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INTEGRATION WITH RESPECT TO VECTOR VALUED RADON POLYMEASURES

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Abstract

Problems dealing with certain functional calculi for systems of non-commuting operators, and ordered calculi for systems of certain types of pseudo-differential operators, can sometimes be treated via the methods of integration with respect to polymeasures. The polymeasures arising in this fashion (called Radon polymeasures) often have additional structure not available in the general theory. This allows for a more extensive class of "integrable" functions than just the product functions allowed in the abstract theory. The purpose here is to further develop special aspects of integration with respect to Radon polymeasures with a particular emphasis on identifying large classes of "integrable" functions.

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Introduction

The purpose of this paper is to further develop special aspects of the theory of integration with respect to a certain class of unbounded, additive set functions called *polymeasures* – these are defined on a family of subsets of a product space. In the case of a product of just two sets, the term *bimeasure* is used. It has been maintained that "bimeasures are of little importance because any reasonably interesting non-negative, $\mathbb{R}$-valued bimeasure determines a measure in the product space" [2, page 129]. Although this viewpoint is valid for non-negative bimeasures, it is no longer the case for bimeasures and polymeasures which may assume negative, or complex values. Such polymeasures are increasingly found in areas as diverse as non-stationary processes [20, 23], harmonic analysis [9, 11, 12, 23], operator theory [15] and quantum physics [4] (as predicted in [17]).

Originally, bimeasures were conceived in order to obtain a representation for certain types of bilinear mappings [19]. In the subsequent theory of integration with respect to bimeasures and polymeasures, as developed by Ylinen [24] and Dobrakov (see [7] and the references therein), attention was restricted to the integration of n-tuples of functions – quite adequate from the view point of obtaining canonical extensions of multilinear mappings and for applications to non-stationary processes. However, there are situations when this class of functions is too small.

The restricted notion of a *Radon polymeasure*, introduced in [13], is well-adapted to deal with problems where the only reason that the polymeasure is not extendible to a measure "lies at infinity". There are many concrete examples of this phenomenon. Since Radon polymeasures behave locally like measures, they admit a more extensive class of integrable functions than arbitrary polymeasures.

For instance, suppose that A_1 and A_2 are linear operators (not necessarily bounded or commuting) in some Banach space E. Suppose also that, for any $\xi_1, \xi_2 \in \mathbb{R}$, the operator $i(\xi_1 A_1 + \xi_2 A_2)$ is densely defined and closable, and its closure generates a 1-parameter group of surjective isometries on E. Then, for every C^∞-function $f : \mathbb{R}^2 \to \mathbb{C}$ with compact support, it is possible to define a bounded operator $f(A_1, A_2)$ on E via the Fourier inversion theorem

$$f(A_1, A_2) = \frac{1}{2\pi} \int_{\mathbb{R}^2} e^{-i(\xi_1 A_1 + \xi_2 A_2)} \hat{f}(\xi_1, \xi_2)\, d\xi_1 d\xi_2.$$

The assignment $f \to f(A_1, A_2)$ is an operator valued distribution, the so called Weyl calculus of (A_1, A_2) [1]. If this distribution is of order zero, then one would expect the existence of a Radon bimeasure W such that $f(A_1, A_2) = \int_{\mathbb{R}^2} f\, dW$, where the "integral" is suitably defined. The larger the class of functions integrable with respect to W, the more functions $f(A_1, A_2)$ of the noncommuting pair (A_1, A_2) can be formed, that is, the richer functional calculus for (A_1, A_2).

As another example, suppose that Q is the spectral measure associated with the operator of "multiplication by x", and P is the spectral measure associated with $i^{-1}d/dx$, both acting in $L^2(\mathbb{R})$. Then the product set function QP is a Radon bimeasure. The "integral" $\int_{\mathbb{R}^2} dQP$, suitably defined, is a pseudo-differential operator. If $f(x, y) = \phi(x)\psi(y)$, $x, y \in \mathbb{R}$, then one would expect $\int_{\mathbb{R}^2} dQP = \int_{\mathbb{R}} \phi dQ . \int_{\mathbb{R}} \psi dP$. However, restricting oneself to the class of product functions produces an artificially small class of pseudo-differential operators in this case.

It is problems of the above kind which suggest that it is worthwhile to further develop the theory of integration with respect to Radon polymeasures.

1. Notation and terminology

By a *semi-algebra* $\mathcal{S}$ of subsets of a set Ω we mean a collection of sets containing Ω and $\emptyset$, closed under finite intersections, and with the property that if $A \in \mathcal{S}$, $B \in \mathcal{S}$, then $A \backslash B$ is equal to the union of a finite family of pairwise disjoint sets U_j, $j = 1, \ldots, k$, belonging to $\mathcal{S}$, which are numbered so that if $U_0 = A \cap B$, then the union $\cup_{j=0}^{m} U_j$ belongs to $\mathcal{S}$ for every $m = 1, \ldots, k$.

A set function $m : \mathcal{S} \to \mathbb{C}$ on the semi-algebra $\mathcal{S}$ is *additive* if $m(A \cup B) = m(A) + m(B)$ for all disjoint sets $A, B \in \mathcal{S}$ such that $A \cup B \in \mathcal{S}$. Let Π denote all finite partitions of the set Ω by elements of the semi-algebra $\mathcal{S}$. The variation $|m| : \mathcal{S} \to [0, \infty]$ of m is the finitely additive extended-real-valued function defined by

$$|m|(A) = \sup \left\{ \sum_{B \in \pi} |m(A \cap B)| : \pi \in \Pi \right\}, \qquad A \in \mathcal{S}.$$

An $\mathcal{S}$*-simple function* is a finite linear combination of characteristic functions of sets from $\mathcal{S}$. The vector space of all $\mathcal{S}$-simple functions is denoted by $\mathrm{sim}\,(\mathcal{S})$. The integral $sm : \mathcal{S} \to \mathbb{C}$ of an $\mathcal{S}$-simple function s with respect to the additive set function m is defined by linearity, as follows. Suppose that s is represented as the sum $\sum_{i=1}^{n} c_i \chi_{A_i}$ with $c_i \in \mathbb{C}$ and $A_i \in \mathcal{S}$ for all $i = 1, \ldots, n$. Then for every $B \in \mathcal{S}$,

$$sm(B) = \sum_{i=1}^{n} c_i m(A_i \cap B).$$

The additivity of the set function m on the semi-algebra $\mathcal{S}$ ensures that the definition of the integral sm does not depend on the particular representation of the simple function s as a linear combination of characteristic functions of sets belonging to $\mathcal{S}$.

The term "locally convex Hausdorff topological vector space" will usually be abbreviated to "lcs". The vector space of all continuous linear functionals on E is denoted by E'. If $\xi \in E'$ and $x \in E$, for convenience we write $\langle x, \xi \rangle$ for the number $\xi(x)$. The *polar* A° of a subset A of E is the set of all $\xi \in E'$ such that $|\langle x, \xi\rangle| \le 1$ for all $x \in A$. Given a topology τ on E, the vector space E endowed with the topology τ is denoted by E_τ.

Let E and F be lcs. The algebraic tensor product of E with F is denoted by $E \otimes F$. The *projective topology* π on $E \otimes F$ is determined by the collection of seminorms $r_{p,q}$ defined, for each $w \in E \otimes F$, by

$$r_{p,q}(w) = \inf\left\{\sum_{j=1}^{n} p(x_j)q(y_j)\right\}$$

as p and q run over collections of seminorms defining the topology of E and F respectively. The supremum is taken over all representations $w = \sum_{j=1}^n x_j \otimes y_j$, $x_j \in E$, $y_j \in F$, $j = 1, \dots, n$, and $n = 1, 2, \dots$. The tensor product $E \otimes F$ endowed with the topology π is written as $E \otimes_\pi F$. The completion of $E \otimes_\pi F$ is denoted by $E \hat{\otimes}_\pi F$. It is a result of Grothendieck [22, Ch. 44] that an element of $E\hat{\otimes}_\pi F$ can be represented as a sum $\sum_{j=1}^\infty \lambda_j x_j \otimes y_j$, where $\{x_j\}_{j=1}^\infty$ is equicontinuous in E, $\{y_j\}_{j=1}^\infty$ is equicontinuous in F, and $\{\lambda_j\}_{j=1}^\infty$ is a summable sequence of real numbers.

If $m : \mathscr{S} \to E$ is an additive set function with values in the lcs E, then for each $\xi \in E'$, the additive scalar-valued set function $\langle m, \xi\rangle$ is defined by $\langle m, \xi\rangle(A) = \langle m(A)\xi\rangle$ for every $A \in \mathscr{S}$. Similarly, if $f : \Omega \to E$ is an E-valued function, the scalar function $\langle f, \xi\rangle : \Omega \to \mathbb{C}$ is defined for each $\xi \in E'$ by $\langle f, \xi\rangle(\omega) = \langle f(\omega), \xi\rangle$ for every $\omega \in \Omega$.

The case where E is the space $\mathscr{L}_s(F)$ of bounded linear operators on a Banach space F (equipped with the strong operator topology) is of special interest. The continuous dual of $\mathscr{L}_s(F)$ is the space of all finite linear combinations of linear functionals of the form $x \otimes \xi$ with $x \in F$ and $\xi \in F'$, where the linear functional $x \otimes \xi$ acts on $\mathscr{L}_s(F)$ by $\langle T, x \otimes \xi\rangle = \langle Tx, \xi\rangle$, $T \in \mathscr{L}(F)$. Suppose that $m : \mathscr{S} \to \mathscr{L}_s(F)$ is an additive set function. For each $x \in F$, the F-valued additive set function $mx : \mathscr{S} \to F$ is defined by $mx(A) = m(A)x$, for every set $A \in \mathscr{S}$. It is clear that for each $x \in F$ and $\xi \in F'$, the identity $\langle mx, \xi\rangle = \langle m, x \otimes \xi\rangle$ holds. Here the brackets on the left refer to the duality between F and F', and on the right, to the duality between $\mathscr{L}_s(F)$ and $\mathscr{L}_s(F)'$.

Now suppose that $\mathscr{S}$ is a σ-algebra of subsets of the set Ω. Let E be a lcs, and suppose that $m : \mathscr{S} \to E$ is a σ-additive set function, that is, an E-valued vector measure. An $\mathscr{S}$-measurable function $f : \Omega \to \mathbb{C}$ is said to be *m-integrable* if for every $\xi \in E'$ the function f is integrable with respect to the scalar measure

$\langle m, \xi\rangle$, and for each $A \in \mathscr{S}$ there exists a vector $fm(A) \in E$ such that

$$\langle fm(A), \xi\rangle = \int_A f d\langle m, \xi\rangle,$$

for every $\xi \in E'$. Then the set function $A \to fm(A)$, $A \in \mathscr{S}$ is a vector measure too, called the *indefinite integral* of f with respect to m. We sometimes write $\int_A f dm$ or $\int_A f(\omega) dm(\omega)$ for the vector $fm(A)$, $A \in \mathscr{S}$, and $m(f)$ for $fm(\Omega)$.

An analogous definition is adopted for the integration of vector-valued functions with respect to a scalar measure. Again suppose that $\mathscr{S}$ is a σ-algebra of subsets of the set Ω and let $\nu : \mathscr{S} \to [0, \infty]$ be a σ-finite measure. Let E be a lcs, and suppose that $f : \Omega \to E$ is an E-valued function. Then f is said to be ν-*integrable* in E if for every $\xi \in E'$, the function $\langle f, \xi\rangle$ is integrable with respect to the scalar measure ν, and for each $A \in \mathscr{S}$ there exists a vector $f\nu(A) \in E$ such that $\langle f\nu(A), \xi\rangle = \int_A \langle f, \xi\rangle d\nu$, for every $\xi \in E'$. Then the set function $A \to f\nu(A)$, $A \in \mathscr{S}$ is a vector measure, called the *indefinite integral* of f with respect to ν. We sometimes write $\int_A f d\nu$ or $\int_A f(\omega) d\nu(\omega)$ for the vector $f\nu(A)$, $A \in \mathscr{S}$ and $\nu(f)$ for $f\nu(\Omega)$.

Sometimes it is necessary to consider other types of integrals for vector-valued functions. If $f : \Omega \to E$ is the norm limit in the Banach space E of E-valued $\mathscr{S}$-simple functions (that is, f is *strongly measurable*), and $\int_\Omega \|f(\omega)\| d\nu(\omega) < \infty$, then f is said to be *Bochner integrable*. It turns out that a Bochner integrable function is integrable in the above sense.

We need to fix some terminology concerning measures defined on the Borel σ-algebra of a Hausdorff topological space X. The relevant results relating to the measures we consider are set out in [21]. The Borel σ-algebra $\mathscr{B}(X)$ of X is the smallest σ-algebra containing every open subset of X.

A σ-additive set function $\mu : \mathscr{B}(X) \to [0, \infty]$ is said to be a *Radon measure on* X if

(i) for every $x \in X$, there exists an open set U containing x such that $\mu(U) < \infty$;

(ii) for every Borel set A contained in X,

$$\mu(A) = \sup\{\mu(K) : K \subseteq A, \quad K \text{ is a compact subset of } X\}$$

The *support* of a Radon measure $\mu : \mathscr{B}(X) \to [0, \infty]$ is defined by

$$\operatorname{supp} \mu = \bigcap \{U^c : U \text{ is open } X, \quad \mu(U) = 0\}.$$

We shall often say that a Borel measure whose variation has property (ii) is *compact inner regular*. A vector measure $m : \mathscr{B}(X) \to E$ with values in a

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lcs E is said to be *compact inner regular* if for every Borel set A and every neighbourhood U of zero in E, there exists a compact subset of K of X such that $K \subseteq A$, and for every set $B \in \mathscr{B}(X)$ we have $m((A \backslash K) \cap B) \in U$. We also call such a vector measure a (vector-valued) *Radon measure*.

It follows that $\nu : \mathscr{B}(X) \to \mathbb{C}$ is a Radon measure if both its real and imaginary parts are Radon measures. Moreover, a σ-additive set function $\nu : \mathscr{B}(X) \to \mathbb{C}$ is a Radon measure if and only if its variation $|\nu| : \mathscr{B}(X) \to [0, \infty)$ is a Radon measure. When it is necessary to emphasise that ν may take complex values, we say that ν is a *scalar* Radon measure.

An important property of Radon measures is that they are determined by their values on compact sets, which may be seen directly from property (ii). It is not difficult to show that on a product space, Radon measures are also determined by their values on compact *product* sets (see the uniqueness for Henry's extension theorem in [21, page 51]), even though the σ-algebra generated by the products of Borel sets need not coincide with the Borel σ-algebra of the product space [21, Chapter 1, Section 9].

2. Radon polymeasures

Let $\Omega_1, \dots, \Omega_n$ be non-empty sets, and suppose that $\mathscr{S}_1, \dots \mathscr{S}_n$ are σ-algebras of subsets of $\Omega_1, \dots, \Omega_n$ respectively. The collection of all product sets $A_1 \times A_2 \times \dots \times A_n$, where $A_1 \in \mathscr{S}_1, \dots, A_n \in \mathscr{S}_n$ is a semi-algebra denoted by $\times_{j=1}^n \mathscr{S}_j$. A *polymeasure* is an additive set function $m : \times_{j=1}^n \mathscr{S}_j \to \mathbb{C}$ such that for each $j = 1, 2, \dots, n$, the set function

$$A_j \to m(A_1 \times \dots \times A_j \times \dots A_n), \qquad A_j \in \mathscr{S}_j$$

is σ-additive for fixed $A_k \in \mathscr{S}_k$, $k \neq j$, and $k = 1, 2, \dots, n$. The *semivariation norm* $\operatorname{sv}(m)$ of a polymeasure $m : \times_{j=1}^n \mathscr{S}_j \to \mathbb{C}$ is defined by

$$\operatorname{sv}(m) = \sup\left\{|m(A)| : A \in \times_{j=1}^n \mathscr{S}_j\right\}.$$

It follows from the Nikodym boundedness theorem, and induction, that $\operatorname{sv}(m) < \infty$ for any polymeasure m [5, page 490]. The space of polymeasures on the semi-algebra $\times_{j=1}^n \mathscr{S}_j$ of product sets is denoted by $\mathscr{P}\mathscr{M}(\times_{j=1}^n \mathscr{S}_j)$. It is endowed with the semivariation norm under which it becomes a Banach space. Furthermore, if a sequence of polymeasures converges setwise on the semi-algebra $\times_{j=1}^n \mathscr{S}_j$, then the limit is again a polymeasure; this is a consequence of the Vitali-Hahn-Saks theorem for measures [3, Chapter 1].

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An additive set function $m : \times_{j=1}^{n} \mathscr{S}_j \to E$ with values in a lcs E called a *polymeasure* if for each $\xi \in E'$, the set function $\langle m, \xi \rangle$ is a polymeasure. It follows from the Orlicz-Pettis lemma [18, I Section 1 Theorem 3] that m is separately σ-additive in the original topology of E. Indeed, if τ is any topology consistent with the duality between E and E', then m is a polymeasure in E_τ.

In the case that $(E, \|\cdot\|)$ is a normed space, the *semivariation norm* $\mathrm{sv}(m)$ of a polymeasure $m : \times_{j=1}^{n} \mathscr{S}_j \to E$ is defined by

$$\mathrm{sv}(m) = \sup \left\{ \|m(A)\| : A \in \times_{j=1}^{n} \mathscr{S}_j \right\}.$$

The range of m on $\times_{j=1}^{n} \mathscr{S}_j$ is a weakly bounded subset of E because, as remarked above, $\mathrm{sv}(\langle m, \xi \rangle) < \infty$ for every $\xi \in E'$. Because weakly bounded subsets of E are automatically norm bounded, $\mathrm{sv}(m) < \infty$.

Let $X_1, X_2 \dots X_n$ be Hausdorff topological spaces. The semi-algebra of products of sets from the Borel σ-algebras of these spaces is denoted by $\times_{j=1}^{n} \mathscr{B}(X_j)$.

An additive set function $m : \times_{j=1}^{n} \mathscr{B}(X_j) \to \mathbb{C}$ is called a *Radon polymeasure* if it is a Radon measure in each component, and its variation is the restriction to $\times_{j=1}^{n} \mathscr{B}(X_j)$ of a (possibly infinite) Radon measure on the product space $\prod_{j=1}^{n} X_j = X_1 \times X_2 \times \dots X_n$. As pointed out at the end of section one, this Radon measure is uniquely determined by its values on compact product sets – it is denoted, again, by $|m|$. It turns out that for a polymeasure on the semi-algebra $\times_{j=1}^{n} \mathscr{B}(X_j)$ to be a Radon polymeasure, it suffices that it be separately compact inner-regular, and each point of $\prod_{j=1}^{n} X_j$ has a neighbourhood on which the variation of m is finite [13, Proposition 1].

An additive set function $m : \times_{j=1}^{n} \mathscr{B}(X_j) \to E$ with values in a lcs E is called a *Radon polymeasure* if for each $\xi \in E'$, the set function $\langle m, \xi \rangle$ is a Radon polymeasure. If τ is any topology consistent with the duality between E and E', then m is a Radon polymeasure in E_τ.

In the following section, we review integration with respect to Radon polymeasures as outlined in [13], where the concept was introduced.

3. Integration with respect to Radon polymeasures

Let $X_1, X_2, \dots, X_n$ be Hausdorff topological spaces and let the set function $m : \times_{j=1}^{n} \mathscr{B}(X_j) \to \mathbb{C}$ be a Radon polymeasure. Given a topological space T, let $\mathscr{C}_T$ denote the family of all compact subsets of T. We have assumed that the

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variation $|m| : \times_{j=1}^{n}\mathscr{B}(X_j) \to [0, \infty]$ of m is the restriction, to the semi-algebra $\times_{j=1}^{n}\mathscr{B}(X_j)$, of a Radon measure defined on the Borel σ-algebra $\mathscr{B}(\prod_{j=1}^{n} X_j)$ of the product space $\prod_{j=1}^{n} X_j$. Therefore, for every $K_j \in \mathscr{C}_{X_j}$, $j = 1, \dots, n$, the restriction $|m|_{K_1\times\dots\times K_n}$ of $|m|$ to the semi-algebra $\times_{j=1}^{n}\mathscr{B}(K_j)$ is the restriction to $\times_{j=1}^{n}\mathscr{B}(K_j)$ of a unique Radon measure defined on $\mathscr{B}(\prod_{j=1}^{n} K_j)$. The uniqueness is a consequence of the Remark at the end of Section 1. There is no danger of confusion if we again denote this Radon measure by $|m|_{K_1\times\dots\times K_n}$. The restriction of m to $\times_{j=1}^{n}\mathscr{B}(K_j)$ is also the restriction of a unique $\mathbb{C}$-valued Radon measure defined on $\mathscr{B}(\prod_{j=1}^{n} K_j)$. We denote this by $m_{K_1\times\dots\times K_n}$.

Set $K = \prod_{j=1}^{n} K_j$, where $K_j \in \mathscr{C}_{X_j}$, $j = 1, \dots, n$, and suppose that $L = \prod_{j=1}^{n} L_j$ is a closed subset of K. The Radon measure m_K is consistent with m_L in the sense that $m_K(A) = m_L(A)$ for every Borel set $A \in \mathscr{B}(\prod_{j=1}^{n} L_j)$. This is ensured by the equality $m(A) = m_K(A) = m_L(A)$, valid for all $A \in \times_{j=1}^{n}\mathscr{B}(L_j)$, because $\times_{j=1}^{n}\mathscr{B}(L_j)$ contains an open base for the topology of $\prod_{j=1}^{n} L_j$. We also have $|m|_K(A) = |m|_L(A)$, for every Borel set $A \in \mathscr{B}(\prod_{j=1}^{n} L_j)$ once we establish the identities $|m_K| = |m|_K$ and $|m_L| = |m|_L$.

For any compact product set $K = \prod_{j=1}^{n} K_j$, the inequality $|m_K|(A) \geq |m|(A) = |m|_K(A)$ is immediate for any $A \in \times_{j=1}^{n}\mathscr{B}(K_j)$, because on the left hand side, the variation is calculated from partitions of A with sets in $\mathscr{B}(\prod_{j=1}^{n} K_j)$, and on the right hand side, with sets in $\times_{j=1}^{n}\mathscr{B}(K_j)$. It follows from the Stone-Weierstrass theorem that every continuous function $f : \prod_{j=1}^{n} K_j \to \mathbb{C}$ is measurable with respect to the σ-algebra generated by $\times_{j=1}^{n}\mathscr{B}(K_j)$, so $|fm_K(A)| \leq |m|\,(|f|\chi_A) = |m|_K\,(|f|\chi_A)$. Moreover, $|m_K|(A) = sup_{\|f\|_\infty\leq 1}$ $|fm_K(A)|$, so the desired equality $|m_K|(A) = |m|_K(A)$ follows. Both $|m_K|$ and $|m|_K$ are finite Radon measures which agree on the base of open product sets for the topology of $\prod_{j=1}^{n} K_j$, so they are equal on $\mathscr{B}(\prod_{j=1}^{n} K_j)$.

If the family of compact product sets K is directed by inclusion, then the Radon measure $|m|$ is the supremum of the increasing family of Radon measures $|m|_K$ in the sense of [21, Proposition 7, p. 56].

A function $f : \prod_{j=1}^{n} X_j \to \mathbb{C}$ is said to be *m-integrable* if for every $K_j \in \mathscr{C}_{X_j}$, $j = 1, \dots, n$, its restriction $f_{K_1\times\dots\times K_n}$ to $\prod_{j=1}^{n} K_j$ is $|m|_{K_1\times\dots\times K_n}$ - integrable, and there exists a Radon polymeasure $fm : \times_{j=1}^{n}\mathscr{B}(X_j) \to \mathbb{C}$ such that

$$fm(K_1 \times \dots \times K_n) = \int_{K_1\times\dots\times K_n} f_{K_1\times\dots\times K_n}\, dm_{K_1\times\dots\times K_n},$$

for all $K_j \in \mathscr{C}_{X_j}$, $j = 1, \dots, n$. The integrability with respect to $|m|_{K_1\times\dots\times K_n}$ of a function defined on $\prod_{j=1}^{n} K_j$ is sufficient to ensure its integrability with respect to the $\mathbb{C}$-valued Radon measure $m_{K_1\times\dots\times K_n}$; the integral above therefore makes

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sense. As usual, we shall sometimes write $\int_A f\,dm$ or $\int_A f(\omega)dm(\omega)$ for the vector $fm(A)$, $A \in \times_{j=1}^n \mathscr{B}(X_j)$, and $m(f)$ for $fm(\prod_{j=1}^n X_j)$.

Now suppose that E is a lcs and that $m : \times_{j=1}^n \mathscr{B}(X_j) \to E$ is an E-valued Radon polymeasure. A function $f : \prod_{j=1}^n X_j \to \mathbb{C}$ is said to be *m-integrable* if for each $\xi \in E'$, f is $\langle m, \xi \rangle$ -integrable, and there exists an E-valued Radon polymeasure $fm : \times_{j=1}^n \mathscr{B}(X_j) \to E$ such that $\langle fm(A), \xi \rangle = \int_A f(\omega)d\langle m, \xi\rangle(\omega)$, for every $\xi \in E'$, and $A \in \times_{j=1}^n \mathscr{B}(X_j)$. We sometimes write $\int_A f\,dm$ or $\int_A f(\omega)dm(\omega)$ for $fm(A)$, and $m(f)$ for $fm(\prod_{j=1}^n X_j)$.

Let F be a Banach space. Suppose that $m : \times_{j=1}^n \mathscr{B}(X_j) \to \mathscr{L}_s(F)$ is a Radon polymeasure. For each $x \in F$ and $\xi \in F'$ the scalar-valued Radon polymeasure $\langle mx, \xi \rangle$ is defined as in Section 1. It follows from the definition above, for the case of $E = \mathscr{L}_s(F)$, that a function $f : \prod_{j=1}^n X_j \to \mathbb{C}$ is m-integrable if for each $x \in F$ and $\xi \in F'$, f is $\langle mx, \xi\rangle$-integrable and there exists an $\mathscr{L}_s(F)$-valued Radon polymeasure $fm : \times_{j=1}^n \mathscr{B}(X_j) \to \mathscr{L}_s(F)$ such that $\langle fm(A)x, \xi\rangle = \int_A f(\omega)d\langle mx, \xi\rangle(\omega)$ for every $x \in F$, $\xi \in F'$ and $A \in \times_{j=1}^n \mathscr{B}(X_j)$.

For the purpose of developing a functional calculus for non-commuting operators, it would be natural to adopt the viewpoint that the indefinite integral of a function with respect to a Radon polymeasure should be another Radon polymeasure. A minimal requirement for this prescription to apply is that f should be m-integrable, as in the definition above.

Let E be a lcs and let $m : \times_{j=1}^n \mathscr{B}(X_j) \to E$ be an E-valued Radon polymeasure. A subset K of $\prod_{j=1}^n X_j$ is said to be *m-null*, if for every $\xi \in E'$, the set K is $|\langle m, \xi\rangle|$-null, that is $|\langle m, \xi\rangle|(K) = 0$. The usual terminology of "m-a.e." is applied to properties which hold off an m-null set. A function which vanishes m-a.e. is said to be *m-null*.

The linear space of all m-integrable functions is denoted by $\mathscr{L}^1(m)$. The quotient space with respect to the linear space of all m-null functions is denoted by $L^1(m)$. For each continuous seminorm p on E, and each $K_1 \in \mathscr{C}_{X_1}, \ldots, K_n \in \mathscr{C}_{X_n}$, we associate a seminorm $P_{K_1\times\ldots\times K_n}$ defined on $L^1(m)$ by

$$P_{K_1\times\ldots\times K_n}(f) = \sup_{\xi\in U_p} |f\langle m, \xi\rangle|_{K_1\times\ldots\times K_n} + \sup_{\xi\in U_p} \mathrm{sv}(f\langle m, \xi\rangle),$$

for every $f \in L^1(m)$. Here U_p is the polar set $\{x : p(x) \le 1\}^\circ$. The collection of seminorms so defined endows $L^1(m)$ with a locally convex topology. If E happens to be the space $\mathscr{L}_s(F)$ of bounded linear operators on a Banach space F and $B_1(F')$ denotes the closed unit ball of F', then the seminorms $p_{x,K_1\times\ldots\times K_n}$

defined on $L^1(m)$ for each $x \in F$, and $K_1 \in \mathscr{C}_{X_1}, \dots, K_n \in \mathscr{C}_{X_n}$ by

$$p_{x,K_1\times\ldots\times K_n}(f) = \sup_{\xi\in B_1(F')} |f\langle mx,\xi\rangle|_{K_1\times\ldots\times K_n} + \sup_{\xi\in B_1(F')} \mathrm{sv}(f\langle mx,\xi\rangle),$$

define the topology of $L^1(m)$. It proves convenient to consider another stronger topology on $L^1(m)$. It follows from the principle of uniform boundedness that for each $K_1 \in \mathscr{C}_{X_1}, \dots, K_n \in \mathscr{C}_{X_n}$ and $f \in L^1(m)$,

$$q_{K_1\times\ldots\times K_n}(f) = \sup_{\xi\in B_1(F)} p_{x,K_1\times\ldots\times K_n}(f) < \infty.$$

The space $L^1(m)$ with the locally convex topology defined by the seminorms $q_{K_1\times\ldots\times K_n}$, for $K_1 \in \mathscr{C}_{X_1}, \dots, K_n \in \mathscr{C}_{X_n}$ is denoted by $L^1_u(m)$. If $X_1, \dots, X_n$ are locally compact and σ-compact, then $L^1_u(m)$ is a Fréchet space.

We now turn to the consideration of classes of functions which are integrable with respect to any Radon polymeasure.

For ease of presentation, the remaining results of this section are stated for Radon bimeasures, but the proofs go over to Radon polymeasures by induction. At this point, we emphasise that bimeasures *do* have some special properties not necessarily shared by polymeasures in more than two variables, see, for example [4, 8, 14]. The proofs below do not appeal to any of these properties.

In the next result, integration with respect to Radon polymeasures is connected to the theory of integration with respect to measures.

LEMMA 1. *Let $m : \mathscr{B}(X_1) \times \mathscr{B}(X_2) \to \mathbb{C}$ be a Radon bimeasure. Suppose that the function $f : X_1 \times X_2 \to \mathbb{R}$ is integrable with respect to the Radon measure $|m|$. Then f is m-integrable, fm is the restriction to $\mathscr{B}(X_1) \times \mathscr{B}(X_2)$ of a Radon measure, and $|m(f)| \le |m|(|f|)$.*

PROOF. For each compact product set $K \subseteq X_1 \times X_2$, the restriction f_K of f to K is integrable with respect to the $\mathbb{C}$-valued Radon measure m_K and $|m_K|(|f_K|) \le |m|(|f|\chi_K) \le |m|(|f|)$. Let $\mathscr{K}$ be the family of compact product sets directed by inclusion. Because f is $|m|$-integrable, for every $\epsilon > 0$, there exists a compact set C such that $|m|(|f|\chi_{C^c}) < \epsilon/2$. We may assume that C is a product set.

Let K, J be compact product sets containing C. Because $\chi_{K\cap J} m_K = \chi_{K\cap J} m_J$, it follows that for every Borel set $A \in \mathscr{B}(X_1 \times X_2)$,

$$|f_K m_K(A) - f_J m_J(A)| = \left|\left[f_K \chi_{K\setminus J}\right].m_K(A) - \left[f_J \chi_{J\setminus K}\right].m_J(A)\right|$$

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$$\begin{aligned} &\le |m_K|\left(\left|f_K\chi_{K\setminus J}\right|\right) + |m_J|\left(\left|f_J\chi_{J\setminus K}\right|\right) \\ &\le |m|\left(\left|f\chi_{K\setminus J}\right|\right) + |m|\left(\left|f\chi_{J\setminus K}\right|\right) \\ &\le 2|m|\left(|f|\chi_{C^c}\right) < \epsilon. \end{aligned}$$

Thus $f_K m_K$, $K \in \mathscr{K}$ converges in variation to a finite Radon measure n. For each $J \in \mathscr{K}$, $f_K m_K(J) = [f_K\chi_J].m_K(J) = f_J m_J(J)$ for all $K \in \mathscr{K}$ such that $J \subseteq K$, so we have $n(J) = f_J m_J(J)$. The function f is therefore m-integrable, and the restriction of n to $\mathscr{B}(X_1) \times \mathscr{B}(X_2)$ is fm.

Given a Hausdorff topological space X, the collection of all bounded Borel measurable functions on X is denoted by $\mathscr{L}^\infty(X)$.

LEMMA 2. *Let E be a Banach space, and suppose that $m : \mathscr{B}(X_1)\times\mathscr{B}(X_2) \to E$ is a Radon bimeasure. For functions $f_1 \in \mathscr{L}^\infty(X_1)$ and $f_2 \in \mathscr{L}^\infty(X_2)$, let $f : X_1 \times X_2 \to \mathbb{C}$ be the function defined by $f(x_1, x_2) = f_1(x_1)f_2(x_2)$ for all $x_1 \in X_1$ and $x_2 \in X_2$. Then f is m-integrable and*

$$\operatorname{sv}(fm) \le 16\operatorname{sv}(m)\|f_1\|_\infty\|f_2\|_\infty. \tag{1}$$

PROOF. Let s_k, $k = 1, 2, \dots$ be $\mathscr{B}(X_1)$-simple functions converging uniformly to f_1 on X_1, and let t_k, $k = 1, 2, \dots$ be $\mathscr{B}(X_2)$-simple functions converging uniformly to f_2 on X_2.

We first define a candidate for the bimeasure fm and then prove that it is indeed the indefinite integral of f with respect to m. Firstly, for each $k, j = 1, 2, \dots$, we define $(s_k \otimes t_j).m : \mathscr{B}(X_1) \times \mathscr{B}(X_2) \to E$ by linearity as the integral of a finite linear combination of characteristic functions of sets from $\mathscr{B}(X_1) \times \mathscr{B}(X_2)$ with respect to the additive set function m. Then $(s_k \otimes t_j).m$ is obviously a bimeasure. It satisfies (1) with $f_1 = s_k$ and $f_2 = t_j$; this may be seen as follows. Fix $A \in \mathscr{B}(X_1)$ and define $M_A : \mathscr{B}(X_2) \to E$ by $M_A(B) = s_k m(A \times B)$ for every $B \in \mathscr{B}(X_2)$. Then $t_j.M_A(B) = (s_k \otimes t_j).m(A \times B)$ for all $B \in \mathscr{B}(X_2)$, and by virtue of Proposition I.1.11 of [3],

$$\sup_{B\in\mathscr{B}(X_2)} \|t_j.M_A(B)\| \le 4\|t_j\|_\infty \sup_{B\in\mathscr{B}(X_2)} \|M_A(B)\| = 4\|t_j\|_\infty \sup_{B\in\mathscr{B}(X_2)} \|s_k.m(\times B)\|.$$

Similarly, fix $B \in \mathscr{B}(X_2)$ and define $m_B : \mathscr{B}(X_1) \to E$ by $m_B(A) = m(A \times B)$ for every $A \in \mathscr{B}(X_1)$. Then

$$\sup_{A\in\mathscr{B}(X_1)} \|s_k.m_B(A)\| \le 4\|s_k\|_\infty \sup_{A\in\mathscr{B}(X_1)} \|m(A \times B)\|,$$

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so that

$$\operatorname{sv}((s_k \otimes t_j).m) = \sup_{\substack{A \in \mathscr{B}(X_1) \\ B \in \mathscr{B}(X_2)}} \|t_j.M_A(B)\| \le 16\|s_k\|_\infty \|t_j\|_\infty \sup_{\substack{A \in \mathscr{B}(X_1) \\ B \in \mathscr{B}(X_2)}} \|m(A \times B)\|.$$

For each $j = 1, 2, \ldots$, set $(f_1 \otimes t_j).m(A \times B) = \lim_{k\to\infty}(s_k \otimes t_j).m(A \times B)$ for every $A \in \mathscr{B}(X_1)$ and $B \in \mathscr{B}(X_2)$. The limit exists because m is a bounded additive set function in the first variable. That $(f_1 \otimes t_j).m$ is a bimeasure follows from the Vitali-Hahn-Saks theorem. Moreover, $(f_1 \otimes t_j).m$ is compact inner-regular in the first variable. Because $(f_1 \otimes 1).m$ is a bounded additive set function in the second variable, the limit

$$(f_1 \otimes f_2).m(A \times B) = \lim_{j\to\infty}(f_1 \otimes t_j).m(A \times B)$$

exists for every $A \in \mathscr{B}(X_1)$ and $B \in \mathscr{B}(X_2)$, and is again a bimeasure which is separately compact inner-regular. The inequality (1) clearly holds.

Now suppose that K_1 and K_2 are compact subsets of X_1 and X_2, respectively. Then by uniform convergence, for each $A \in \mathscr{B}(K_1) \times \mathscr{B}(K_2)$,

$$\begin{aligned}(f_1 \otimes f_2).m(A) &= \lim_{j\to\infty}\lim_{k\to\infty}(s_k \otimes t_j).m(A) \\ &= \lim_{j\to\infty}\lim_{k\to\infty}\int_A (s_k \otimes t_j)_{K_1\times K_2} dm_{K_1\times K_2} \\ &= \int_A (f_1 \otimes f_2)_{K_1\times K_2} dm_{K_1\times K_2},\end{aligned}$$

which proves that the bimeasure $(f_1 \otimes f_2).m$ is indeed a Radon bimeasure – the indefinite integral of f with respect to the Radon bimeasure m.

REMARK 1. The proof of Lemma 2 shows that for *any* bimeasure m, a natural meaning can be attached to the indefinite integral $(f \otimes g).m$ of the product of bounded measurable functions f and g with respect to m. In other words, integrate with respect to each variable separately. The proof also shows that if f and g are as above, then $\int_{X_1\times X_2} f \otimes g\, dm = \int_{X_1} f\, d\mu_g$, where μ_g is the E-valued Borel measure on X_1 defined by $\mu_g(A) = m(\chi_A \otimes g)$.

A similar proof shows that if E is a sequentially complete lcs, the tensor product $f = f_1 \otimes f_2$ of two functions $f_1 \in \mathscr{L}^\infty(X_1)$ and $f_2 \in \mathscr{L}^\infty(X_2)$ is m-integrable. In particular, it applies to the case where $E = \mathscr{L}_s(F)$ for a Banach space F. If for an additive set function $m : \mathscr{B}(X_1) \times \mathscr{B}(X_2) \to \mathscr{L}(F)$ bounded on the semi-algebra $\mathscr{B}(X_1) \times \mathscr{B}(X_2)$, we define the semivariation $\mathrm{sv}(m)$ by

$$\mathrm{sv}(m) = \sup\{\|m(A_1 \times A_2)\| : A_1 \in \mathscr{B}(X_1),\ A_2 \in \mathscr{B}(X_2)\},$$

where the norm is the operator norm on $\mathscr{L}(F)$, then when m is an $\mathscr{L}_s(F)$-valued Radon bimeasure, the inequality $\mathrm{sv}(fm) \leq 16\mathrm{sv}(m)\|f_1\|_\infty \|f_2\|_\infty$ holds. An appeal to the principle of uniform boundedness implies that $\mathrm{sv}(m) < \infty$, even though m is not necessarily a bimeasure for the topology of uniform convergence on $\mathscr{L}(F)$.

PROPOSITION 3. *Let E be a Banach space, and suppose that $m : \mathscr{B}(X_1) \times \mathscr{B}(X_2) \to E$ is a Radon bimeasure. Any function belonging to the complete projective tensor product $\mathscr{L}^\infty(X_1)\hat{\otimes}_\pi \mathscr{L}^\infty(X_2)$ of the spaces $\mathscr{L}^\infty(X_1)$ and $\mathscr{L}^\infty(X_2)$ is m-integrable.*

PROOF. Suppose that $f \in \mathscr{L}^\infty(X_1)\hat{\otimes}_\pi \mathscr{L}^\infty(X_2)$. Then there exist bounded Borel measurable functions $f_n : X_1 \to \mathbb{C}$ and $g_n : X_2 \to \mathbb{C}$, and numbers $\lambda_n \geq 0$, $n = 1, 2 \ldots$ such that

$$f(x_1, x_2) = \sum_{n=1}^{\infty} \lambda_n f_n(x_1) g_n(x_2)$$

for all $x_1 \in X_1$ and $x_2 \in X_2$, with $\|f_n\|_\infty \leq 1$ and $\|g_n\|_\infty \leq 1$ for all $n = 1, 2, \ldots$, and $\sum_{n=1}^\infty \lambda_n < \infty$ [22, Theorem 45.1]. Let b be the bimeasure defined by

$$b(A_1 \times A_2) = \sum_{n=1}^{\infty} \lambda_n [(f_n \otimes g_n)m](A_1 \times A_2)$$

for each $A_1 \in \mathscr{B}(X_1)$ and $A_2 \in \mathscr{B}(X_2)$. According to Lemma 2, $f_n \otimes g_n$ is m-integrable for each $n = 1, 2, \ldots$. The sum converges absolutely because

$$\sum_{n=1}^{\infty} \|\lambda_n [(f_n \otimes g_n)m](A_1 \times A_2)\| \leq 16\mathrm{sv}(m) \sum_{n=1}^{\infty} \lambda_n,$$

for all $A_1 \in \mathscr{B}(X_1)$, $A_2 \in \mathscr{B}(X_2)$, so that b is an E-valued bimeasure. Each of the bimeasures $(f_n \otimes g_n)m$ is separately compact inner-regular, so that b is too.

Let K be a compact product set in $X_1 \times X_2$, and let $\xi \in E'$. Then for every $A \in \mathscr{B}(X_1) \times \mathscr{B}(X_2)$,

$$\langle b, \xi\rangle(K \cap A) = \sum_{n=1}^{\infty} \lambda_n [(f_n \otimes g_n).\langle m, \xi\rangle](K \cap A)$$

$$= \sum_{n=1}^{\infty} \lambda_n \, [(f_n \otimes g_n)_K . \langle m, \xi \rangle_K] \, (A)$$
$$= \int_A f_K \, d\langle m, \xi \rangle_K .$$

Thus, $|\langle b, \xi \rangle|(K) \leq \|f\|_\infty |\langle m, \xi \rangle|(K)$. The same inequality holds if K is replaced by any product set U for which $|\langle m, \xi \rangle|(U)\rangle < \infty$. Because m is a Radon bimeasure, it follows that b is a Radon bimeasure. Consequently, f is m-integrable, and $fm = b$.

REMARK 2. Not all bounded Borel measurable functions are integrable with respect to a given bimeasure. For example, suppose that ϕ, $\psi \in L^2(\mathbb{R}) \backslash L^1(\mathbb{R})$, and $m : \mathscr{B}(\mathbb{R}^2) \to \mathbb{C}$ is the bimeasure defined by

$$m(A \times B) = \lim_{n \to \infty} \int_{A \cap [-n,n]} \int_{B \cap [-n,n]} \overline{\psi(y)} e^{-ixy} \phi(x) \, dx dy$$

for all A, $B \in \mathscr{B}(\mathbb{R})$. Clearly, $e^{i\alpha xy}$ is integrable with respect to m for all $\alpha \in \mathbb{R}$, $\alpha \neq 1$, but e^{ixy} is not m-integrable. This also shows that dominated convergence does not hold for bimeasures and functions which are not products of pairs of functions.

We state here a basic convergence theorem for polymeasures (but formulated just for bimeasures for ease of presentation). It is a simple application of dominated convergence for vector measures, and Remark 1 (see also [20, Theorem 1, p. 45]).

PROPOSITION 4. *Let E be a Banach space, and let $m : \mathscr{B}(X_1) \times \mathscr{B}(X_2) \to E$ be a Radon bimeasure. Suppose that for each $n = 1, 2, \dots$, $f_n : X_1 \to \mathbb{C}$ and $g_n : X_2 \to \mathbb{C}$ are bounded Borel measurable functions. Suppose also that there exists $C > 0$ such that $|f_n(x_1)| \leq C$ for all $x_1 \in X_1$, and $|g_n(x_2)| \leq C$ for all $x_2 \in X_2$, for every $n = 1, 2, \dots$.*

If $\lim_{n\to\infty} f_n = f$ pointwise on X_1 and $\lim_{n\to\infty} g_n = g$ pointwise on X_2, then the iterated limit

$$\lim_{m\to\infty} \lim_{n\to\infty} m(f_n \otimes g_m) = m(f \otimes g).$$

A stronger version of the bounded convergence theorem has been proved by Dobrakov [6, Theorem 3]. Furthermore, Dobrakov has given a general

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dominated convergence theorem for polymeasures with respect to n-tuples of functions [7, Theorem 10 and Corollary to Theorem 5]. (Note, however, that the proof of Theorem 5 is incomplete; it is given in article XIV of Dobrakov's sequence of papers).

The next convergence result is a consequence of the Vitali-Hahn-Saks theorem. If μ is a non-negative Radon measure on a Hausdorff space X, then $L^1_{\mathrm{loc}}(\mu)$ denotes the collection of all μ-equivalence classes of functions f such the $\mu(|f|\chi_K) < \infty$ for every compact set K in X. We give $L^1_{\mathrm{loc}}(\mu)$ the locally convex topology defined by the seminorms $f \to \mu(|f|\chi_K)$, $f \in L^1_{\mathrm{loc}}(\mu)$, with K a compact subset of X.

PROPOSITION 5. *Let X_1, X_2 be locally compact Hausdorff spaces. Let $m : \mathscr{B}(X_1) \times \mathscr{B}(X_2) \to \mathbb{C}$ be a Radon bimeasure. Suppose that for each $n = 1, 2, \dots$, $f_n : X_1 \times X_2 \to \mathbb{C}$ is an m-integrable function. If $f \in L^1_{\mathrm{loc}}(|m|)$, $\lim_{n\to\infty} f_n = f$ in $L^1_{\mathrm{loc}}(|m|)$, and $f_n m(A \times B)$ converges for every $A \in \mathscr{B}(X_1)$, $B \in \mathscr{B}(X_2)$, then f is m-integrable and $fm(A \times B) = \lim_{n\to\infty} f_n m(A \times B)$ for every $A \in \mathscr{B}(X_1)$, $B \in \mathscr{B}(X_2)$.*

We now look at a class of functions which are integrable with respect to every Radon bimeasure defined on certain product spaces. For Euclidean groups this class is often associated with functional calculi for systems of operators [1, 15]. Let G be a locally compact abelian group with dual group Γ. If $\mu : \mathscr{B}(G) \to \mathbb{C}$ is a measure, then the *Fourier-Stieltjes transform* $\hat{\mu}$ of μ is defined by

$$\hat{\mu}(\gamma) = \int_G \langle -g, \gamma \rangle \, d\mu(g), \qquad \text{for all } \gamma \in \Gamma.$$

The total variation $|\mu|(G)$ is denoted by $\|\mu\|$, and we have the estimate $\|\hat{\mu}\|_\infty \leq \|\mu\|$.

THEOREM 6. *Let G_1, G_2 be locally compact abelian groups, let $G = G_1 \times G_2$ and let $\mu : \mathscr{B}(G) \to \mathbb{C}$ be a Radon measure. Suppose that $m : \mathscr{B}(\Gamma_1) \times \mathscr{B}(\Gamma_2) \to E$ is a Radon bimeasure with values in the Banach space E. Then $\hat{\mu}$ is m-integrable and* $\mathrm{sv}(\hat{\mu} m) \leq 16 \mathrm{sv}(m) \|\mu\|$.

PROOF. For each $g \in G$, we have $\langle -g, \cdot \rangle \in \mathscr{L}^\infty(\Gamma_1) \otimes \mathscr{L}^\infty(\Gamma_2)$, so by Lemma 2 the function $\langle -g, \cdot \rangle$ is integrable with respect to the Radon bimeasure m, and $\mathrm{sv}(\langle -g, \cdot \rangle m) \leq 16 \mathrm{sv}(m)$. Now for every compact product set K in $\Gamma = \Gamma_1 \times \Gamma_2$, for every $\xi \in E'$, and every $A \in \mathscr{B}(\Gamma_1) \times \mathscr{B}(\Gamma_2)$,

$$\big\langle \big[\langle -g, \cdot \rangle m\big](K \cap A), \xi \big\rangle = \int_{K \cap A} \langle -g, \gamma \rangle \, d\langle m, \xi \rangle_K(\gamma).$$

Because $\langle m, \xi\rangle_K$ is a measure, the function $g \to \langle [\langle -g, \cdot\rangle m](K \cap A), \xi\rangle$, $g \in G$ is continuous. We now establish that the E-valued function $g \to [\langle -g, \cdot\rangle m](A)$, $g \in G$ is *strongly μ-measurable* for every set $A \in \mathscr{B}(\Gamma_1) \times \mathscr{B}(\Gamma_2)$; that is to say, it is the limit μ-a.e. in E of a sequence of E-valued $\mathscr{B}(\Gamma)$-simple functions.

Suppose that a function $f : G \to E$ is weakly continuous. Then $\mu \circ f^{-1}$ is a Radon measure for a weak topology $\sigma(E, E')$ of E, which is Radon-equivalent to the norm topology of E [21, Theorem 3, page 162], so there exist compact subsets K_n, $n = 1, 2, \ldots$ of G such that $|\mu \circ f^{-1}| \left(G \setminus \bigcup_{n=1}^{\infty} K_n\right) = 0$. Thus, f has μ-essentially separable range in E and it is scalarly measurable, so it is strongly μ-measurable [3, II. 1.2, page 42]. Therefore, it follows that the function $g \to [\langle -g, \cdot\rangle m](K \cap A)$, $g \in G$ is strongly measurable for each compact product set K in Γ, and each $A \in \mathscr{B}(\Gamma_1) \times \mathscr{B}(\Gamma_2)$.

Now let K_2 be a compact subset of Γ_2, and let $A_2 \in \mathscr{B}(\Gamma_2)$. The collection of all sets $A_1 \in \mathscr{B}(\Gamma_1)$ such that the function

$$g \to \big[\langle -g, \cdot\rangle m\big](A_1 \times (A_2 \cap K_2)), \qquad g \in G$$

is strongly measurable is a monotone class $\mathscr{M}_1$, because for each $g \in G$ the function $\langle -g, \cdot\rangle$ is integrable with respect to the Radon bimeasure m. Furthermore, $\mathscr{M}_1$ contains every set $A \cap K$ with $A \in \mathscr{B}(\Gamma_1)$ and with K a compact subset of Γ_1. Consequently, $\mathscr{M}_1 = \mathscr{B}(\Gamma_1)$.

Similarly, for fixed $A_1 \in \mathscr{B}(\Gamma_1)$, the collection of all sets $A_2 \in \mathscr{B}(\Gamma_2)$ such that the function

$$g \to \big[\langle -g, \cdot\rangle m\big](A_1 \times A_2), \qquad g \in G$$

is strongly measurable is a monotone class $\mathscr{M}_2$ containing every set $A \cap K$ with $A \in \mathscr{B}(\Gamma_2)$ and with K a compact subset of Γ_2. Consequently, $\mathscr{M}_2 = \mathscr{B}(\Gamma_2)$.

Therefore, the function $g \to [\langle -g, \cdot\rangle m]\,(A)$, $g \in G$ is strongly measurable for each set $A \in \mathscr{B}(\Gamma_1) \times \mathscr{B}(\Gamma_2)$, and it is bounded by $16\mathrm{sv}(m)$.

For every compact product set K contained in Γ and every $\xi \in E'$, we have, by Fubini's theorem for finite measures,

$$\begin{aligned}\int_G \big\langle \big[\langle -g, \cdot\rangle m\big](K \cap A), \xi\big\rangle\, d\mu(g) &= \int_G \big[\big(\langle -g, \cdot\rangle\big)_K \langle m, \xi\rangle_K\big](K \cap A)\, d\mu(g) \\ &= \int_{K\cap A} \left(\int_G \langle -g, \cdot\rangle d\mu(g)\right)_K d\langle m, \xi\rangle_K \\ &= \int_{K\cap A} (\hat{\mu})_K\, d\langle m, \xi\rangle_K .\end{aligned}$$

The set function $A \to \int_G \langle [\langle -g, \cdot \rangle m](A), \xi \rangle d\mu(g)$, $A \in \mathscr{B}(\Gamma_1) \times \mathscr{B}(\Gamma_2)$ is therefore bounded on the algebra generated by $(\mathscr{B}(\Gamma_1) \times \mathscr{B}(\Gamma_2)) \cap K$. It is a bimeasure, by dominated convergence, and it follows that $\hat{\mu}$ is $\langle m, \xi \rangle$-integrable and

$$\left(\hat{\mu}\langle m, \xi \rangle\right)(A) = \int_G \left\langle \left[\langle -g, \cdot \rangle m\right](A), \xi \right\rangle d\mu(g)$$

for every $A \in \mathscr{B}(\Gamma_1) \times \mathscr{B}(\Gamma_2)$. A bounded strongly measurable Banach space-valued function is Bochner integrable with respect to a finite measure, so for every $A \in \mathscr{B}(\Gamma_1) \times \mathscr{B}(\Gamma_2)$, the E-valued function $g \to [\langle -g, \cdot \rangle m](A)$, $g \in G$ is μ-integrable in E. Therefore, $\hat{\mu}$ is m-integrable and $\hat{\mu}m(A) = \int_G [\langle -g, \cdot \rangle m](A) d\mu(g)$ for every $A \in \mathscr{B}(\Gamma_1) \times \mathscr{B}(\Gamma_2)$.

Suppose that $G_1 = G_2 = \mathbb{R}$. It is well known that the Fourier transform maps the space of rapidly decreasing functions $\mathscr{S}(\mathbb{R}^2)$ on $\mathbb{R}^2$ into itself, and every rapidly decreasing function is integrable. It follows that any rapidly decreasing function can be represented as the Fourier-Stieltjes transform of a Borel measure. In particular, rapidly decreasing functions are m-integrable for any Banach space-valued Radon bimeasure on $\mathscr{B}(\mathbb{R}) \times \mathscr{B}(\mathbb{R})$. This also follows from Proposition 3, the representation $\mathscr{S}(\mathbb{R}^2) = \mathscr{S}(\mathbb{R}) \hat{\otimes}_\pi \mathscr{S}(\mathbb{R})$ [22, Theorem 51.6] and the continuous inclusion of the (metrizable) lcs $\mathscr{S}(\mathbb{R}) \otimes_\pi \mathscr{S}(\mathbb{R})$ into the normed space $\mathscr{L}^\infty(\mathbb{R}) \otimes_\pi \mathscr{L}^\infty(\mathbb{R})$.

There are other directions in which the technique above could be applied. For example, one could look at the integrals of functions $f : \Omega \to \mathscr{L}^\infty(\mathbb{R}) \hat{\otimes}_\pi \mathscr{L}^\infty(\mathbb{R})$ with respect to a finite measure μ on a σ-algebra of subsets of Ω.

If E is a Banach space, and m is an $\mathscr{L}_s(E)$-valued bimeasure on $\mathscr{B}(\mathbb{R}) \times \mathscr{B}(\mathbb{R})$, then it also follows the $\hat{\mu}$ is m-integrable, and $\mathrm{sv}(\hat{\mu}m) \leq 16 \mathrm{sv}(m) \|\mu\|$.

Part of the assumptions concerning a scalar Radon polymeasure m is that its variation should be the restriction of a Radon measure. On $\mathbb{R}^n$, this means that m is associated with a distribution of order zero. We now turn to a consideration of the nature of this association in the vector-valued setting.

Let $m : \mathscr{B}(X_1) \times \mathscr{B}(X_2) \to \mathbb{C}$ be a Radon bimeasure. Suppose that ϕ is a function on $X_1 \times X_2$ which is zero off the compact product set K, and continuous on K. According to Lemma 1, ϕ is m-integrable and $|\phi m(A)| \leq |m|(|\phi|) \leq |m|(K)\|\phi\|_\infty$ for all $A \in \mathscr{B}(X_1) \times \mathscr{B}(X_2)$. A slight modification is needed in the case that $m : \mathscr{B}(X_1) \times \mathscr{B}(X_2) \to E$ is a Radon bimeasure when E is infinite dimensional, because we have not assumed that there exists a *vector measure* $m_K : \mathscr{B}(K) \to E$ such that $\langle m, \xi \rangle|_{\mathscr{B}(K)} = \langle m_K, \xi \rangle$ for all $\xi \in E'$.

PROPOSITION 7. *Let K be a compact product subset of $X = X_1 \times X_2$. Let p be a continuous seminorm on the sequentially complete lcs E. Let $m : \mathscr{B}(X_1) \times \mathscr{B}(X_2) \to E$ be a Radon bimeasure.*

Suppose that f is a function on $X_1 \times X_2$ vanishing off K, and continuous on K. Then f is m-integrable. Moreover, there exists $\beta_{K,p} > 0$ such that $p(gm(A)) \le \beta_{K,p}\|g\|_\infty$ for all functions g vanishing off K and continuous on K, and all $A \in \mathscr{B}(X_1) \times \mathscr{B}(X_2)$.

PROOF. For any compact product set C, let m_C denote the E-valued Radon polymeasure defined by $m_C(A) = m(A \cap C)$ for all $A \in \mathscr{B}(X_1) \times \mathscr{B}(X_2)$.

For each $\xi \in E'$, the scalar-valued Radon bimeasure $\langle m_C, \xi\rangle$ is additive. Because $|\langle m_C, \xi\rangle|(C) = |\langle m, \xi\rangle|(C) < \infty$, it is the restriction to the semi-algebra $\mathscr{B}(X_1) \times \mathscr{B}(X_2)$ of a bounded additive set function on the algebra $[\mathscr{B}(X_1) \times \mathscr{B}(X_2)]$ generated by $\mathscr{B}(X_1) \times \mathscr{B}(X_2)$.

It follows that for any $\mathscr{B}(X_1) \times \mathscr{B}(X_2)$-simple function s, we have

$$(2)\quad |\langle m_C(s), \xi\rangle| \le 4 \sup\left\{|\langle m(A\cap C), \xi\rangle| : A \in [\mathscr{B}(X_1) \times \mathscr{B}(X_2)]\right\}\|\chi_C s\|_\infty.$$

A weakly bounded subset of E is bounded in the original topology of E, so

$$\beta_{C,p} = \sup\{p(m(A \cap C)) : A \in [\mathscr{B}(X_1) \times \mathscr{B}(X_2)]\} < \infty.$$

Taking the supremum of both sides of equation (2) as ξ ranges over the polar set $\{x \in E : p(x) \le 1\}^\circ$, we have

$$(3)\qquad p(m(\chi_C s)) = p(m_C(s)) \le 4\beta_{C,p}\|\chi_C s\|_\infty.$$

According to Lemma 2, the product $f_1 \otimes f_2$ of bounded Borel measurable functions f_1 and f_2 (on X_1 and X_2, respectively) is integrable. Moreover, from the proof it follows that if s_k, $k = 1, 2, \dots$ are $\mathscr{B}(X_1)$-simple functions and t_k, $k = 1, 2, \dots$ are $\mathscr{B}(X_2)$-simple functions such that $s_k \to f$ and $t_k \to g$ uniformly as $k \to \infty$, then for every $A \in \mathscr{B}(X_1) \times \mathscr{B}(X_2)$, the equality $(f_1 \otimes f_2).m(A) = \lim_{j\to\infty} \lim_{k\to\infty}(s_k \otimes t_j).m(A)$ holds. The estimate (3) therefore extends to all finite linear combinations of such $f_1 \otimes f_2$.

For each $j = 1, 2$ let $\pi_j : X \to X_j$ be the natural projection map. Let $K = K_1 \times K_2$ be a compact product set. Let $\mathscr{D}_K$ be the family of all finite linear combinations of functions of the form $(g_1 \circ \pi_1)(g_2 \circ \pi_2) : K \to \mathbb{C}$ with $g_1 : K_1 \to \mathbb{C}$ and $g_2 : K_2 \to \mathbb{C}$ bounded and continuous functions. By the Stone-Weierstrass theorem, $\mathscr{D}_K$ is dense in $C(K)$.

Now suppose that $g : X \to \mathbb{C}$ vanishes off K and $g|_K$ is continuous. There exist functions $g_j : X \to \mathbb{C}$ such that g_j vanishes off K, $g_j|_K \in \mathscr{D}_K$, for all $j = 1, 2, \ldots$, and $g_j \to g$ uniformly on K as $j \to \infty$. By the estimate (3), $p(m(\chi_A(g_k - g_j))) \leq 4\beta_{K,p}\|g_k - g_j\|_\infty$ for all $k, j = 1, 2, \ldots$, for any continuous seminorm p on E, and any set $A \in \mathscr{B}(X_1) \times \mathscr{B}(X_2)$, so it follows that g is m-integrable, and $p(m(g)) \leq 4\beta_{K,p}\|g\|_\infty$ for every $k = 1, 2, \ldots$.

REMARK 3. It follows from the above proposition, and the principle of uniform boundedness, that a Radon polymeasure on $\mathbb{R}^n$ with values in the space $\mathscr{L}_s(F)$ of bounded linear operators on a Banach space F is necessarily a distribution of order zero, in the sense that any smooth function of compact support is m-integrable, and for every compact subset K of $\mathbb{R}^n$, there exists $\beta_K > 0$ such that

$$\|m(f)\| \leq \beta_K \|f\|_\infty$$

for all smooth functions f on $\mathbb{R}^n$ with compact support contained in K.

The next result gives a condition for the integrability of a function in terms of distributions of order zero.

PROPOSITION 8. *Let X_1, X_2 be locally compact spaces and let $X = X_1 \times X_2$. Let $m : \mathscr{B}(X_1) \times \mathscr{B}(X_2) \to E$ be a Radon bimeasure with values in the lcs E.*

Then a function $f : X \to \mathbb{C}$ is m-integrable if and only if for each $\phi \in C_c(X)$, the function $f\phi$ is m-integrable, and there exists a Radon bimeasure $u_f : \mathscr{B}(X_1) \times \mathscr{B}(X_2) \to E$ such that

$$u_f(\phi) = \int_X f\phi \, dm$$

for all $\phi \in C_c(X)$.

PROOF. Suppose first that a Radon bimeasure u_f exists satisfying $u_f(\phi) = \int_X f\phi dm$ for all $\phi \in C_c(X)$. Let $\xi \in E'$. Let K be a compact product subset of X. Because X is locally compact, we can find $\phi \in C_c(X)$ such that $\phi(x) = 1$ for all $x \in K$. By assumption, the function $f\phi$ is m-integrable, which implies that the restriction $(f\phi)_K$ of $f\phi$ to the compact set K is integrable with respect to the measure $\langle m, \xi\rangle_K$. But $(f\phi)_K = f_K$, so it follows that $\int_K |f_K| \, d|\langle m, \xi\rangle_K| < \infty$.

Now suppose that K is the closure of a relatively compact open set U. Because

$$\langle u_f(\psi), \xi\rangle = \int_X f\psi \, d\langle m, \xi\rangle = \int_K f_K \psi \, d\langle m, \xi\rangle_K$$

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for all $\psi \in C_c(X)$ with supp $\psi \subseteq U$, it follows that $\langle u_f,\xi\rangle_K(A) = \int_A f_K d\langle m,\xi\rangle_K$ for all Borel subsets A of U. Here we have appealed to the fact that two Borel measures μ and ν on a locally compact space are equal, if $\mu(\psi) = \nu(\psi)$ for every continuous function ψ with compact support. In particular,

$$\langle u_f(L), \xi\rangle = \langle u_f, \xi\rangle_K(L) = \int_L f_K \, d\langle m, \xi\rangle_K = \int_L f_L \, d\langle m, \xi\rangle_L$$

for any compact product subset L of U. Because any compact product subset L of X is contained in some relatively compact open set U, we have $\langle u_f(L), \xi\rangle = \int_L f_L d\langle m, \xi\rangle_L$. This equality is true for any $\xi \in E'$, so f is m-integrable, and $u_f = fm$.

Now suppose that f is m-integrable. Let ϕ be a continuous function on X with support contained in the compact product set K. Let $\xi \in E'$. It follows from the argument before Proposition 7 that $\langle fm, \xi\rangle(\phi\chi_C) = \int_{K\cap C} \phi \, d\langle fm, \xi\rangle_K$ for each compact product set C. However, $\langle fm, \xi\rangle_K = f_K\langle m, \xi\rangle_K$ so that

$$\langle fm,\xi\rangle(\phi\chi_C) = \int_{K\cap C} \phi f_K \, d\langle m,\xi\rangle_K = \int_{K\cap C} (\phi f)_K \, d\langle m,\xi\rangle_K = \int_C (\phi f)_C \, d\langle m,\xi\rangle_C,$$

which proves that $f\phi$ is m-integrable, and $fm(\phi) = m(f\phi)$.

Suppose that we define the *support*, supp m, of a Radon polymeasure $m : \times_{j=1}^n \mathscr{B}(X_j) \to \mathbb{C}$ to be the support of $|m|$, where $|m|$ is the unique extension of the variation of m to a Radon measure on the Borel σ-algebra of $X = \prod_{j=1}^n X_j$. In other words, supp m is the smallest closed set whose complement is m-null. If $K =$ supp m is compact, then it is clear that the Radon polymeasure m is actually the restriction to $\times_{j=1}^n \mathscr{B}(X_j)$ of a scalar Radon measure on $\mathscr{B}(X)$. This follows from the estimate $|m(A)| \leq |m|(A \cap K)$, which is true for all A in the *algebra* generated by $\times_{j=1}^n \mathscr{B}(X_j)$.

Suppose now that the product X is locally compact and $C_c(X)$ denotes the space of continuous functions with compact support in X. If U is any open subset of X such that $m(f) = 0$ for any $f \in C_c(X)$ with supp $f \subseteq U$, then supp $m \subseteq U^c$. To see this, take a compact subset K of U. Then for any $f \in C_c(X)$ with support in K, $m_K(f) = m(f) = 0$, where m_K is the unique Radon measure on $\mathscr{B}(X)$ such that $m_K(A) = m(A\cap K)$ for all $A \in \times_{j=1}^n \mathscr{B}(X_j)$. It follows that $|m|(K) = |m_K|(K) = 0$. Therefore,

$$|m|(U) = \sup\{|m|(K) : K \subset U\} = 0,$$

proving that $\operatorname{supp} m \subseteq U^c$. If U is any open subset of X such that $|m|(U) = 0$, then for any $f \in C_c(X)$ with $\operatorname{supp} f \subseteq U$, it is clear that $|m(f)| \leq |m|(|f|) = 0$. Combining the two inclusions establishes the following result.

LEMMA 9. *Let $X = \prod_{j=1}^n X_j$ be the product of locally compact Hausdorff spaces $X_1, \ldots, X_n$ and let $m : \times_{j=1}^n \mathscr{B}(X_j) \to \mathbb{C}$ be a Radon polymeasure. Then*

(4) $\operatorname{supp} m = \bigcap\{U^c : m(f) = 0 \text{ for any } f \in C_c(X) \text{ with } \operatorname{supp} f \subseteq U\}.$

REMARK 4. A similar argument applies if $C_c(X)$ is replaced by other spaces. For example, if X is a product of smooth finite dimensional manifolds, then $C_c(X)$ can be replaced by the space of smooth functions with compact support defined on X.

For a Radon polymeasure $m : \times_{j=1}^n \mathscr{B}(X_j) \to E$ with values in a lcs E, we define $\operatorname{supp} m = \bigcap_{\xi \in E'} \operatorname{supp} \langle m, \xi \rangle$. The equality (4) holds in this case too.

It follows from Proposition 7 and Lemma 8, that if E is a sequentially complete lcs and $m : \times_{j=1}^n \mathscr{B}(\mathbb{R}) \to E$ is a Radon polymeasure, then the assignment $f \to \int_{\mathbb{R}^n} f\,dm$, $f \in C_c^\infty(\mathbb{R}^n)$ is a distribution of order zero (in E) whose support is precisely $\operatorname{supp} m$.

The final result illustrates the point that Radon polymeasures differ from genuine measures on the product space only because of their "behaviour at infinity". We say that a lcs E contains a copy of c_0 if and only if there exists a closed subspace F of E and an isometric isomorphism of c_0 onto F with its relative topology. Spaces which do *not* have this property include all weakly sequentially complete spaces (hence reflexive spaces) and others; see [10], for example.

PROPOSITION 10. *Let X_1, X_2 be Hausdorff topological spaces and let $X = X_1 \times X_2$. Let E be a sequentially complete lcs which does not contain a copy of c_0. If $m : \mathscr{B}(X_1) \times \mathscr{B}(X_2) \to E$ is a Radon bimeasure with compact support, then m is the restriction to $\mathscr{B}(X_1) \times \mathscr{B}(X_2)$ of an E-valued Radon measure on $\mathscr{B}(X)$.*

PROOF. Let K be a compact product set containing the support of m. Let $C(K)$ denote the collection of all continuous functions on K. Let $\mathscr{D}_K$ denote the linear space of all finite linear combinations of continuous product functions on K. By the Stone-Weierstrass theorem, $\mathscr{D}_K$ is dense in $C(K)$. According to Lemma 2, every function $f \in \mathscr{D}_K$ is m-integrable.

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For every $\xi \in E'$ we have $|\langle m, \xi\rangle|(K) < \infty$, so m is weakly bounded on the algebra $[K \cap (\mathscr{B}(X_1) \times \mathscr{B}(X_2))]$ generated by $\{K \cap A : A \in \mathscr{B}(X_1) \times \mathscr{B}(X_2)\}$. Then m is bounded in E on $[K \cap (\mathscr{B}(X_1) \times \mathscr{B}(X_2))]$. It follows that for every neighbourhood U of zero in E, $\sup_{\xi \in U^\circ} |\langle m, \xi\rangle|(K) < \infty$. Define a linear map $\Phi : \mathscr{D}_K \to E$ by $\Phi(f) = m(f)$ for all $f \in \mathscr{D}_K$.

If p is a continuous seminorm on E, set $U_p = \{x \in E : p(x) \leq 1\}$. Then $p(\Phi(f)) \leq \sup_{\xi \in U_p^\circ} |\langle m, \xi\rangle|(K)\|f\|_\infty$ for every $f \in \mathscr{D}_K$. Here $\|\cdot\|_\infty$ is the sup norm on $C(K)$. By virtue of the sequential completeness of E, there exists a unique continuous linear extension $\bar{\Phi} : C(K) \to E$ of Φ. Because E contains no copy of c_0, there exists a unique E-valued Radon measure $m_K : \mathscr{B}(K) \to E$ such that $m_K(f) = \bar{\Phi}(f)$ for all $f \in C(K)$, [16].

For all $f \in \mathscr{D}_K$ and $\xi \in E'$, $\langle m_K(f), \xi\rangle = \langle m(f), \xi\rangle = \langle m, \xi\rangle_K(f)$ from which it follows that $\langle m_K(f), \xi\rangle = \langle m, \xi\rangle_K(f)$ for all $f \in C(K)$, so that $\langle m_K(A), \xi\rangle = \langle m, \xi\rangle_K(A)$ for all $A \in \mathscr{B}(K)$. In particular, $\langle m_K(A), \xi\rangle = \langle m, \xi\rangle_K(A), = \langle m(A), \xi\rangle$ for all $A \in \mathscr{B}(X_1) \times \mathscr{B}(X_2)$.

If $A \in \mathscr{B}(X_1) \times \mathscr{B}(X_2)$, then $A \backslash K$ is the union of pairwise disjoint sets $V_j \in \mathscr{B}(X_1) \times \mathscr{B}(X_2)$, $j = 1, \dots, n$, all disjoint from K. Because K contains the support of m, $|m(V_j)| \leq |m|(V_j) \leq |m|(K^c) = 0$, for all $j = 1, \dots, n$. The additivity of m ensures that

$$m(A) = m(A \cap K) + \sum_{j=1}^{n} m(V_j) = m(A \cap K) = m_K(A \cap K),$$

where $B \to m_K(B \cap K)$, $B \in \mathscr{B}(X)$ is an E-valued Radon measure on X.

REMARK 5. The requirement that E contains no copy of c_0 is essential. Indeed, let $E = c_0$ and $X = X_1 \times X_2$, where $X_j = [-\pi, \pi]$, $j = 1, 2$. For each integer $n \geq 1$, let $f_n(x_1) = n\chi_{[0,n^{-1}]}(x_1)$, $x_1 \in X_1$, and $g_n(x_2) = e^{-inx_2}$, $x_2 \in X_2$. Define a set function $\nu : \mathscr{B}(X) \to \ell^\infty$ by

$$\nu(C) = \left\{\int_C f_n(x_1)g_n(x_2)\,dx_1dx_2\right\}_{n=1}^{\infty}, \qquad C \in \mathscr{B}(X).$$

It follows from the integrability (with respect to Lebesgue measure on X) of the function $\psi = \sum_{n=1}^\infty \xi_n f_n \otimes g_n$, whenever $\xi = \{\xi_n\}_{n=1}^\infty \in \ell^1$, that $\langle \nu(C), \xi\rangle = \int_C \psi(x_1, x_2)dx_1dx_2$, for all $C \in \mathscr{B}(X)$. This shows that ν is σ-additive for the weak*-topology $\sigma(\ell^\infty, \ell^1)$ on ℓ^∞, and hence, the range of the set function ν is a bounded subset of ℓ^∞.

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Now suppose that A and B are Borel sets in $[-\pi, \pi]$. Let $\hat{\chi}_B$ denote the Fourier transform of χ_B. It follows that $|\nu(A \times B)_n| \leq |\hat{\chi}_B(n)|$, for all $n = 1, 2, \ldots$, and hence, $\nu(A \times B) \in c_0$ by the Riemann-Lebesgue lemma. Accordingly, the range of the restriction of ν to the algebra $[\mathscr{B}(X_1) \times \mathscr{B}(X_2)]$ is contained in the subspace c_0 of ℓ^∞, and it is bounded there.

The restriction m of ν to $\mathscr{B}(X_1) \times \mathscr{B}(X_2)$ is a compactly supported Radon bimeasure in c_0 which is σ-addiitive on $[\mathscr{B}(X_1) \times \mathscr{B}(X_2)]$, and it has bounded range in c_0. However, the claim is that m is *not* the restriction of an E-valued measure on $\mathscr{B}(X)$. To see this, assume the contrary, that is, suppose that the range of ν on $\mathscr{B}(X)$ is contained in c_0. Then, for any sequence of Borel sets $E_r \subseteq [-\pi, \pi]$, $r = 1, 2, \ldots$, decreasing to the empty set, it would follow that $\lim_{r\to\infty} \sup\{\|m(E_r \times F)\|_\infty : F \in \mathscr{B}([-\pi, \pi])\} = 0$ [18, II Section 1 Lemma 3].

Now for the sets $E_r = [0, r^{-1}]$ and $F_r = \{x_2 \in X_2 : \mathrm{Re}\,(g_r(x_2)) = \cos(rx_2) \geq 1/\sqrt{2}\}$, $r = 1, 2, \ldots$, it follows that $\int_{E_r} f_n(x_1)dx_1 = 1$, for all $n \geq r$, so we have (for r fixed)

$$
\begin{aligned}
\|m(E_r \times F_r)\|_\infty &= \sup_{n\geq 1} \left| \int_{E_r} f_n(x_1)\,dx_1 \right| \cdot \left| \int_{F_r} g_n(x_2)\,dx_2 \right| \\
&\geq \sup_{n\geq r} \left| \int_{F_r} g_n(x_2)\,dx_2 \right| \\
&\geq \left| \int_{F_r} g_r(x_2)\,dx_2 \right| \geq \frac{1}{\sqrt{2}}.\lambda(F_r)
\end{aligned}
$$

for each $r = 1, 2, \ldots$, where λ is the Lebesgue measure on $[-\pi, \pi]$. Since $\lim_{r\to\infty} \lambda(F_r) = \pi/2$, the range of ν on $\mathscr{B}(X)$ cannot be contained in c_0.

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