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2-Local Isometries on Spaces of Lipschitz Functions

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Abstract. Let (X, d) be a metric space, and let $\mathrm{Lip}(X)$ denote the Banach space of all scalar-valued bounded Lipschitz functions f on X endowed with one of the natural norms

$$\|f\| = \max\{\|f\|_\infty, L(f)\} \quad \text{or} \quad \|f\| = \|f\|_\infty + L(f),$$

where $L(f)$ is the Lipschitz constant of f. It is said that the isometry group of $\mathrm{Lip}(X)$ is canonical if every surjective linear isometry of $\mathrm{Lip}(X)$ is induced by a surjective isometry of X. In this paper we prove that if X is bounded separable and the isometry group of $\mathrm{Lip}(X)$ is canonical, then every 2-local isometry of $\mathrm{Lip}(X)$ is a surjective linear isometry. Furthermore, we give a complete description of all 2-local isometries of $\mathrm{Lip}(X)$ when X is bounded.

1 Introduction

In [14], Šemrl introduced the following concept. A map Φ of an algebra A into itself is a 2-*local automorphism* (respectively, 2-*local derivation*) if for every $a, b \in A$ there is an automorphism (respectively, derivation) $\Phi_{a,b}\colon A \to A$, depending on a and b, such that $\Phi_{a,b}(a) = \Phi(a)$ and $\Phi_{a,b}(b) = \Phi(b)$. Šemrl [14] proved that every 2-local automorphism of the algebra $B(H)$ of all bounded linear operators on an infinite-dimensional separable Hilbert space H is an automorphism, and a similar assertion was stated concerning the 2-local derivations.

Motivated by these results, Molnár [10] extended the notion of 2-locality to isometries as follows. Given a Banach space X, it is said that a map $\Phi\colon X \to X$ is a 2-*local isometry* if for every $x, y \in X$ there is a surjective linear isometry $\Phi_{x,y}\colon X \to X$, which depends on x and y, such that $\Phi_{x,y}(x) = \Phi(x)$ and $\Phi_{x,y}(y) = \Phi(y)$ (no linearity or surjectivity of Φ is assumed). Molnár [10] showed that every 2-local isometry of $B(H)$ is a surjective linear isometry. Numerous papers on 2-locality have since appeared [9, 11, 17], and more recently [1, 5–8, 18].

Furthermore, Molnár [10] introduced the study of 2-locality for function algebras. In this direction, Győry [2] showed that if X is a first countable σ-compact Hausdorff space and $\mathcal{C}_0(X)$ is the Banach space of all scalar-valued continuous functions on X vanishing at infinite endowed with the uniform norm, then every 2-local isometry of $\mathcal{C}_0(X)$ is a surjective linear isometry. Recently, Hatori et al. [3] considered 2-local isometries on uniform algebras including certain algebras of holomorphic functions

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of one and two complex variables. In this paper, we shall study 2-local isometries on spaces of Lipschitz functions.

Let X be a metric space, and let $\mathrm{Lip}(X)$ be the Banach space of all scalar-valued bounded Lipschitz functions on X, equipped either with the maximum norm $\|f\| = \max\{\|f\|_\infty, L(f)\}$ or with the sum norm $\|f\| = \|f\|_\infty + L(f)$, where

$$\|f\|_\infty = \sup\{|f(x)| : x \in X\}$$

is the uniform norm of f, and

$$L(f) = \sup\left\{ \frac{|f(x) - f(y)|}{d(x, y)} : x, y \in X,\ x \neq y \right\}$$

is the Lipschitz constant of f.

The surjective linear isometries of $\mathrm{Lip}(X)$ have been the subject of considerable study [4, 13, 15, 16]. It is easy to see that if τ is a scalar of modulus 1 and φ is a surjective isometry of X, then the weighted composition operator

$$\Phi(f) = \tau(f \circ \varphi), \quad \forall f \in \mathrm{Lip}(X),$$

is a surjective linear isometry of $\mathrm{Lip}(X)$. If every surjective linear isometry of $\mathrm{Lip}(X)$ is of the above-mentioned form, we shall say, for brevity, that the isometry group of $\mathrm{Lip}(X)$ is canonical. In general, the isometry group of $\mathrm{Lip}(X)$ is not canonical (see an example in [16]). However, Rao and Roy [12] proved that the isometry group of $\mathrm{Lip}[0, 1]$ endowed with the sum norm is canonical. On the other hand, with the maximum norm on $\mathrm{Lip}(X)$, the same conclusion was obtained, among others, by Roy [13] when X is compact and connected with diameter at most 1; and, independently, by Vasavada [15] when X is compact and satisfies certain separation conditions.

We now describe the matter of this paper. In Section 2, under the condition that X is a bounded metric space and the isometry group of $\mathrm{Lip}(X)$ is canonical, we shall prove that every 2-local isometry $\Phi\colon \mathrm{Lip}(X) \to \mathrm{Lip}(X)$ is essentially a weighted composition operator of the form

$$\Phi(f)\,|_{X_0} = \tau(f \circ \varphi), \quad \forall f \in \mathrm{Lip}(X),$$

where X_0 is a subset of X, τ is a unimodular scalar and $\varphi\colon X_0 \to X$ is a Lipschitz bijection.

As we have seen above, the main problem concerning 2-local isometries on Banach spaces is to answer the question whether the 2-local isometries are surjective linear isometries. In Section 3, we shall give a positive answer for 2-local isometries of $\mathrm{Lip}(X)$. Namely, when X is, in addition, separable, we shall show that $X_0 = X$ and φ is a Lipschitz homeomorphism, and therefore Φ is a surjective linear isometry of $\mathrm{Lip}(X)$.

2 Representation of 2-Local Isometries

Let (X, d) be a metric space. Throughout this paper we shall frequently use the following functions. For $x \in X$ and $\delta > 0$, define $h_{x,\delta}\colon X \to [0,1]$ by

$$h_{x,\delta}(z) = \max\left\{0, 1 - \frac{d(z,x)}{\delta}\right\}.$$

Clearly, $h_{x,\delta} \in \mathrm{Lip}(X)$ with $L(h_{x,\delta}) \leq 1/\delta$; $h_{x,\delta}(z) = 0$ if $d(z,x) \geq \delta$, and $h_{x,\delta}(z) = 1$ if and only if $z = x$.

As usual, $\mathbb{K}$ will denote the field of real or complex numbers, and $S_{\mathbb{K}}$ the set of all unimodular scalars of $\mathbb{K}$. Given $\alpha \in \mathbb{K}$, $\widehat{\alpha}$ will stand for the function constantly equal α on X.

Theorem 2.1 *Let X be a bounded metric space, and let Φ be a 2-local isometry of $\mathrm{Lip}(X)$ whose isometry group is canonical. Then there exists a subset X_0 of X, a unimodular scalar τ and a bijective Lipschitz map $\varphi\colon X_0 \to X$ such that*

$$\Phi(f)|_{X_0} = \tau(f \circ \varphi), \quad \forall f \in \mathrm{Lip}(X).$$

Proof Let $g \in \mathrm{Lip}(X)$. Since Φ is a 2-local isometry of $\mathrm{Lip}(X)$, there exists a surjective linear isometry $\Phi_{\hat{1},g}$ of $\mathrm{Lip}(X)$ such that $\Phi(\hat{1}) = \Phi_{\hat{1},g}(\hat{1})$ and $\Phi(g) = \Phi_{\hat{1},g}(g)$. Because the isometry group of $\mathrm{Lip}(X)$ is canonical, we have

$$\Phi_{\hat{1},g}(f) = \tau_{\hat{1},g}(f \circ \varphi_{\hat{1},g}), \quad \forall f \in \mathrm{Lip}(X),$$

where $\tau_{\hat{1},g} \in S_{\mathbb{K}}$ and $\varphi_{\hat{1},g}$ is a surjective isometry of X. Obviously, $\Phi(\hat{1}) = \Phi_{\hat{1},g}(\hat{1}) = \hat{\tau}_{\hat{1},g}$ and, since g is arbitrary, $\Phi(\hat{1}) = \hat{\tau}_{\hat{1},\hat{1}}$. Define $\Phi_0 = \overline{\tau}_{\hat{1},\hat{1}}\Phi$. Clearly,

$$\Phi_0(g) = \overline{\tau}_{\hat{1},\hat{1}}\tau_{\hat{1},g}(g \circ \varphi_{\hat{1},g}) = g \circ \varphi_{\hat{1},g}$$

since $\overline{\tau}_{\hat{1},\hat{1}}\tau_{\hat{1},g} = \overline{\tau}_{\hat{1},\hat{1}}\tau_{\hat{1},\hat{1}} = 1$. By the surjectivity of $\varphi_{\hat{1},g}$, it follows that

$$\Phi_0(g)(X) = g(X).$$

For $x \in X$ and $f \in \mathrm{Lip}(X)$, define

$$E_{x,f} = \{y \in X : \Phi_0(f)(y) = f(x)\}.$$

Next we show that $E_{x,f}$ is nonempty and $\bigcap_{f \in \mathrm{Lip}(X)} E_{x,f}$ is a singleton. Notice that $\Phi_0(h_{x,1})(X) = h_{x,1}(X) \subset [0,1]$. Furthermore,

$$\Phi_0(h_{x,1}) = \overline{\tau}_{\hat{1},\hat{1}}\Phi(h_{x,1}) = \overline{\tau}_{\hat{1},\hat{1}}\tau_{h_{x,1},f}(h_{x,1} \circ \varphi_{h_{x,1},f}),$$

$$\Phi_0(f) = \overline{\tau}_{\hat{1},\hat{1}}\Phi(f) = \overline{\tau}_{\hat{1},\hat{1}}\tau_{h_{x,1},f}(f \circ \varphi_{h_{x,1},f}),$$

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where $\tau_{h_{x,1},f} \in S_{\mathbb{K}}$ and $\varphi_{h_{x,1},f}$ is a surjective isometry of X. Therefore

$$\begin{aligned} E_{x,h_{x,1}} &= \{y \in X : |\Phi_0(h_{x,1})(y)| = 1\} = \{y \in X : h_{x,1}(\varphi_{h_{x,1},f}(y)) = 1\} \\ &= \{y \in X : \varphi_{h_{x,1},f}(y) = x\}. \end{aligned}$$

This last set has a unique point b_x, since $\varphi_{h_{x,1},f}$ is bijective. Hence $E_{x,h_{x,1}} = \{b_x\}$. It follows that

$$\Phi_0(f)(b_x) = \overline{\tau}_{\hat{1},\hat{1}} \tau_{h_{x,1},f} f(\varphi_{h_{x,1},f}(b_x)) = \overline{\tau}_{\hat{1},\hat{1}} \tau_{h_{x,1},f} f(x),$$

and since $\overline{\tau}_{\hat{1},\hat{1}} \tau_{h_{x,1},f} = \Phi_0(h_{x,1})(b_x) = 1$, we have $\Phi_0(f)(b_x) = f(x)$. This means that $b_x \in E_{x,f}$, and thus $E_{x,h_{x,1}} = \{b_x\} \subset E_{x,f}$. Since f is arbitrary, we conclude that $\bigcap_{f \in \mathrm{Lip}(X)} E_{x,f} = \{b_x\}$.

Hence we can define a function $\psi \colon X \to X$ by

$$\{\psi(x)\} = \bigcap_{f \in \mathrm{Lip}(X)} E_{x,f}.$$

Now we see that ψ is injective. Let $x, y \in X, x \neq y$. Since $\psi(x) \in E_{x,h_{x,1}}$ and $\psi(y) \in E_{y,h_{x,1}}$, we have $\Phi_0(h_{x,1})(\psi(x)) = h_{x,1}(x) = 1$ and $\Phi_0(h_{x,1})(\psi(y)) = h_{x,1}(y) \neq 1$, which implies $\psi(x) \neq \psi(y)$.

Put $X_0 = \psi(X)$ and let φ be the bijection $\psi^{-1} \colon X_0 \to X$. Let $y \in X_0$. Clearly, $y = \psi(\varphi(y)) \in \bigcap_{f \in \mathrm{Lip}(X)} E_{\varphi(y),f}$, and therefore, for every $f \in \mathrm{Lip}(X)$, we have $y \in E_{\varphi(y),f}$, that is, $f(\varphi(y)) = \Phi_0(f)(y) = \overline{\tau}_{\hat{1},\hat{1}} \Phi(f)(y)$, which yields $\Phi(f)(y) = \tau_{\hat{1},\hat{1}} f(\varphi(y))$.

Taking $\tau = \tau_{\hat{1},\hat{1}}$, then $|\tau| = 1$ and so we have shown that

$$\Phi(f)(y) = \tau f(\varphi(y)), \quad \forall y \in X_0, \ \forall f \in \mathrm{Lip}(X).$$

It remains to prove that φ is Lipschitz. For each $x \in X$, define

$$f_x(z) = d(z, x), \quad \forall z \in X.$$

Clearly, $f_x \in \mathrm{Lip}(X)$ and $\|f_x\| \leq \|f_x\|_\infty + L(f_x) \leq \mathrm{diam}(X) + 1$, where $\mathrm{diam}(X)$ denotes the diameter of X. Put $k = \mathrm{diam}(X) + 1$. Then

$$\|\Phi(f_x)\| = \|\Phi_{f_x,\hat{1}}(f_x)\| = \|f_x\| \leq k,$$

since $\Phi(f_x) = \Phi_{f_x,\hat{1}}(f_x)$ and $\Phi_{f_x,\hat{1}}$ is an isometry of $\mathrm{Lip}(X)$.

Let $x, y \in X_0$. Since $L(\Phi(f_{\varphi(y)})) \leq \|\Phi(f_{\varphi(y)})\|$, we have

$$|\Phi(f_{\varphi(y)})(x) - \Phi(f_{\varphi(y)})(y)| \leq k d(x, y).$$

Taking into account that

$$\begin{aligned} \Phi(f_{\varphi(y)})(x) &= \tau f_{\varphi(y)}(\varphi(x)) = \tau d(\varphi(x), \varphi(y)), \\ \Phi(f_{\varphi(y)})(y) &= \tau f_{\varphi(y)}(\varphi(y)) = \tau d(\varphi(y), \varphi(y)) = 0, \end{aligned}$$

we conclude that $d(\varphi(x), \varphi(y)) \leq k d(x, y)$. ■

3 A Problem of Algebraic Reflexivity for 2-Local Isometries

In this section we shall prove our main result: every 2-local isometry of $\mathrm{Lip}(X)$ is a surjective linear isometry, when X is a separable bounded metric space and the isometry group of $\mathrm{Lip}(X)$ is canonical. To prepare the proof of this fact, we begin with the following.

Lemma 3.1 *Let X be a metric space, and let $R = \{r_n : n \in \mathbb{N}\}$ be a countable set of pairwise distinct points of X. Then there exists a Lipschitz function $f : X \to [0, 1]$ with $L(f) \le 1$ satisfying the following properties:*

(a) $0 < f(r_j)$ *for all* $j \in \mathbb{N}$ *and* $f(r_i) \neq f(r_j)$ *if* $i, j \in \mathbb{N}$ *with* $i \neq j$.

(b) *For each* $j \in \mathbb{N}$, *there exists* $t_j \in]0, 1]$ *such that*

$$d(x, r_j) < t_j/4 \quad \Rightarrow \quad f(x) \le f(r_j) - d(x, r_j)/2.$$

Hence f has a strict local maximum at r_j.

(c) $f(r_1) = 1$ *and* $f(x) < 1$ *if* $x \neq r_1$.

Proof We define two sequences $\{t_n\}$ and $\{s_n\}$ of positive scalars and a sequence $\{f_n\}$ of non-negative functions on X as follows:

$$t_1 = 1, \quad s_1 = 1, \quad f_1 = h_{r_1,1},$$

and for each positive integer n,

$$t_{n+1} = \min\left(\left\{ \frac{t_n}{3} \right\} \cup \left\{ \frac{f_n(r_j) - f_n(r_{n+1})}{4} : f_n(r_{n+1}) < f_n(r_j),\ j \in \{1, \dots, n\} \right\} \right),$$

$$s_{n+1} = t_{n+1} + f_n(r_{n+1}),$$

$$f_{n+1} = \max\{f_n, s_{n+1} h_{r_{n+1}, s_{n+1}}\}.$$

A plain argument by induction allows us to see that, for each $n \in \mathbb{N}$, $f_n \in \mathrm{Lip}(X)$, $L(f_n) \le 1$, $f_n(r_1) = 1$ and $0 \le f_n(x) < 1$ if $x \neq r_1$.

Next we prove the following.

Claim For each $n \in \mathbb{N}$,

(i) $s_1, \dots, s_n$ are pairwise distinct;

(ii) If $s_j < s_n$ for some $j \in \{1, \dots, n\}$, then $s_n\, h_{r_n, s_n}(r_j) \le f_{j-1}(r_j) + \sum_{i=j+1}^{n} t_i$;

(iii) $f_n(r_j) = s_j$ for all $j \in \{1, \dots, n\}$.

The proof is by induction on n. Assertions (i), (ii), and (iii) are trivial for $n = 1$. Assume (i), (ii), and (iii) hold for $1, \dots, n$; we shall prove it for $n + 1$.

To see that $s_1, \dots, s_n, s_{n+1}$ are pairwise distinct, let $j \in \{1, \dots, n\}$. Using (iii), if $f_n(r_j) \le f_n(r_{n+1})$, we have

$$s_j = f_n(r_j) < f_n(r_{n+1}) + t_{n+1} = s_{n+1};$$

and if $f_n(r_{n+1}) < f_n(r_j)$,

$$s_{n+1} = t_{n+1} + f_n(r_{n+1}) \leq \frac{f_n(r_j) - f_n(r_{n+1})}{4} + f_n(r_{n+1}) < f_n(r_j) = s_j.$$

In any case, $s_{n+1} \neq s_j$, and we have finished.

To prove (ii) for $n+1$, let $j \in \{1, \dots, n+1\}$ and suppose that $s_j < s_{n+1}$ (which implies $j < n+1$). Clearly, $s_{n+1} < s_1$, therefore $j > 1$. We distinguish two cases.

Case 1. If $f_n(r_{n+1}) = f_{n-1}(r_{n+1}) = \cdots = f_j(r_{n+1})$, then $f_j(r_{n+1}) = f_{j-1}(r_{n+1})$. In the contrary case we have $f_j(r_{n+1}) = s_j\, h_{r_j,s_j}(r_{n+1})$, and therefore

$$f_n(r_{n+1}) = f_j(r_{n+1}) < s_j = f_n(r_j).$$

Hence $s_{n+1} < f_n(r_j) = s_j < s_{n+1}$, a contradiction. Thus $f_j(r_{n+1}) = f_{j-1}(r_{n+1})$. Since $L(f_{j-1}) \leq 1$, we have

$$\begin{aligned} s_{n+1}&\left(1 - \frac{d(r_{n+1}, r_j)}{s_{n+1}}\right) \\ &= t_{n+1} + f_n(r_{n+1}) - d(r_{n+1}, r_j) = t_{n+1} + f_{j-1}(r_{n+1}) - d(r_{n+1}, r_j) \\ &\leq t_{n+1} + f_{j-1}(r_j) \leq \sum_{i=j+1}^{n+1} t_i + f_{j-1}(r_j), \end{aligned}$$

and therefore

$$s_{n+1}\, h_{r_{n+1},s_{n+1}}(r_j) \leq \sum_{i=j+1}^{n+1} t_i + f_{j-1}(r_j).$$

Case 2. Suppose Case 1 does not hold. Then $j < n$, and there is $i \in \{j+1, \dots, n\}$ such that

$$f_n(r_{n+1}) = f_{n-1}(r_{n+1}) = \cdots = f_i(r_{n+1}) = s_i\, h_{r_i,s_i}(r_{n+1}).$$

If $s_i \leq s_j$, using (iii), we have

$$f_n(r_{n+1}) = s_i\, h_{r_i,s_i}(r_{n+1}) < s_i = f_n(r_i),$$

and therefore

$$s_{n+1} = t_{n+1} + f_n(r_{n+1}) \leq \frac{f_n(r_i) - f_n(r_{n+1})}{4} + f_n(r_{n+1}) < f_n(r_i) = s_i \leq s_j < s_{n+1},$$

which is impossible. Hence $s_j < s_i$. Then

$$\begin{aligned} s_{n+1}\left(1 - \frac{d(r_{n+1}, r_j)}{s_{n+1}}\right) &= t_{n+1} + s_i\, h_{r_i,s_i}(r_{n+1}) - d(r_{n+1}, r_j) \leq t_{n+1} + s_i\, h_{r_i,s_i}(r_j) \\ &\leq t_{n+1} + \sum_{l=j+1}^{i} t_l + f_{j-1}(r_j) \leq \sum_{l=j+1}^{n+1} t_l + f_{j-1}(r_j) \end{aligned}$$

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and, in consequence,

$$s_{n+1}\, h_{r_{n+1},s_{n+1}}(r_j) \le \sum_{l=j+1}^{n+1} t_l + f_{j-1}(r_j).$$

Finally, we check (iii) for $n+1$. Let $j \in \{2, \dots, n\}$. If $s_j < s_{n+1}$, as proved above, we have

$$s_{n+1}\, h_{r_{n+1},s_{n+1}}(r_j) \le \sum_{l=j+1}^{n+1} t_l + f_{j-1}(r_j) \le t_j \sum_{l=1}^{n+1-j} \frac{1}{3^l} + f_{j-1}(r_j) < s_j = f_n(r_j)$$

and thus $f_{n+1}(r_j) = f_n(r_j) = s_j$. If $s_{n+1} \le s_j$, since $s_{n+1}\, h_{r_{n+1},s_{n+1}}(r_j) < s_{n+1}$, it is clear that $f_{n+1}(r_j) = s_j$. For $j = n+1$, $f_{n+1}(r_{n+1}) = s_{n+1}$ follows easily, and this completes the proof of our claims. ∎

Now we find f. Pick $n \in \mathbb{N}$. We shall prove that $\|f_{n+1} - f_n\|_\infty \le t_{n+1}$. Given $x \in X$, we have either

$$f_{n+1}(x) - f_n(x) = 0 < t_{n+1}$$

or

$$\begin{aligned} 0 < f_{n+1}(x) - f_n(x) &= s_{n+1} h_{r_{n+1},s_{n+1}}(x) - f_n(x) \\ &= s_{n+1}\left(1 - \frac{d(x, r_{n+1})}{s_{n+1}}\right) - f_n(x) \\ &= t_{n+1} + f_n(r_{n+1}) - d(x, r_{n+1}) - f_n(x) \le t_{n+1}. \end{aligned}$$

Therefore,

$$\|f_{n+1} - f_n\|_\infty \le t_{n+1} \le \frac{t_1}{3^n} = \frac{1}{3^n}, \quad \forall n \in \mathbb{N}. \tag{3.1}$$

Hence $\{f_n\}$ is a Cauchy sequence in $(\mathcal{C}_b(X), \|\cdot\|_\infty)$, where $\mathcal{C}_b(X)$ denotes the space of all scalar-valued bounded continuous functions on X. Then there exists $f \in \mathcal{C}_b(X)$ such that $\lim_{n\to\infty} \|f_n - f\|_\infty = 0$, and it is immediate to check that $f \in \operatorname{Lip}(X)$ and $L(f) \le 1$.

Next we prove (a), (b), and (c). Since $f(r_j) = s_j$ for all $j \in \mathbb{N}$ and the scalars s_j are positive and pairwise distinct, (a) follows.

To prove (b), let $j \in \mathbb{N}$ and $x \in X$ be such that $0 < d(x, r_j) < t_j/4$. For $j > 1$ observe that

$$f_{j-1}(x) \le f_{j-1}(r_j) + d(x, r_j) < s_j - d(x, r_j) = s_j h_{r_j,s_j}(x),$$

and therefore $f_j(x) = s_j\, h_{r_j,s_j}(x)$. Moreover, $f_1 = s_1\, h_{r_1,s_1}$.

If $f_n(x) = f_j(x)$ for all $n > j$, using that $s_j = f(r_j)$ we have

$$f(x) = f_j(x) = s_j h_{r_j,s_j}(x) = s_j - d(x, r_j) < f(r_j) - \frac{d(x, r_j)}{2},$$

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which proves (b) in this case.

Suppose now $\{n \in \mathbb{N} : n > j,\ f_n(x) > f_j(x)\} \neq \varnothing$, and let

$$m = \min\{n \in \mathbb{N} : n > j,\ f_n(x) > f_j(x)\}.$$

Then $f_m(x) = s_m h_{r_m,s_m}(x)$. We shall prove that $s_m \leq s_j$. If $s_j < s_m$, then $j > 1$ and, by applying (ii) of the claim, we have

$$s_m h_{r_m,s_m}(r_j) \leq \sum_{i=j+1}^{m} t_i + f_{j-1}(r_j) \leq t_j \sum_{i=1}^{m-j} \frac{1}{3^i} + f_{j-1}(r_j) < \frac{t_j}{2} + f_{j-1}(r_j) = s_j - \frac{t_j}{2}.$$

It follows that

$$\begin{aligned} f_m(x) = s_m h_{r_m\, s_m}(x) &\leq d(x, r_j) + s_m\, h_{r_m,s_m}(r_j) < d(x, r_j) + s_j - \tfrac{t_j}{2} \\ &< s_j - \tfrac{t_j}{4} < s_j - d(x, r_j) = f_j(x), \end{aligned}$$

which contradicts the definition of m. Hence $s_m \leq s_j$. Then

$$f_{m-1}(r_m) < t_m + f_{m-1}(r_m) = s_m \leq s_j = f_{m-1}(r_j),$$

and therefore

$$\begin{aligned} t_m + \frac{s_m}{3} = \frac{4t_m}{3} + \frac{f_{m-1}(r_m)}{3} &\leq \frac{4}{3}\, \frac{f_{m-1}(r_j) - f_{m-1}(r_m)}{4} + \frac{f_{m-1}(r_m)}{3} \\ &= \frac{f_{m-1}(r_j)}{3} = \frac{s_j}{3}. \end{aligned}$$

In consequence,

$$t_m \leq \frac{s_j - s_m}{3} \leq \frac{s_j - f_m(x)}{3} < \frac{s_j - f_j(x)}{3}.$$

On the other hand, using inequality (3.1) we have

$$\begin{aligned} f_{n+m-1}(x) - f_j(x) &= f_{n+m-1}(x) - f_{m-1}(x) \\ &= f_{n+m-1}(x) - f_{n+m-2}(x) + \cdots + f_m(x) - f_{m-1}(x) \\ &\leq t_{n+m-1} + \cdots + t_m \leq \frac{t_m}{3^{n-1}} + \cdots + \frac{t_m}{3^0} < \frac{3}{2}\, t_m \end{aligned}$$

for every $n \in \mathbb{N}$. Hence,

$$f(x) \leq f_j(x) + \frac{3}{2}\, t_m < f_j(x) + \frac{s_j - f_j(x)}{2} = s_j - \frac{d(x, r_j)}{2} = f(r_j) - \frac{d(x, r_j)}{2},$$

and this completes the proof of (b).

Finally, we show (c). Since $f_n(r_1) = 1$ for all n, we have $f(r_1) = 1$. Let $x \in X \backslash \{r_1\}$. Notice that $f_n \geq f_1$ for all n. If $f_n(x) = f_1(x)$ for all $n > 1$, then $f(x) = f_1(x) < 1$. Otherwise we can suppose $\{n \in \mathbb{N} : n > 1,\ f_n(x) > f_1(x)\} \neq \varnothing$ and let

$$m = \min\{n \in \mathbb{N} : n > 1,\ f_n(x) > f_1(x)\}.$$

Clearly, $f_m(x) = s_m h_{r_m,s_m}(x)$. As $f_{m-1}(r_m) < 1 = f_{m-1}(r_1)$, then, reasoning as in the proof of (b), we obtain

$$t_m < \frac{s_1 - f_1(x)}{3}; \quad f_{n+m-1}(x) - f_1(x) < \frac{3}{2} t_m, \quad \forall n \in \mathbb{N}$$

and thus

$$f(x) \leq f_1(x) + \frac{s_1 - f_1(x)}{2} = \frac{s_1 + f_1(x)}{2} < 1 = f(r_1). \qquad \blacksquare$$

In order to prove our central result, we also need the following proposition, which is interesting in itself. It is a version of a result of Győry [2] for Lipschitz functions. A detailed reading of its proof shows that the adaptation is far from simple.

Proposition 3.2 *Let X be a metric space, and let $R = \{r_n : n \in \mathbb{N}\}$ be a countable set of pairwise distinct points of X. Then there exist Lipschitz functions $f, g \colon X \to [0, 1]$ such that f has a strict local maximum at every point of R and*

$$\{z \in X : (f(z), g(z)) = (f(r_n), g(r_n))\} = \{r_n\}, \quad \forall n \in \mathbb{N}.$$

Proof Let f and $\{t_n\}$ be as in Lemma 3.1. We prepare the proof in three steps.

First, we show that for each $n \in \mathbb{N}$, there exists $g_n \in \mathrm{Lip}(X)$ satisfying the following conditions.

(i) $0 \leq g_n \leq 1/2$, $g_n(r_j) > 0$ for all $j \in \mathbb{N}$ and $g_n(x) < g_n(r_1)$ if $x \neq r_1$;

(ii) For each $j \in \mathbb{N}$ there exist scalars $\delta_{n,j} \in]0, 1]$ and $\alpha_{n,j} > 0$ such that

$$d(x, r_j) < \delta_{n,j} \quad \Rightarrow \quad g_n(x) \leq g_n(r_j) - \alpha_{n,j} d(x, r_j).$$

Hence, g_n has a strict local maximum at r_j.

(iii) $g_n(r_j) \notin g_n(f^{-1}(\{f(r_j)\}) \backslash \{r_j\})$ for all $j = 1, \ldots, n$.

(iv) The set $g_n(f^{-1}(\{f(r_j)\}))$ is finite for all $j \in \mathbb{N}$.

(v) $L(g_n) \leq \sum_{k=1}^{n} \frac{1}{2^k} < 1$.

We prove it by induction. Define $g_1 = f/2$. Using the properties of f, it is easy to check that g_1 satisfies properties (i) to (v). Assume there is $g_n \in \mathrm{Lip}(X)$ satisfying properties (i) to (v).

Taking into account (ii), the fact that f has a strict local maximum at r_{n+1} and that $f(r_{n+1}) \neq f(r_j)$ for all $j \in \{1, \ldots, n\}$, the continuity of f permits us to choose a scalar $\rho_{n+1} \in]0, \delta_{n,n+1}[$ such that $f(x) \neq f(r_j)$ for all $j \in \{1, \ldots, n\}$, $f(x) < f(r_{n+1})$ and $g_n(x) \leq g_n(r_{n+1}) - \alpha_{n,n+1} d(x, r_{n+1})$ for all $x \in X$ for which $0 < d(x, r_{n+1}) < \rho_{n+1}$. Put

$$A = \{x \in X : \rho_{n+1} \leq d(x, r_{n+1}) < \delta_{n,n+1}\},$$

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and define the scalar $\beta_{n+1} = \sup\{g_n(x) : x \in A\}$ if $A \neq \varnothing$ and $\beta_{n+1} = 0$, otherwise. If $x \in A$, 2 yields

$$\beta_{n+1} \leq g_n(r_{n+1}) - \alpha_{n,n+1}\rho_{n+1} < g_n(r_{n+1}).$$

Let $\gamma_{n+1} \in]\beta_{n+1}, g_n(r_{n+1})[$ and define

$$U_{n+1} = \{x \in X : g_n(x) > \gamma_{n+1},\ d(x, r_{n+1}) < \rho_{n+1}\}.$$

Clearly, U_{n+1} is an open neighbourhood of r_{n+1}. Furthermore,

$$f^{-1}(\{f(r_j)\}) \cap U_{n+1} = \begin{cases} \varnothing, & j = 1, \dots, n, \\ \{r_{n+1}\}, & j = n+1, \end{cases}$$

and therefore

$$f^{-1}(\{f(r_{n+1})\}) \setminus \{r_{n+1}\} \subset X \setminus U_{n+1}. \tag{3.2}$$

From property (i) we deduce that g_n is not constant, and therefore $L(g_n) > 0$. Since $g_n(f^{-1}(\{f(r_{n+1})\}))$ is finite by (iv), we can take a scalar ε_{n+1} in the set

(3.3)
$$\Big]0, \frac{\delta_{n,n+1} - \rho_{n+1}}{2^{n+1}} L(g_n)\Big[\setminus \left\{ \frac{s - g_n(r_{n+1})}{\gamma_{n+1} - g_n(r_{n+1})} \ : \ s \in g_n\Big(f^{-1}\big(\{f(r_{n+1})\}\big)\Big) \right\}.$$

Then $\varepsilon_{n+1} < 1/2^{n+1}$. Let $g_{n+1} : X \to \mathbb{R}$ be defined by

$$g_{n+1}(x) = \begin{cases} (1 - \varepsilon_{n+1})g_n(x) + \varepsilon_{n+1}\gamma_{n+1}, & x \in U_{n+1}, \\ g_n(x), & x \in X \setminus U_{n+1}. \end{cases}$$

Observe that $g_{n+1} \leq g_n$. Next we show that g_{n+1} satisfies properties (i) to (v).

Property(v) Let $x, y \in X$. Assume, for instance, $g_{n+1}(y) \leq g_{n+1}(x)$. If $x, y \in U_{n+1}$, it is clear that

$$\begin{aligned} g_{n+1}(x) - g_{n+1}(y) &= (1 - \varepsilon_{n+1})g_n(x) - (1 - \varepsilon_{n+1})g_n(y) \\ &\leq (1 - \varepsilon_{n+1})L(g_n)d(x, y) \leq \left(\sum_{k=1}^{n+1} \frac{1}{2^k}\right) d(x, y). \end{aligned}$$

Likewise, the same conclusion can be drawn for $x \in U_{n+1}$, $y \in X \setminus U_{n+1}$, and for $x, y \in X \setminus U_{n+1}$. Finally, if $x \in X \setminus U_{n+1}$ and $y \in U_{n+1}$, we have

$$\begin{aligned} \gamma_{n+1} &= (1 - \varepsilon_{n+1})\gamma_{n+1} + \varepsilon_{n+1}\gamma_{n+1} < (1 - \varepsilon_{n+1})g_n(y) + \varepsilon_{n+1}\gamma_{n+1} \\ &= g_{n+1}(y) \leq g_{n+1}(x) = g_n(x). \end{aligned}$$

Since $x \in X \setminus U_{n+1}$, it follows that $\rho_{n+1} \leq d(x, r_{n+1})$. Moreover, $\delta_{n,n+1} \leq d(x, r_{n+1})$, since otherwise we have $x \in A$ and $g_n(x) \leq \beta_{n+1} < \gamma_{n+1}$, a contradiction. Hence,

$$\delta_{n,n+1} - \rho_{n+1} < d(x, r_{n+1}) - d(y, r_{n+1}) \leq d(x, y),$$

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and thus

$$\varepsilon_{n+1}g_n(y) - \varepsilon_{n+1}\gamma_{n+1} < \varepsilon_{n+1}g_n(y) < \varepsilon_{n+1} < \frac{\delta_{n,n+1} - \rho_{n+1}}{2^{n+1}}L(g_n) < \frac{d(x,y)}{2^{n+1}}L(g_n).$$

In this way,

$$\begin{aligned} g_{n+1}(x) - g_{n+1}(y) &= g_n(x) - g_n(y) + \varepsilon_{n+1}g_n(y) - \varepsilon_{n+1}\gamma_{n+1} \\ &< L(g_n)d(x,y) + \frac{L(g_n)}{2^{n+1}}d(x,y) < (\sum_{k=1}^{n}\frac{1}{2^k} + \frac{1}{2^{n+1}})d(x,y) = (\sum_{k=1}^{n+1}\frac{1}{2^k})d(x,y). \end{aligned}$$

Hence $g_{n+1} \in \mathrm{Lip}(X)$ and $L(g_{n+1}) \le \sum_{k=1}^{n+1}\frac{1}{2^k}$.

Property (i) It is a simple matter to see that g_{n+1} satisfies this property.

Property (ii) Let $j \in \mathbb{N}$. If $r_j \in X\backslash U_{n+1}$, then $g_{n+1}(r_j) = g_n(r_j)$. Let us take, in this case, $\delta_{n+1,j} = \delta_{n,j} \in]0,1[$ and $\alpha_{n+1,j} = \alpha_{n,j} > 0$. If $d(x,r_j) < \delta_{n+1,j}$, we have

$$g_{n+1}(x) \le g_n(x) \le g_n(r_j) - \alpha_{n,j}d(x,r_j) = g_{n+1}(r_j) - \alpha_{n+1,j}d(x,r_j).$$

If $r_j \in U_{n+1}$, we can choose a $\delta_{n+1,j} \in]0,\delta_{n,j}[$ such that

$$\{x \in X : d(x,r_j) < \delta_{n+1,j}\} \subset U_{n+1}.$$

Put $\alpha_{n+1,j} = (1-\varepsilon_{n+1})\alpha_{n,j} > 0$. If $d(x,r_j) < \delta_{n+1,j}$,

$$\begin{aligned} g_{n+1}(x) &= (1-\varepsilon_{n+1})g_n(x) + \varepsilon_{n+1}\gamma_{n+1} \\ &\le (1-\varepsilon_{n+1})g_n(r_j) - (1-\varepsilon_{n+1})\alpha_{n,j}d(x,r_j) + \varepsilon_{n+1}\gamma_{n+1} \\ &= g_{n+1}(r_j) - \alpha_{n+1,j}d(x,r_j). \end{aligned}$$

Property (iii) Since ε_{n+1} belongs to the set (3.3), we have

$$g_{n+1}(r_{n+1}) = (1-\varepsilon_{n+1})g_n(r_{n+1}) + \varepsilon_{n+1}\gamma_{n+1} \notin g_n(f^{-1}(\{f(r_{n+1})\})),$$

and, by inclusion (3.2),

$$\begin{aligned} g_{n+1}(f^{-1}(\{f(r_{n+1})\})\backslash\{r_{n+1}\}) &= g_n(f^{-1}(\{f(r_{n+1})\})\backslash\{r_{n+1}\}) \\ &\subset g_n(f^{-1}(\{f(r_{n+1})\})). \end{aligned}$$

Hence, $g_{n+1}(r_{n+1}) \notin g_{n+1}(f^{-1}(\{f(r_{n+1})\})\backslash\{r_{n+1}\})$.

Let $j \in \{1,\dots,n\}$. Since $f^{-1}(\{f(r_j)\}) \cap U_{n+1} = \varnothing$, then $r_j \in X\backslash U_{n+1}$ and $f^{-1}(\{f(r_j)\})\backslash\{r_j\} \subset X\backslash U_{n+1}$. Therefore

$$g_{n+1}(r_j) = g_n(r_j) \notin g_n(f^{-1}(\{f(r_j)\})\backslash\{r_j\}) = g_{n+1}(f^{-1}(\{f(r_j)\})\backslash\{r_j\}).$$

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Property (iv) Let $j \in \mathbb{N}$. It is sufficient to observe that the sets

$$g_{n+1}(f^{-1}(\{f(r_j)\}) \cap U_{n+1}) = \{(1-\varepsilon_{n+1})g_n(x) + \varepsilon_{n+1}\gamma_{n+1} : x \in f^{-1}(\{f(r_j)\}) \cap U_{n+1}\}$$

and

$$g_{n+1}(f^{-1}(\{f(r_j)\}) \cap (X\backslash U_{n+1})) = g_n(f^{-1}(\{f(r_j)\}) \cap (X\backslash U_{n+1}))$$

are finite by the property (iv) of g_n.

Secondly, it is straightforward to check that $\{g_n\}$ converges to $g \in \operatorname{Lip}(X)$ in $(\mathcal{C}_b(X), \|\cdot\|_\infty)$. Moreover, $L(g) \leq 1$ and $0 \leq g \leq 1/2$.

Thirdly, we prove that $\{x \in X : (f(x), g(x)) = (f(r_j), g(r_j))\} = \{r_j\}$, for each $j \in \mathbb{N}$. Indeed, we show easily by induction that for each $n \in \mathbb{N}$,

$$g_n(x) = g_j(x), \quad \forall x \in f^{-1}(\{f(r_j)\}), \quad \forall j \in \{1, \dots, n\}.$$

Given $j \in \mathbb{N}$, assume $(f(x), g(x)) = (f(r_j), g(r_j))$ for some $x \in X\backslash\{r_j\}$. Then $r_j, x \in f^{-1}(\{f(r_j)\})$ and, as proved above, we have $g_{n+j}(x) = g_j(x)$ and $g_{n+j}(r_j) = g_j(r_j)$ for all $n \in \mathbb{N}$. It follows that

$$g_j(r_j) = g(r_j) = g(x) = g_j(x) \in g_j(f^{-1}(\{f(r_j)\})\backslash\{r_j\}),$$

but this contradicts property (iii) of g_j. ■

We have now gathered all the ingredients to prove our main theorem.

Theorem 3.3 *Let X be a separable bounded metric space. If the isometry group of* $\operatorname{Lip}(X)$ *is canonical, then every* 2*-local isometry of* $\operatorname{Lip}(X)$ *is a surjective linear isometry.*

Proof By Theorem 2.1, there exist $X_0 \subset X$, $\tau \in S_{\mathbb{K}}$ and a Lipschitz bijection $\varphi\colon X_0 \to X$ such that $\Phi(f)|_{X_0} = \tau(f \circ \varphi)$ for all $f \in \operatorname{Lip}(X)$.

If X is finite, it is clear that $X_0 = X$ and $\varphi^{-1}\colon X \to X$ is Lipschitz.

Let us suppose now that X is nonfinite. Due to the separability of X, there exists a countable dense subset $R = \{r_n : n \in \mathbb{N}\}$ of X, where the points r_n are pairwise distinct. Let $f, g\colon X \to [0,1]$ be as in Proposition 3.2. We can write $\Phi(f) = \tau_{f,g}(f \circ \varphi_{f,g})$ and $\Phi(g) = \tau_{f,g}(g \circ \varphi_{f,g})$ for some $\tau_{f,g} \in S_{\mathbb{K}}$ and some surjective isometry $\varphi_{f,g}$ of X. For each natural n we have

$$\tau_{f,g} f(\varphi_{f,g}(\varphi^{-1}(r_n))) = \Phi(f)(\varphi^{-1}(r_n)) = \tau f(r_n),$$
$$\tau_{f,g} g(\varphi_{f,g}(\varphi^{-1}(r_n))) = \Phi(g)(\varphi^{-1}(r_n)) = \tau g(r_n).$$

Since f and g are non-negative and $|\tau_{f,g}| = |\tau| = 1$, it follows that

$$f(\varphi_{f,g}(\varphi^{-1}(r_n))) = f(r_n), \quad g(\varphi_{f,g}(\varphi^{-1}(r_n))) = g(r_n).$$

Then Proposition 3.2 yields $\varphi_{f,g}(\varphi^{-1}(r_n)) = r_n$, and so

$$\varphi^{-1}(z) = {\varphi_{f,g}}^{-1}(z), \quad \forall z \in R.$$

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Let $x \in X_0$. Since R is dense in X, there is a sequence $\{z_n\}$ in R converging to $\varphi_{f,g}(x)$. Then $\{\varphi_{f,g}{}^{-1}(z_n)\}$ converges to x, but $\varphi_{f,g}{}^{-1}(z_n) = \varphi^{-1}(z_n)$ for all $n \in \mathbb{N}$, and so $\{\varphi^{-1}(z_n)\}$ converges to x. It follows that $\{z_n\}$ converges to $\varphi(x)$. In consequence, $\varphi_{f,g}(x) = \varphi(x)$ for every $x \in X_0$.

Let $y \in X$. Since φ is surjective, there is $x \in X_0$ for which $\varphi(x) = \varphi_{f,g}(y)$. Then $\varphi_{f,g}(x) = \varphi_{f,g}(y)$, as proved above, which implies $y = x$ by the injectivity of $\varphi_{f,g}$ and so $y \in X_0$. Hence $X = X_0$. Therefore $\varphi = \varphi_{f,g}$, and thus φ is a surjective isometry of X.

In both cases we have $\Phi(f) = \tau(f \circ \varphi)$ for all $f \in \mathrm{Lip}(X)$, where $\varphi^{-1}\colon X \to X$ is Lipschitz. Consequently, Φ is surjective and linear. From the definition of 2-local isometry, it is easy to deduce that Φ is an isometry. We conclude that Φ is a surjective linear isometry of $\mathrm{Lip}(X)$. ■

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