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STRONG CONVERGENCE OF SOME ALGORITHMS FOR λ-STRICT PSEUDO-CONTRACTIONS IN HILBERT SPACE

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Abstract

Two algorithms have been constructed for finding the minimum-norm fixed point of a λ-strict pseudo-contraction T in Hilbert space. It is shown that the proposed algorithms strongly converge to the minimum-norm fixed point of T.

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1. Introduction

Let C be a nonempty closed convex subset of a real Hilbert space H. Recall that a mapping $T : C \to C$ is said to be strictly pseudo-contractive if there exists a constant $0 \le \lambda < 1$ such that

$$\|Tx - Ty\|^2 \le \|x - y\|^2 + \lambda\|(I - T)x - (I - T)y\|^2, \quad \forall x, y \in C. \tag{1.1}$$

In such a case we also say that T is a λ-strictly pseudo-contractive mapping. It is clear that, in a real Hilbert space H, (1.1) is equivalent to

$$\langle Tx - Ty, x - y\rangle \le \|x - y\|^2 - \frac{1 - \lambda}{2}\|(I - T)x - (I - T)y\|^2, \quad \forall x, y \in C. \tag{1.2}$$

We use $\mathrm{Fix}(T)$ to denote the set of fixed points of T.

It is clear that the class of strictly pseudo-contractive mappings strictly includes the class of nonexpansive mappings, which are mappings T on C such that

$$\|Tx - Ty\| \le \|x - y\|, \quad \forall x, y \in C.$$

Iterative methods for nonexpansive mappings have been extensively investigated in the literature; see [1, 8, 9, 16, 19, 23, 25, 33, 35, 37] and the references therein. Related

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work can be found in [2–7, 10–15, 17, 18, 20–22, 24, 26–32, 34, 36]. However, iterative methods for strictly pseudo-contractive mappings are far less developed than those for nonexpansive mappings, although Browder and Petryshyn [2] initiated their work in 1967. Strictly pseudo-contractive mappings have more powerful applications than nonexpansive mappings in solving inverse problems (see Scherzer [21]), so it is of interest to develop algorithms for strictly pseudo-contractive mappings.

On the other hand, in many problems, it is required to find a solution with minimum norm. In an abstract way, we may formulate such problems as finding a point $x^\dagger$ with the property

$$x^\dagger \in C \quad \text{and} \quad \|x^\dagger\|^2 = \min_{x\in C} \|x\|^2,$$

where C is a nonempty closed convex subset of a real Hilbert space H. In other words, $x^\dagger$ is the (nearest point or metric) projection of the origin onto C,

$$x^\dagger = P_C(0),$$

where P_C is the metric (or nearest point) projection from H onto C. A typical example is the least-squares solution to the constrained linear inverse problem

$$\begin{cases} Ax = b, \\ x \in C, \end{cases}$$

where A is a bounded linear operator from H to another real Hilbert space H_1 and b is a given point in H_1. Related work for finding the minimum-norm solution (or fixed point) has been considered by some authors; see [7, 11, 30–32, 34].

In the present paper, two algorithms have been constructed for finding the minimum-norm fixed point of a λ-strict pseudo-contraction T in Hilbert space. It is shown that the proposed algorithms strongly converge to the minimum-norm fixed point of T.

2. Preliminaries

Let C be a nonempty closed convex subset of H. For every point $x \in H$, there exists a unique nearest point in C, denoted by $P_C x$, such that

$$\|x - P_C x\| \le \|x - y\|, \quad \forall y \in C.$$

The mapping P_C is called the metric projection of H onto C. It is well known that P_C is a nonexpansive mapping and is characterised by the following property:

$$\langle x - P_C x, y - P_C x\rangle \le 0, \quad \forall x \in H,\ y \in C. \tag{2.1}$$

In order to prove our main results, we need the following well-known lemmas.

LEMMA 2.1. *Let C be a nonempty, closed and convex subset of a real Hilbert space H. Let $T : C \to C$ be a λ-strictly pseudo-contractive mapping. Then $I - T$ is demi-closed at 0, that is, if $x_n \rightharpoonup x \in C$ and $x_n - Tx_n \to 0$, then $x = Tx$.*

LEMMA 2.2. *Assume that* $\{a_n\}$ *is a sequence of nonnegative real numbers such that* $a_{n+1} \le (1-\gamma_n)a_n + \gamma_n\delta_n$, $n \ge 0$, *where* $\{\gamma_n\}$ *is a sequence in* $(0, 1)$ *and* $\{\delta_n\}$ *is a sequence in* $\mathbb{R}$ *such that:*

(i) $\sum_{n=0}^{\infty} \gamma_n = \infty$;
(ii) $\limsup_{n\to\infty} \delta_n \le 0$ *or* $\sum_{n=0}^{\infty} |\delta_n\gamma_n| < \infty$.

Then $\lim_{n\to\infty} a_n = 0$.

3. Main results

Let C be a nonempty closed convex subset of a real Hilbert space H. Let $T : C \to C$ be a λ-strict pseudo-contraction. Let $k \in (0, 1-\lambda)$ be a constant. For each $t \in (0, 1)$, we consider the mapping T_t given by

$$T_t x = P_C((1-k-t)x + kTx), \quad \forall x \in C.$$

It is easy to check that $T_t : C \to C$ is a contraction for a small enough t. As a matter of fact, from (1.1) and (1.2),

$$\begin{aligned} \|T_t x - T_t y\|^2 &= \|P_C((1-k-t)x + kTx) - P_C((1-k-t)y + kTy)\|^2 \\ &\le \|(1-k-t)(x-y) + k(Tx-Ty)\|^2 \\ &= (1-k-t)^2\|x-y\|^2 + k^2\|Tx-Ty\|^2 \\ &\quad + 2(1-k-t)k\langle Tx-Ty, x-y\rangle \\ &\le (1-k-t)^2\|x-y\|^2 + k^2(\|x-y\|^2 + \lambda\|(I-T)x-(I-T)y\|^2) \\ &\quad + 2(1-k-t)k\Big(\|x-y\|^2 - \frac{1-\lambda}{2}\|(I-T)x-(I-T)y\|^2\Big) \\ &= (\lambda k^2 - (1-\lambda)(1-k-t)k)\|(I-T)x-(I-T)y\|^2 + (1-t)^2\|x-y\|^2 \\ &= k(k-(1-t)(1-\lambda))\|(I-T)x-(I-T)y\|^2 + (1-t)^2\|x-y\|^2. \end{aligned} \tag{3.1}$$

We can choose a small enough t such that $k \le (1-t)(1-\lambda)$. Then, from (3.1),

$$\|T_t x - T_t y\| \le (1-t)\|x-y\|, \quad \forall x, y \in C, \tag{3.2}$$

which implies that T_t is a contraction. Using the Banach contraction principle, there exists a unique fixed point x_t of T_t in C, that is,

$$x_t = P_C((1-k-t)x_t + kTx_t). \tag{3.3}$$

THEOREM 3.1. *Suppose that* $\mathrm{Fix}(T) \ne \emptyset$. *Then, as* $t \to 0$, *the net* $\{x_t\}$ *generated by* (3.3) *converges strongly to the minimum-norm fixed point of* T.

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Proof. First, we prove that $\{x_t\}$ is bounded. Take $u \in \mathrm{Fix}(T)$. From (3.3) and (3.2),

$$\begin{aligned}\|x_t - u\| &= \|P_C((1-k-t)x_t + kTx_t) - P_C u\| \\ &\le \|(1-k-t)(x_t-u) + k(Tx_t - u) - tu\| \\ &\le (1-t)\|x_t - u\| + t\|u\|,\end{aligned}$$

that is, $\|x_t - u\| \le \|u\|$, which implies that $\{x_t\}$ is bounded and so is $\{Tx_t\}$.

From (3.3),

$$\begin{aligned}\|x_t - Tx_t\| &= \|P_C((1-k-t)x_t + kTx_t) - P_C Tx_t\| \\ &\le \|(1-k)(x_t - Tx_t) - tx_t\| \\ &\le (1-k)\|x_t - Tx_t\| + t\|x_t\|.\end{aligned}$$

It follows that

$$\|x_t - Tx_t\| \le \frac{t}{k}\|x_t\| \to 0. \tag{3.4}$$

Next we show that $\{x_t\}$ is relatively norm-compact as $t \to 0$. Assume that $\{t_n\} \subset (0, 1)$ is such that $t_n \to 0$ as $n \to \infty$. Put $x_n := x_{t_n}$. From (3.4),

$$\|x_n - Tx_n\| \to 0. \tag{3.5}$$

Setting $y_t = (1-k-t)x_t + kTx_t$, we then have $x_t = P_C y_t$, and, for any $u \in \mathrm{Fix}(T)$,

$$\begin{aligned}x_t - u &= x_t - y_t + y_t - u \\ &= x_t - y_t + (1-k-t)(x_t - u) + k(Tx_t - u) - tu.\end{aligned} \tag{3.6}$$

Using the property (2.1) of the metric projection,

$$\langle x_t - y_t, x_t - u\rangle \le 0. \tag{3.7}$$

Combining (3.6) and (3.7),

$$\begin{aligned}\|x_t - u\|^2 &= \langle x_t - y_t, x_t - u\rangle + \langle (1-k-t)(x_t - u) + k(Tx_t - u), x_t - u\rangle - t\langle u, x_t - u\rangle \\ &\le \|(1-k-t)(x_t - u) + k(Tx_t - u)\|\,\|x_t - u\| - t\langle u, x_t - u\rangle \\ &\le (1-t)\|x_t - u\|^2 - t\langle u, x_t - u\rangle.\end{aligned}$$

Hence, $\|x_t - u\|^2 \le \langle u, u - x_t\rangle$. In particular,

$$\|x_n - u\|^2 \le \langle u, u - x_n\rangle, \quad u \in \mathrm{Fix}(T). \tag{3.8}$$

Since $\{x_n\}$ is bounded we may assume, without loss of generality, that $\{x_n\}$ converges weakly to a point $x^* \in C$. Noting (3.5), we can use Lemma 2.1 to get $x^* \in \mathrm{Fix}(T)$. Therefore we can substitute x^* for u in (3.8) to get

$$\|x_n - x^*\|^2 \le \langle x^*, x^* - x_n\rangle.$$

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Consequently, the weak convergence of $\{x_n\}$ to x^* actually implies that $x_n \to x^*$ strongly. This proves the relative norm-compactness of the net $\{x_t\}$ as $t \to 0$.

To show that the entire net $\{x_t\}$ converges to x^*, assume that $x_{s_n} \to \tilde{x} \in \mathrm{Fix}(T)$, where $s_n \to 0$. In (3.8), we take $u = \tilde{x}$ to get

$$\|x^* - \tilde{x}\|^2 \le \langle \tilde{x}, \tilde{x} - x^* \rangle. \tag{3.9}$$

Interchange x^* and $\tilde{x}$ to obtain

$$\|\tilde{x} - x^*\|^2 \le \langle x^*, x^* - \tilde{x} \rangle. \tag{3.10}$$

Adding (3.9) and (3.10) yields

$$2\|x^* - \tilde{x}\|^2 \le \|x^* - \tilde{x}\|^2,$$

which implies that $\tilde{x} = x^*$.

Finally, we return to (3.8) and take the limit as $n \to \infty$ to get

$$\|x^* - u\|^2 \le \langle u, u - x^* \rangle, \quad u \in \mathrm{Fix}(T).$$

Equivalently,

$$\|x^*\|^2 \le \langle x^*, u \rangle, \quad u \in \mathrm{Fix}(T).$$

This clearly implies that

$$\|x^*\| \le \|u\|, \quad u \in \mathrm{Fix}(T).$$

Therefore, x^* is a minimum-norm fixed point of T. This completes the proof. □

COROLLARY 3.2. *Suppose that* $\mathrm{Fix}(T) \neq \emptyset$ *and the origin* 0 *belongs to* C. *Then, as* $t \to 0+$, *the net* $\{x_t\}$ *generated by the algorithm*

$$x_t = (1 - k - t)x_t + kTx_t$$

converges strongly to the minimum-norm fixed point of T.

Now we propose the following iterative algorithm which is the discretisation of the implicit method (3.3). For given $x_0 \in C$, chosen arbitrarily, let the sequence $\{x_n\}$ be generated iteratively by

$$x_{n+1} = P_C((1 - k - \alpha_n)x_n + kTx_n), \quad n \ge 0, \tag{3.11}$$

where $\{\alpha_n\}$ is a real sequence in $(0, 1)$.

THEOREM 3.3. *Suppose that* $\mathrm{Fix}(T) \neq \emptyset$ *and the following conditions are satisfied:*

(i) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=0}^{\infty} \alpha_n = \infty$;
(ii) $\lim_{n\to\infty}(\alpha_n/\alpha_{n+1}) = 1$.

Then the sequence $\{x_n\}$ *generated by (3.11) strongly converges to the minimum-norm fixed point* x^* *of* T.

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Proof. First, we prove that the sequence $\{x_n\}$ is bounded. Take $x^* \in \mathrm{Fix}(T)$. From (3.11),

$$\begin{aligned} \|x_{n+1} - x^*\| &= \|P_C((1-k-\alpha_n)x_n + kTx_n) - x^*\| \\ &\le \|(1-k-\alpha_n)(x_n - x^*) + k(Tx_n - x^*)\| + \alpha_n\|x^*\|. \end{aligned} \tag{3.12}$$

From (3.2), we note that

$$\|(1-k-\alpha_n)(x_n - x^*) + k(Tx_n - x^*)\| \le (1-\alpha_n)\|x_n - x^*\|. \tag{3.13}$$

It follows from (3.12) and (3.13) that

$$\begin{aligned} \|x_{n+1} - x^*\| &\le (1-\alpha_n)\|x_n - x^*\| + \alpha_n\|x^*\| \\ &\le \max\{\|x_n - x^*\|, \|x^*\|\} \\ &\le \max\{\|x_0 - x^*\|, \|x^*\|\}. \end{aligned}$$

Hence, $\{x_n\}$ is bounded and so is $\{Tx_n\}$.

We now estimate $\|x_{n+1} - x_n\|$. From (3.11),

$$\begin{aligned} \|x_{n+1} - x_n\| &= \|P_C((1-k-\alpha_n)x_n + kTx_n) \\ &\qquad - P_C((1-k-\alpha_{n-1})x_{n-1} + kTx_{n-1})\| \\ &\le \|(1-k-\alpha_n)(x_n - x_{n-1}) + k(Tx_n - Tx_{n-1}) \\ &\qquad + (\alpha_{n-1} - \alpha_n)x_{n-1}\| \\ &\le \|(1-k-\alpha_n)(x_n - x_{n-1}) + k(Tx_n - Tx_{n-1})\| \\ &\qquad + |\alpha_{n-1} - \alpha_n|\|x_{n-1}\| \\ &\le (1-\alpha_n)\|x_n - x_{n-1}\| + |\alpha_{n-1} - \alpha_n|M, \end{aligned} \tag{3.14}$$

where $M > 0$ is a constant such that $\sup_n\{\|x_n\|\} \le M$. Using Lemma 2.2 with (3.14), we conclude that

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0.$$

We observe that

$$\begin{aligned} \|x_n - Tx_n\| &\le \|x_n - x_{n+1}\| + \|x_{n+1} - Tx_n\| \\ &\le \|x_n - x_{n+1}\| + (1-k)\|x_n - Tx_n\| + \alpha_n\|x_n\|, \end{aligned}$$

that is,

$$\|x_n - Tx_n\| \le \frac{1}{k}\{\|x_{n+1} - x_n\| + \alpha_n M\} \to 0.$$

Let the net $\{x_t\}$ be defined by (3.3). By Theorem 3.1, $x_t \to x^*$ as $t \to 0$. Next we prove that $\limsup_{n\to\infty}\langle x^*, x^* - x_n\rangle \le 0$.

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Set $y_t = (1-k-t)x_t + kTx_t$. It follows that

$$\begin{aligned}\|x_t - x_n\|^2 &= \langle x_t - y_t, x_t - x_n\rangle + \langle y_t - x_n, x_t - x_n\rangle \\ &\le \langle y_t - x_n, x_t - x_n\rangle \\ &= \langle (1-k-t)(x_t - x_n) + k(Tx_t - Tx_n), x_t - x_n\rangle \\ &\quad + k\langle Tx_n - x_n, x_t - x_n\rangle - t\langle x_n, x_t - x_n\rangle \\ &\le (1-t)\|x_t - x_n\|^2 + k\|Tx_n - x_n\|\,\|x_t - x_n\| \\ &\quad - t\langle x_n - x_t, x_t - x_n\rangle - t\langle x_t, x_t - x_n\rangle \\ &= \|x_t - x_n\|^2 + k\|Tx_n - x_n\|\,\|x_t - x_n\| - t\langle x_t, x_t - x_n\rangle,\end{aligned}$$

and hence that

$$\langle x_t, x_t - x_n\rangle \le \frac{k}{t}\|Tx_n - x_n\|\,\|x_t - x_n\|.$$

Therefore,

$$\limsup_{t\to 0}\limsup_{n\to\infty}\langle x_t, x_t - x_n\rangle \le 0. \tag{3.15}$$

Note that the two limits $\limsup_{t\to 0}$ and $\limsup_{n\to\infty}$ are interchangeable. In fact,

$$\begin{aligned}\langle x^*, x^* - x_n\rangle &= \langle x^*, x^* - x_t\rangle + \langle x^* - x_t, x_t - x_n\rangle + \langle x_t, x_t - x_n\rangle \\ &\le \langle x^*, x^* - x_t\rangle + \|x^* - x_t\|\,\|x_t - x_n\| + \langle x_t, x_t - x_n\rangle \\ &\le \langle x^*, x^* - x_t\rangle + \|x^* - x_t\|M + \langle x_t, x_t - x_n\rangle.\end{aligned}$$

This, together with $x_t \to x^*$ and (3.15), implies that

$$\limsup_{n\to\infty}\langle x^*, x^* - x_n\rangle \le 0.$$

Finally, we show that $x_n \to x^*$. Set $y_n = (1-k-\alpha_n)x_n + kTx_n$ for all $n \ge 0$. From (3.11),

$$\begin{aligned}\|x_{n+1} - x^*\|^2 &= \langle x_{n+1} - y_n, x_{n+1} - x^*\rangle + \langle y_n - x^*, x_{n+1} - x^*\rangle \\ &\le \langle y_n - x^*, x_{n+1} - x^*\rangle \\ &= \langle (1-k-\alpha_n)(x_n - x^*) + k(Tx_n - x^*), x_{n+1} - x^*\rangle \\ &\quad + \alpha_n\langle x^*, x^* - x_{n+1}\rangle \\ &\le \|(1-k-\alpha_n)(x_n - x^*) + k(Tx_n - x^*)\|\,\|x_{n+1} - x^*\| \\ &\quad + \alpha_n\langle x^*, x^* - x_{n+1}\rangle \\ &\le (1-\alpha_n)\|x_n - x^*\|\,\|x_{n+1} - x^*\| + \alpha_n\langle x^*, x^* - x_{n+1}\rangle \\ &\le \frac{1-\alpha_n}{2}(\|x_n - x^*\|^2 + \|x_{n+1} - x^*\|^2) + \alpha_n\langle x^*, x^* - x_{n+1}\rangle.\end{aligned}$$

It follows that

$$\|x_{n+1} - x^*\|^2 \le (1-\alpha_n)\|x_n - x^*\|^2 + \frac{2\alpha_n}{1+\alpha_n}\langle x^*, x^* - x_{n+1}\rangle.$$

We can check that all assumptions of Lemma 2.2 are satisfied. Therefore, $x_n \to x^*$. This completes the proof. □

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COROLLARY 3.4. *Suppose that* $\mathrm{Fix}(T) \neq \emptyset$ *and the origin* 0 *belongs to* C. *Assume that the following conditions are satisfied:*

(i) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=0}^{\infty} \alpha_n = \infty$;

(ii) $\lim_{n\to\infty}(\alpha_n/\alpha_{n+1}) = 1$.

Then the sequence $\{x_n\}$ *generated by the algorithm*

$$x_{n+1} = (1 - k - \alpha_n)x_n + kTx_n, \quad n \geq 0,$$

converges strongly to the minimum-norm fixed point x^* *of* T.

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