \{0, 0.5\}$. However, by weak regularity, $\emptyset \not\succeq \{0.5\}$. So, by additivity, $\{0\} \not\succeq \{0, 0.5\}$.

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it seems there is no reason to accept countable additivity yet reject monotone continuity.

In recent work, DiBella (2018) shows that any qualitative probability relation that satisfies constraints analogous to nonnegativity, additivity, transitivity, and weak regularity and that represents a fair lottery over a countably infinite set of outcomes violates monotone continuity. Given the aforementioned connections between monotone continuity and countable additivity, this result is perhaps unsurprising. After all, it is well known that any numerical probability representation of such a lottery violates countable additivity. What is perhaps more surprising is that any qualitative probability representation of the *continuous uniform distribution* violates monotone continuity as well. I prove this fact in the next section.

4. The continuous uniform distribution and monotone continuity

Let me begin with a clarification. When I speak of a qualitative probability relation that *represents* the continuous uniform distribution, I use the term *represents* not in the sense of the aforementioned representation theorems but rather in the informal sense in which a given mathematical model might be said to represent some collection of pretheoretic ideas. For example, it is plausible that the concept of a *set* in Zermelo–Fraenkel set theory represents at least some pretheoretic conception of *collection* (for an argument to this effect, see Boolos 1971). As I explained in section 2, the standard numerical representation of the continuous uniform distribution readily satisfies the pretheoretic constraints (U1) and (U2) that characterize the distribution. I now explain how these constraints can be satisfied by a qualitative probability relation—and why any such relation violates monotone continuity.

Suppose $\succeq$ is some qualitative probability relation that represents the continuous uniform distribution. Clearly $\succeq$ must satisfy the following constraint:

Fairness. For any $i, j \in [0, 1]$, $\{i\} \approx \{j\}$.

If $\succeq$ satisfies fairness, then (U1) is readily satisfied. Indeed, fairness plausibly encapsulates what it *means* for a probability distribution to be fair, namely, that each outcome is equiprobable.

We might also stipulate that $\succeq$ satisfies the following constraint:

Length ordering. For any $[a_1, b_1), [a_2, b_2) \subseteq [0, 1]$, $[a_1, b_1) \succeq [a_2, b_2)$ if and only if $(b_1 - a_1) \geq (b_2 - a_2)$.

Although length ordering does not explicitly make reference to "probability ratios"—so it is not immediately obvious whether (U2) is satisfied—there are methods of making sense of probability ratios purely in terms of qualitative probability such that (U2) is plausibly satisfied if length ordering is satisfied. In particular, note that any two half-open subintervals of equal length are equiprobable according to length ordering. For example, $[0, 1/4) \approx [1/4, 1/2)$. Because $[0, 1/2)$ is simply the union of these two

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pairwise disjoint subintervals, it seems intuitively plausible that $[0, 1/2)$ is *twice as probable* as each of $[0, 1/4)$ and $[1/4, 1/2)$—which is exactly what (U2) says.[9]

In the appendix, I construct a qualitative probability structure that satisfies nonnegativity, additivity, transitivity, weak regularity, fairness, and length ordering. This construction shows that there indeed exists a qualitative probability representation of the continuous uniform distribution (in the aforementioned sense).

That said, the stipulation that $\succeq$ satisfy length ordering is unnecessary to yield a violation of monotone continuity. It suffices merely that (1) $\succeq$ satisfies nonnegativity, additivity, transitivity, weak regularity, and fairness and that (2) the algebra $\mathcal{F}$ over which $\succeq$ is defined satisfies the following weak "measurability" constraint:

Measurability. $\mathcal{F}$ contains every singleton and half-open interval of the form $[a, b)$ included in $[0, 1]$.

To see how monotone continuity is violated, first note that the following constraint is a simple consequence of nonnegativity and additivity:

Monotonicity. If $A \subseteq B$, then $B \succeq A$.[10]

I will sometimes employ monotonicity in what follows. Here is the main result of this section.

Theorem 1. Consider a qualitative probability structure $\langle [0, 1], \mathcal{F}, \succeq \rangle$. If this structure satisfies nonnegativity, additivity, transitivity, weak regularity, fairness, and measurability, then $\succeq$ violates monotone continuity.

Proof. For every positive integer i, let $p_i = \frac{2^i - 1}{2^i}$ and $A_i = [0, p_i)$. Also, let $A = \cup_i A_i = [0, 1)$ and $B = (0, 1)$. Note that $A_1 \subseteq A_2 \subseteq \ldots$ I now show that $B \succeq A_i$ for every i yet $B \not\succeq A$.

Let i be an arbitrary positive integer. Note that $0 < p_i < 1$. So, $(0, p_i] \subseteq (0, 1)$. Thus, by monotonicity, $(0, 1) \succeq (0, p_i]$. Moreover, by fairness, $\{p_i\} \succeq \{0\}$. So, by additivity, $(0, p_i] = [\{p_i\} \cup (0, p_i)] \succeq [\{0\} \cup (0, p_i)] = [0, p_i)$. Thus, by transitivity, $(0, 1) \succeq [0, p_i)$. That is, $B \succeq A_i$.

Next, by weak regularity, $\emptyset \not\succeq \{0\}$. Thus, by additivity, $\emptyset \cup B \not\succeq \{0\} \cup B = A$. That is, $B \not\succeq A$—in violation of monotone continuity. $\square$

The preceding result suggests that the continuous uniform distribution and fair lotteries over countably infinite sets of outcomes are not as dissimilar as previous writers on fair infinite lotteries have thought. Once we view such lotteries from a

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[9] This is the reasoning Stefánsson (2018) provides to motivate his "ratio principle," according to which event A is twice as probable as event B just in case there is some event C such that (1) $B \approx C$, (2) $B \cap C = \emptyset$, and (3) $A = B \cup C$. Elliott (2020) shows how we can understand rational probability ratio comparisons more generally in terms of qualitative probability via his "general ratio principle." DiBella (forthcoming) generalizes still further to make sense of arbitrary real-valued probability ratios in qualitative terms.

[10] *Proof.* Suppose $A \subseteq B$. Then, $(B \backslash A) \cap A = \emptyset$ and $(B \backslash A) \cup A = B$. By nonnegativity, $(B \backslash A) \succeq \emptyset$. So, by additivity, $[(B \backslash A) \cup A] \succeq (\emptyset \cup A)$. That is, $B \succeq A$.

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qualitative perspective—and, crucially, assume weak regularity—each fares the same with respect to monotone continuity.

Additionally, as I argued in the previous section, if we should accept countable additivity as a constraint on numerical probability, then we should accept monotone continuity as a constraint on qualitative probability. Thus, if one accepts countable additivity and thereby rejects the conceptual coherence of fair lotteries over countably infinite sets of outcomes, then one should accept monotone continuity and thereby reject the conceptual coherence of the continuous uniform distribution. At least, the latter claim follows *if* one should also accept the assumptions of theorem 1. As I said earlier, the most controversial among these is weak regularity. So, whether one should have differing attitudes toward fair lotteries over countably infinite sets of outcomes and the continuous uniform distribution largely hinges on the status of weak regularity. In section 6, I argue that one *should* accept weak regularity—at least, in the special case in which qualitative probability is interpreted as rational comparative confidence. However, before I do that, I will prove a generalization of theorem 1.

5. Arbitrary fair infinite lotteries and monotone continuity

Consider a fair lottery over an arbitrary infinite set Ω of outcomes. Clearly any qualitative probability relation $\succeq$ that represents such a lottery must satisfy the following constraint:

Generalized fairness. For any $\omega, \omega' \in \Omega$: $\{\omega\} \approx \{\omega'\}$.[11]

To yield a violation of monotone continuity in this more general case, I will assume that the algebra $\mathcal{F}$ over which $\succeq$ is defined satisfies the following constraint:

Measurability*. $\mathcal{F}$ is a σ-algebra on Ω that contains every singleton included in Ω.

I now prove the following generalization of theorem 1.

Theorem 2. Consider a qualitative probability structure $\langle \Omega, \mathcal{F}, \succeq \rangle$, where Ω is infinite. Assume the Axiom of Choice. If this structure satisfies nonnegativity, additivity, transitivity, weak regularity, generalized fairness, and measurability*, then $\succeq$ violates monotone continuity.

Proof. Because Ω is infinite, the Axiom of Choice entails that Ω has a countably infinite subset $A = \{\omega_1, \omega_2, \ldots\}$. For every positive integer i, let $A_i = \{\omega_1, \ldots, \omega_i\}$. Also, let $B = A \backslash \{\omega_1\} = \{\omega_2, \omega_3, \ldots\}$. Note that $A_1 \subseteq A_2 \subseteq \ldots$ and $A = \cup_i A_i = \{\omega_1, \omega_2, \ldots\}$. I now show that $B \succeq A_i$ for every i, yet $B \not\succeq A$.

First, let i be an arbitrary positive integer. Note that $(A_{i+1} \backslash \{\omega_1\}) = \{\omega_2, \ldots, \omega_{i+1}\} \subseteq B$. So, by monotonicity, $B \succeq (A_{i+1} \backslash \{\omega_1\})$. Next, by generalized fairness, $\{\omega_{i+1}\} \succeq \{\omega_1\}$. Thus, by additivity, $[\{\omega_{i+1}\} \cup \{\omega_2, \ldots, \omega_i\}] \succeq [\{\omega_1\} \cup \{\omega_2, \ldots, \omega_i\}]$. That is, $(A_{i+1} \backslash \{\omega_1\}) \succeq A_i$. Hence, by transitivity, $B \succeq A_i$.

[11] Given an infinite set Ω of outcomes, Parker (2020) constructs a qualitative probability relation $\succeq$ that satisfies nonnegativity, additivity, transitivity, weak regularity, and generalized fairness. Although Parker does not specify the algebra $\mathcal{F}$ on which $\succeq$ is defined, it is clear from his construction that $\mathcal{F}$ can be taken to be any algebra on Ω that contains every singleton included in Ω.

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Next, by weak regularity, $\emptyset \not\succeq \{\omega_1\}$. Thus, by additivity, $\emptyset \cup B \not\succeq \{\omega_1\} \cup B = A$. That is $B \succeq A$—in violation of monotone continuity.[12] □

Although this result is not a full generalization of theorem 1—because it requires qualitative probability to be defined over a σ-algebra—it does show that violations of monotone continuity are quite typical among qualitative probability representations of fair infinite lotteries (if the Axiom of Choice is true). As with theorem 1, the key assumption being made here about qualitative probability is that it satisfies weak regularity. I now defend weak regularity.

6. In defense of weak regularity

Although the qualitative constraints of regularity and weak regularity have not received much attention in recent philosophical literature, the following constraint on any *numerical* probability function P has:

> **Quantitative regularity.** If A is possible, then $P(A) > 0$ (see, e.g., Williamson 2007; Hájek, forthcoming; Pruss 2013b; Easwaran 2014; Howson 2017; Benci, Horsten, and Wenmackers 2018; Parker 2019).

In what follows, I refer to the qualitative constraints as *qualitative regularity* and *weak qualitative regularity* to distinguish them from *quantitative regularity*.[13]

Different versions of quantitative regularity vary with respect to different types of modality as well as different types of probability. For example, the following is an epistemic version of quantitative regularity:

> **Quantitative epistemic regularity.** If A is epistemically possible for agent S, then S's credence function P should be such that $P(A) > 0$.

One can also consider versions of quantitative regularity that are formulated with respect to physical possibility (and objective chance), logical possibility (and logical probability), and so on (for further discussion, see Hájek, forthcoming, sec. 2).

Although quantitative regularity may seem to be an intuitively plausible principle, several objections have been leveled against it in cases in which a given numerical probability function is defined over an infinite set of outcomes (for a review of such objections, see Benci, Horsten, and Wenmackers 2018, sec. 4). Though fewer objections have been leveled specifically against qualitative regularity or weak qualitative regularity, perhaps the most prominent such objection is Williamson's (2007) "infinite coin toss" argument.[14] That said, I will not defend weak qualitative regularity by offering new criticisms of Williamson's argument here (for criticisms of Williamson's argument, see Weintraub 2008; Howson 2017). Instead, I will defend weak qualitative

[12] For an alternative proof of this result (formulated somewhat differently), see Villegas's (1964) lemma 1 and its corollary. Note that Villegas implicitly assumes that every infinite set has a countably infinite subset, which is entailed by the Axiom of Choice.

[13] Note that there is no distinctive quantitative analogue of weak qualitative regularity: if $0 \not\succeq P(A)$, then $P(A) > 0$.

[14] Williamson argues against quantitative regularity, qualitative regularity, *and* weak qualitative regularity. For an additional argument against weak qualitative regularity, see Pruss (2013a).

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regularity by offering a new *argument* for it. Then, I will discuss the import of my argument to Wiliamson's.

By providing an argument for weak qualitative regularity, I also aim to fill a notable gap in the literature, namely, that virtually all extant pro-regularity arguments have concerned quantitative regularity rather than either of its qualitative counterparts.[15] Nonetheless, as I later explain, it is perfectly consistent to accept weak qualitative regularity (or even qualitative regularity) yet to reject quantitative regularity.

6.1 In defense of weak qualitative epistemic regularity

In this section, I defend a specific *epistemic* version of weak qualitative regularity. It is not the only conceivable epistemic interpretation of the principle, but I argue that it is one that holds independent philosophical interest.

Let $\langle \Omega, \mathcal{F}, \succeq \rangle$ be a given qualitative probability structure. When such a structure is taken to represent some scenario, it is standard to regard the collection of "possibilities" (relevant to the scenario at hand) as being represented by Ω, any possible proposition as being represented by a nonempty subset of Ω, and any impossible proposition as being represented by the empty set. Thus a less representation-laden formulation of weak qualitative regularity is as follows:

> **Weak qualitative regularity.** No impossible proposition is at least as probable as some possible proposition.

Now, just as we may consider different versions of quantitative regularity, so we may consider different versions of the preceding principle.

For the purposes of this section, I take $\succeq$ to represent the attitudes of *comparative confidence* that a given agent S should have. That is, I interpret $A \succeq B$ as meaning that S should be *at least as confident* in A as in B. Additionally, I take Ω to represent the collection of "epistemic possibilities" for S (more on this soon) and $\mathcal{F}$ to represent the collection of propositions toward which S has attitudes of comparative confidence. These interpretations lead to the following epistemic version of weak qualitative regularity:

> **Weak qualitative epistemic regularity.** For any epistemically possible A and epistemically impossible B, it is *not* the case that one should be at least as confident in B as in A.[16]

[15] The only argument for either qualitative regularity or weak qualitative regularity of which I am aware is Koopman's (1940) proof of his theorem 1, part of which can be interpreted as a version of qualitative regularity. However, while the mathematical legitimacy of Koopman's result is unassailable, it is difficult to extract a precise philosophical interpretation of his result. For example, Koopman's discussion leaves open what exactly Ω ("1" in Koopman's terminology)—and, hence, $\emptyset$ in the statement that $A \succ \emptyset$—represents. Presumably, it is some collection of epistemic possibilities for a given agent, though Koopman does not clarify what such possibilities amount to. I will make clarifications of this sort in my argument for weak qualitative regularity, and the formal structure of my argument will be somewhat different from Koopman's.

[16] I take the operative sense of "should" here, and in what follows, to be that of epistemic rationality.

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My argument for weak qualitative epistemic regularity relies on assuming that the attitudes of comparative confidence an agent should adopt satisfy additivity and that the following "quasi-definitional" claims are true:

(A1). A is epistemically possible for one if and only if it is not the case that one should be certain in $\neg A$.

(A2). If one should be certain in A and one should be at least as confident in B as in A, then one should be certain in B.

I call these assumptions "quasi-definitional" because I regard (A1) simply as a theoretically fruitful stipulative definition and (A2) as being all but analytic. I defend these assumptions shortly. However, before I do so, it is worth clarifying how I am interpreting qualitative probability in the present epistemic context.

When I say that $A \succeq B$ means that one should be at least as confident in A as in B, I do not assume that there is a *unique* rationally permissible comparative confidence relation for one to adopt. Rather, $\succeq$ should be understood as representing that unique relation that contains all and only those attitudes of comparative confidence that are *common* to the comparative confidence relations that are rationally permissible for one to adopt. Note that this interpretation of qualitative probability has distinctive consequences for how the constraints of qualitative probability are to be interpreted as rationality requirements. In particular, on this interpretation, the claim that $\succeq$ satisfies additivity amounts to the following:

Epistemic additivity. Suppose $(A \cap C) = (B \cap C) = \emptyset$. One should be at least as confident in A as in B if and only if one should be at least as confident in $(A \cup C)$ as in $(B \cup C)$.

Note that this principle is a "doubly narrow-scope" normative interpretation of additivity: the "should" scopes twice, narrowly, over the antecedent and the consequent. That said, this is not the only conceivable interpretation of additivity as a rationality requirement. We can also consider the following "wide-scope" interpretation:

Epistemic additivity$_{WS}$. Suppose that $(A \cap C) = (B \cap C) = \emptyset$. One should be such that [one is at least as confident in A as in B if and only if one is at least as confident in $(A \cup C)$ as in $(B \cup C)$].

More generally, to extract an alternative normative interpretation of qualitative probability, we can interpret $\succeq$ as an actual agent's comparative confidence relation and then discuss the properties that $\succeq$ *should* satisfy. This approach strikes me as perfectly coherent, and indeed both of the foregoing normative interpretations of additivity strike me as independently plausible. Nonetheless, the normative interpretation of qualitative probability I adopt will simply be more useful in my argument for weak qualitative epistemic regularity.

6.1.1 In defense of (A1) and (A2)

As I said earlier, I regard (A1) simply as a useful stipulative definition. Although the concept of "epistemic possibility" is often understood in terms of *knowledge*, there is

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clearly an analogous concept that involves rational certainty.[17] In particular, I suspect that (A1)—or something close to it—is the conception of epistemic possibility implicitly assumed by most Bayesian epistemologists (and other broadly probabilistic epistemologists) who are concerned with the question of what credences a given agent should adopt. For what are the "possibilities" associated with Ω that are meant to figure in our model of such an agent? A natural answer, for starters, is that the "impossible" propositions are those that the agent should be certain are false. Intuitively, these are propositions that are completely "ruled out." Equivalently, any "possible" proposition included in Ω is one for which it is *not* the case that the agent should be certain it is false. Given that Bayesians tend to theorize primarily in terms of rational certainty (and degrees thereof) rather than knowledge, then, it seems plausible that (A1) is implicitly assumed by many such philosophers. However, even if this sociological hypothesis is false, it still seems quite plausible that (A1) would be a theoretically fruitful definition for many Bayesians to adopt, as it would constitute a convenient item of theoretical vocabulary with which to investigate the question of what credences an agent should adopt. So, it seems reasonable to adopt (A1) at least as a stipulative definition in broadly probabilistic contexts and to explore the consequences of doing so—for example, whether (A1) leads to a plausible version of weak qualitative regularity.

Next, (A2) strikes me as being all but analytic.[18] If it were to turn out to be false, then I would simply have to confess to being unable to understand what the words employed in (A2) meant. That said, it is important to bear in mind that (A2) explicitly concerns *rational doxastic attitudes*—specifically, rationally required attitudes of certainty and comparative confidence. It is not a claim about our mathematical *representations* of such attitudes. In particular, it should not be interpreted under the assumption that the phrase "one should be certain in A" *means* (1) that $A = \Omega$, (2) that one should be equally confident in A and Ω, or (3) that one's credence function P should be such that $P(A) = P(\Omega) = 1$. To be sure, the latter claims are familiar consequences of our standard mathematical representations of "rationally required certainty." However, because they all involve the representation-laden notion of Ω, it is hard to see how any of them could spell out the *meaning* of "rationally required certainty"—at least, not as we might pretheoretically understand the term. Nonetheless, I take it that we do have at least some pretheoretic understanding of the concept *should be certain*, and I regard (A2) merely as expressing a pretheoretic platitude about this concept (as well as that of *should be at least as confident as*). Thus any legitimate assessment of (A2) must ultimately concern the rationally required attitudes of certainty and comparative confidence themselves—not merely our mathematical representations of them. Let me illustrate this point by describing an excessively representation-laden objection to (A2).

[17] Note that (A1) is a rational certainty–based analogue of the "permissive" conception of epistemic possibility described by Gendler and Hawthorne (2002, 3).

[18] I say "all but analytic" rather than "analytic" lest it be worried whether there are any genuinely analytic claims. Nonetheless, I regard (A2) as being epistemically on par with seemingly paradigmatically analytic claims like "All bachelors are unmarried."

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Consider a fair throw of an infinitely thin dart at the interval $[0, 1]$, and let A be the proposition that the dart will *not* land at the point 0.2.[19] For the sake of the present objection, let us assume that quantitative epistemic regularity is false. It then follows that your credence function P should be such that $P(A) = 1$. Because you should have maximal credence in A, one might therefore think that you should be at least as confident in A as in every proposition. In particular, one might think that you should be at least as confident in A as in the proposition that the dart will land *somewhere* in $[0, 1]$—which, by assumption, you should be certain in. By (A2), it then follows that you should be certain in A. However, it seems clear that this is *not* the case; at best, you should be "almost" certain in A. Hence one might think that (A2) fails.

The problem with this objection is that it smuggles in the representation-theoretic assumption that if you should have $P(p) \geq P(q)$, then you should be at least as confident in p as in q.[20] (This assumption licenses the inference from the claim that you should have $P(A) = 1$ to the claim that you should be at least as confident in A as in every proposition.) However, as is the case with all representation-theoretic claims that connect rational comparative confidence and rational numerical credence, whether this assumption holds depends on what specific properties are satisfied by rational comparative confidence. In particular, whether this assumption holds depends on whether weak qualitative epistemic regularity is true. Indeed, if the latter (along with epistemic additivity) is true, then it is *not* the case that you should be at least as confident in A as in the proposition that the dart will land somewhere in $[0, 1]$. So, even if we grant the falsity of *quantitative* epistemic regularity, the foregoing objection requires the additional assumption that weak *qualitative* epistemic regularity is false—which is precisely to beg the question in the present context.

6.1.2 The argument for weak qualitative epistemic regularity

I now present the argument for weak qualitative epistemic regularity.

Theorem 3. Suppose (A1), (A2), and epistemic additivity are true. Then, weak qualitative epistemic regularity is true.

Proof. Suppose that A is epistemically possible for S and that B is epistemically impossible for S. We need to show that $B \not\succeq A$.

Suppose for reductio that $B \succeq A$. By (A1), S should be certain in $\neg B$. Also, because B is epistemically impossible, $B = \emptyset$. So, $\emptyset \succeq A$. Furthermore, by epistemic additivity, $\neg A \succeq (A \cup \neg A) = \Omega = \neg B$. Thus, by (A2), S should be certain in $\neg A$. However, by (A1), it is not the case that S should be certain in $\neg A$. Contradiction.

6.2 Discussion

I will now make a few remarks about the import of the foregoing argument.

First, I have only argued that we should accept an *epistemic* version of weak qualitative regularity. I have not argued that we should accept weak qualitative regularity

[19] I assume here, for the sake of argument, that such a scenario is conceptually coherent. If that is not the case, then this objection cannot be coherently formulated.

[20] That is, the objection commits what Easwaran (2014) calls the "numerical fallacy."

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in an unqualified manner or that we should accept other versions of weak qualitative regularity (e.g., one formulated in terms of physical possibility and objective chance). It might be that the preceding argument can be modified to support other versions of weak qualitative regularity, but I have not attempted to do so here.

Second, note that weak qualitative epistemic regularity is extremely weak. It only states that it is *not* the case that one should be at least as confident in some epistemically impossible proposition as in some epistemically possible proposition. This is weaker than the claim that one *should* be such that one is not at least as confident in some epistemically impossible proposition as in some epistemically possible proposition, that is, that it is rationally *impermissible* for one to be at least as confident in some epistemically impossible proposition as in some epistemically possible proposition. As before, I leave open whether my argument for weak qualitative epistemic regularity can be modified to support this stronger claim.

Third, let us see how weak qualitative epistemic regularity bears on the epistemic significance of theorem 1. Let us first suppose that monotone continuity, nonnegativity, additivity, and transitivity are indeed constraints on rational comparative confidence. Then, theorem 1 admits of at least two epistemic interpretations:

(i) If there is some continuum-sized lottery in whose outcomes one *should* be equally confident, then one should be at least as confident in some epistemically impossible proposition as in some epistemically possible proposition.
(ii) If there is some continuum-sized lottery in whose outcomes it is *permissible* to be equally confident, then it is permissible for one to be at least as confident in some epistemically impossible proposition as in some epistemically possible proposition.[21]

On the first interpretation, the lottery in question is one such that one *should* be equally confident in every epistemically possible outcome (and one should be certain that some such outcome obtains). On the second interpretation, the lottery in question is one such that it is merely *permissible* to be equally confident in every epistemically possible outcome (and one should be certain that some such outcome obtains). Because the consequent of (i)—but not that of (ii)—conflicts with weak qualitative epistemic regularity, my argument for the latter only has bearing on the first interpretation of theorem 1. Thus I do not take the epistemic upshot of theorem 1 to be that it is *impermissible* for one to be equally confident in each outcome of any continuum-sized lottery. Rather, I only take the epistemic upshot of the theorem to be that there is no continuum-sized lottery in whose outcomes one *should* be equally confident. While one might think that the latter claim is prima facie implausible, the claim that there *is* some continuum-sized lottery in whose outcomes one should be equally confident does not quite seem to have the plausibility of the quasi-definitional claims (A1) and (A2) that lead to weak qualitative epistemic regularity. Indeed, the claim that there is some continuum-sized lottery in whose outcomes one should be equally confident seems

[21] Per measurability, I assume on each of these interpretations that one has attitudes of comparative confidence toward every proposition involving such a lottery that can be parameterized as a singleton or half-open interval of the form $[a, b)$ included in $[0, 1]$. Analogous interpretations—involving fair infinite lotteries in general—are also available for theorem 2, provided the Axiom of Choice is true.

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no more prima facie plausible than the claim that there is some *countably* infinite lottery in whose outcomes one should be equally confident—which, of course, the defender of countable additivity must reject. So, if one accepts monotone continuity—and recall that, earlier, I argued that *if* one should accept countable additivity, *then* one should accept monotone continuity—it would seem that the most principled response for the defender of monotone continuity is simply to follow suit by rejecting the claim that there is some continuum-sized lottery in whose outcomes one should be equally confident.

Finally, I have not argued that we should accept any version of *quantitative* regularity. In fact, accepting weak qualitative regularity (or even qualitative regularity) is perfectly compatible with rejecting quantitative regularity. Accepting weak qualitative regularity (or qualitative regularity) yet rejecting quantitative regularity merely amounts to denying that any weakly regular (or fully regular) qualitative probability relation is perfectly representable via numerical probability.[22] Moreover, not all arguments against quantitative regularity can be straightforwardly transformed into arguments against qualitative regularity or weak qualitative regularity. For example, Hájek's (forthcoming) and Pruss's (2013b) "cardinality" arguments against quantitative regularity have no obvious qualitative analogue, as the arguments turn essentially on the relation between the cardinality of the set of possible outcomes and the cardinality of the range of values of numerical probability functions. However, qualitative probability relations have no "range" of numerical values. Nonetheless, Williamson (2007) argues against quantitative regularity *and* its qualitative counterparts. So, I now turn specifically to his argument against weak qualitative regularity.[23]

6.3 Williamson's argument against weak qualitative regularity

Williamson formulates his arguments against the various regularity principles as being against all versions of these principles—including physical and epistemic versions.[24] In each argument, the key premise that Williamson employs is the following:

> **Isomorphism principle.** If two events are isomorphic (in the relevant sense), then they are equiprobable.[25]

[22] That said, my argument for weak qualitative epistemic regularity can be transformed into an argument for quantitative epistemic regularity if we add the assumption that one's credence function P always represents one's comparative confidence relation $\succeq$ (in the sense described in section 3). Then, because it is not the case that $\succeq$ should be such that $\emptyset \succeq A$ for any epistemically possible A, it is not the case that P should be such that $P(\emptyset) \geq P(A)$. So, P should be such that $P(A) > P(\emptyset) = 0$. By the same lights, this representational assumption also leads to an argument for quantitative epistemic regularity if qualitative epistemic regularity is true. However, to satisfy quantitative regularity, one generally needs to allow probability functions to take on infinitesimal values in addition to real values (see Benci, Horsten, and Wenmackers 2018). So, if one has independent objections to infinitesimal probability, but not to qualitative probability, one may embrace weak qualitative regularity—or even qualitative regularity—while rejecting quantitative regularity.

[23] Williamson's rejection of weak qualitative regularity is the "similar conclusion" he reaches at step 17 (2007, 178).

[24] Williamson: "Henceforth, we need not specify what kind of probability is in play, because the argument [against regularity] is the same for all kinds" (2007, 174).

[25] Here I follow the terminology of Parker's (2019) reconstruction of Williamson's argument.

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Construed epistemically, the principle becomes the following:

> **Epistemic isomorphism principle.** If two events are isomorphic (in the relevant sense), then one should be equally confident in them.

An example will serve to clarify the "relevant sense" of isomorphism here. Suppose that a fair coin is tossed infinitely many times at one-second intervals, starting at some time t, where each toss is independent of the others. Let $H(1\ldots)$ be the event that the coin comes up heads on teach toss, and let $H(2\ldots)$ be the event that the coin comes up heads each time from one second after t. Williamson regards these events as relevantly isomorphic, because "we can map the constituent single-toss events of $H(1\ldots)$ one-one onto the constituent single-toss events of $H(2\ldots)$ in a natural way that preserves the physical structure of the set-up just by mapping each toss to its successor" (2007, 175). So, by the epistemic isomorphism principle, one should be equally confident in $H(1\ldots)$ and $H(2\ldots)$.

To spell out Williamson's argument against weak qualitative regularity, let $H(1)$ be the event that the coin lands heads on the first toss. Because the tosses are fair and mutually independent, $[\neg H(1) \cap H(2\ldots)] \succeq H(1\ldots)$. Moreover, by the isomorphism principle, $H(1\ldots) \succeq H(2\ldots)$. So, by transitivity, $[\neg H(1) \cap H(2\ldots)] \succeq H(2\ldots)$. Equivalently, $[\neg H(1) \cap H(2\ldots)] \succeq [H(1\ldots) \cup (\neg H(1) \cap H(2\ldots)]$. Thus, by additivity, $\emptyset \succeq H(1\ldots)$—in violation of weak qualitative regularity. Because Williamson intends his argument to apply to any version of weak qualitative regularity, it constitutes an argument against weak qualitative epistemic regularity when '$\succeq$' is interpreted as rationally required comparative confidence.

Let us now compare the assumptions of Williamson's argument with the assumptions of my argument for weak qualitative epistemic regularity. Both arguments assume an epistemic version of additivity. Williamson's argument additionally assumes the epistemic isomorphism principle and an epistemic version of transitivity, and my argument additionally assumes (A1) and (A2). Because the two arguments yield inconsistent conclusions, we cannot rationally embrace all these assumptions. For the sake of the present discussion, let us assume that we should accept the uncontroversial additivity and transitivity. Should we also accept Williamson's epistemic isomorphism principle or the pair of (A1) and (A2)?

I have already defended (A1) and (A2), so let us consider the epistemic isomorphism principle. In his elaboration of Williamson's argument, Parker (2019) motivates the epistemic isomorphism principle via the following physical construal of the isomorphism principle:

> **Physical isomorphism principle.** If two events are isomorphic, then they are objectively equiprobable (i.e., have the same chance).

Parker argues that this principle—in conjunction with Lewis's (1980) principal principle—implies the epistemic isomorphism principle. Moreover, Parker takes one of the motivating ideas behind the physical isomorphism principle to be that "the chance of an event is determined by the physical laws and local, qualitative circumstances" (2019, 4). However, although this claim has an air of intuitive plausibility, it is also a substantive metaphysical thesis that is far from obviously true. Consider the following competing hypotheses about the nature of chance:

Chance absolutism. Chance is fundamentally a monadic property of events (which may be numerically measured using the resources of probability theory).

Chance comparativism. Chance is fundamentally a binary relation among events and only derivatively a monadic property of individual events. For example, claims of the sort "Event *A* is *more objectively probable than* event *B*" are more fundamental than claims of the sort "The chance of *A* is *x*."

No doubt chance absolutism is, as a matter of sociological fact, more commonly accepted among philosophers and scientists than chance comparativism. Moreover, if we accept chance absolutism, then the claim to which Parker appeals in motivating the physical isomorphism principle appears intuitively plausible. After all, if chance is fundamentally monadic, what could determine *the* chance of *an* event but its local, qualitative circumstances? Nonetheless, there are reasons to doubt chance absolutism. In particular, though he does not discuss chance specifically, Dasgupta (2013) provides a compelling argument for a fully general comparativism about quantity—of which chance comparativism appears to be a special case. However, if we accept chance comparativism, then there seems to be little reason why the chance of an event must be determined entirely by its local, qualitative circumstances. If chance is fundamentally relational, then the chance of some event *A* could conceivably be partly determined by nonlocal circumstances, namely, by comparative chance relations that *A* bears to events not concerning the specific spatiotemporal locations with which *A* is concerned. However, if this is the case, then we seem to lose our motivation for the physical isomorphism principle and, by extension, the epistemic isomorphism principle.

All of that said, I do not claim that chance comparativism is true and that chance absolutism is false. Rather, my point is that the epistemic isomorphism principle seems to inherit its intuitive plausibility partly from the nontrivial metaphysical claim that chance absolutism is true—at least, if we follow Parker's approach in motivating the epistemic isomorphism principle. By contrast, as I have argued, (A1) is merely a fruitful stipulative definition, and (A2) is all but analytic. They are not substantive metaphysical claims, nor are they even substantive claims about the requirements of epistemic rationality (as, e.g., epistemic additivity is). As such, they appear to be on firmer footing than that which Parker has provided for the epistemic isomorphism principle.

Nonetheless, even if we don't take Parker's detour through chance, accepting the epistemic isomorphism principle still seems to presuppose that how confident one should be in the occurrence of a physical event is entirely determined by intrinsic features of that event and not at all by that event's relations to *other* events. As with chance, this is a nontrivial assumption to make—in this case, about the requirements (or grounds) of epistemic rationality. However, it is difficult to see how this assumption could follow from stipulative definitions like (A1) or claims with comparable epistemic status to all-but-analytic claims like (A2). So, if we had to choose between the epistemic isomorphism principle and the pair of (A1) and (A2), my bet would be firmly on the latter—and, thus, on weak qualitative epistemic regularity.

7. Conclusion

I have shown that a qualitative analogue of the standard argument against the conceptual coherence of fair lotteries over countably infinite sets of outcomes generalizes to an argument against the conceptual coherence of a much wider class of fair infinite lotteries—including the continuous uniform distribution. As we saw, the crucial assumption of this argument was that qualitative probability satisfies weak qualitative regularity. So, along the way, I provided a new argument for an epistemic version of weak qualitative regularity. I will close by making a few remarks about the import of the foregoing discussion.

First, because the standard numerical representation of the continuous uniform distribution is employed in various statistical and scientific applications, it might be worried whether the generalized argument against its conceptual coherence threatens the legitimacy of this representation for one who accepts countable additivity. I do not believe that it does. Though the arguments of this article have bearing on the philosophical interpretation of this representation, they need not affect its statistical or scientific utility. One can continue to accept countable additivity yet hold that the standard numerical representation of the continuous uniform distribution is merely a useful mathematical tool that does not perfectly represent anything in the real world (or in the world of ideally rational epistemic agents). Indeed, many theoretically fruitful mathematical representations have such a character—physical theories that involve mathematical singularities, theories that represent discrete quantities as continuous, and so on. If one wishes to defend countable additivity, then one need only add the standard numerical representation of the continuous uniform distribution to this long list.

Second, I have not argued that one *should* reject the conceptual coherence of the continuous uniform distribution. I have only argued that, *if* one rejects the conceptual coherence of fair lotteries over countably infinite sets of outcomes on grounds of countable additivity, *then* one should reject the conceptual coherence of the continuous uniform distribution on grounds of monotone continuity. However, I have argued neither that one should accept countable additivity as a constraint on numerical probability nor that one should accept monotone continuity as a constraint on qualitative probability. Whether one should indeed accept these constraints is a question beyond the scope of this article.

Finally, I have only discussed how one allegedly problematic feature of fair lotteries over countably infinite sets of outcomes—namely, the failure of countable additivity—generalizes to arbitrary fair infinite lotteries. I have not discussed whether, and how, other such features generalize to arbitrary fair infinite lotteries. For example, I have not discussed whether qualitative probability relations that represent arbitrary fair infinite lotteries are susceptible to a qualitative analogue of nonconglomerability or any decision-theoretic defects (cf. section 1). Nonetheless, at least with respect to the former, there are reasons to suspect that they are. In particular, assuming standard axioms of qualitative *conditional* probability, DiBella (2018) shows that any qualitative conditional probability relation that represents a fair lottery over a countably infinite set of outcomes must be nonconglomerable (in a precise qualitative sense). Because (assuming the Axiom of Choice) any fair infinite lottery includes a fair countably infinite sublottery, it seems reasonable to suspect more generally that any qualitative probability relation that represents a fair infinite lottery must also be nonconglomerable. That said, a detailed discussion of qualitative nonconglomerability requires considering the nuances of qualitative *conditional* probability—and not, as I have done in this article, merely qualitative *unconditional* probability—so I cannot say more about it here. Nonetheless, these issues merit further investigation.

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Appendix: Qualitative representation of the continuous uniform distribution

Here I construct a qualitative probability structure $\langle \Omega, \mathcal{F}, \succeq \rangle$ that represents the continuous uniform distribution in the sense that it satisfies nonnegativity, additivity, transitivity, weak regularity, measurability, fairness, and length ordering.

First, let $\Omega = [0,1]$, and let $\mathcal{F}$ be the set that contains every subset A of Ω of the following form for nonnegative integers l, m, n:

$$\begin{aligned} A &= (A_1 \backslash A_2) \cup A_3 \\ &= (([a_1, b_1] \cup \ldots \cup [a_l, b_l]) \backslash \{c_1, \ldots, c_m\}) \cup \{d_1, \ldots, d_n\}, \end{aligned}$$

where $A_1 = ([a_1, b_1] \cup \ldots \cup [a_l, b_l])$, $A_2 = \{c_1, \ldots, c_m\} \subseteq A_1$, $A_3 = \{d_1, \ldots, d_n\}$, $a_i \neq b_i$, $[a_i, b_i] \cap [a_j, b_j] = \emptyset$ if $i \neq j$, and $(A_1 \cap A_3) = \emptyset$. (If $l = 0$, $m = 0$, or $n = 0$, then A_1, A_2, or A_3 is empty, respectively.) It is straightforward to show that $\mathcal{F}$ is an algebra on Ω and that every member of $\mathcal{F}$ admits of a unique decomposition of the foregoing form. Clearly $\mathcal{F}$ satisfies measurability.

Next, given A's decomposition of the preceding form, let $L(A) = (b_1 - a_1) + \ldots + (b_l - a_l)$ and $N(A) = l - m + n$ if $A \neq \emptyset$; if $A = \emptyset$, let $L(A) = N(A) = 0$. Also, for any $A, B \in \mathcal{F}$, let $A \succeq B$ if and only if one of the following conditions holds:

(i) $L(A) > L(B)$
(ii) $L(A) = L(B)$ and $N(A) \geq N(B)$.

It is straightforward (but tedious) to verify that $\succeq$ satisfies nonnegativity, additivity, transitivity, weak regularity, fairness, and length ordering. For illustration, I only demonstrate that $\succeq$ satisfies additivity here; the other cases are similar.

Suppose $(A \cap C) = (B \cap C) = \emptyset$. Clearly $L(A \cup C) = L(A) + L(C)$, $L(B \cup C) = L(B) + L(C)$, $N(A \cup C) = N(A) + N(C)$, and $N(B \cup C) = N(B) + N(C)$. For the left-to-right direction, suppose $A \succeq B$. There are two cases to consider: (1) $L(A) > L(B)$ or (2) $L(A) = L(B)$ and $N(A) \geq N(B)$. In each case, it readily follows that $(A \cup C) \succeq (B \cup C)$. For the right-to-left direction, suppose $(A \cup C) \succeq (B \cup C)$. There are two cases to consider: (1) $L(A \cup C) > L(B \cup C)$ or (2) $L(A \cup C) = L(B \cup C)$ and $N(A \cup C) \geq N(B \cup C)$. In case 1, it follows that $L(A) > L(B)$. In case 2, it follows that $L(A) = L(B)$ and $N(A) \geq N(B)$. Thus, in each case, it follows that $A \succeq B$.

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