the star number of n sides, respectively, for $n \in \mathbb{N}$, the following identity holds:

$$\tau(n) - \sigma(n) = n - 1.$$

Proof: The proof is demonstrated for $n = 6$.

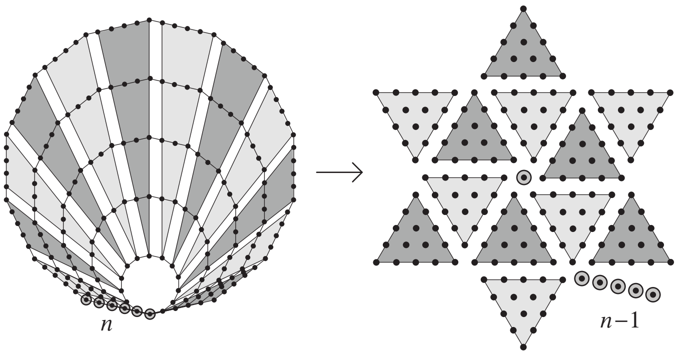

FIGURE 2

In general, where $p(n, k)$ and $c(n, k)$ are expressed in terms of triangular numbers $T_n = \frac{1}{2}n(n + 1)$ via $p(n, k) = n + (k - 2)T_{n-1}$ and $c(n, k) = 1 + kT_{n-1}$ respectively, we simply deduce

$$p(n,k) - c(n,k-2) = n + (k-2)T_{n-1} - \left[1 + (k-2)T_{n-1}\right] = n - 1.$$

Reference

1. G. Caglayan, Visualising the star number - centred dodecagonal number isomorphism, *Math. Gaz.*, **104** (November 2020) pp. 543.

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107.27 The discrete renewal theorem with bounded inter-event times

Probabilistic Sequence

The purpose of this Note is to prove the celebrated Discrete Renewal Theorem in a common special case, using only very elementary methods.

To introduce the problem, consider a class of board games in which a player's counter makes a sequence of moves in a fixed direction along a line of squares S_n, $n \geqslant 0$. The counter starts from S_0, with the sizes of successive moves determined by the roll of a die (or multiple dice), which may be biased.

For the n-th square S_n, it is natural to ask for the probability that the counter ever lands on S_n, denoted by u_n. This is especially valuable where S_n is a square on which the player gains some reward or pays some penalty. Note that $u_0 = 1$ and we have allowed the line of squares to be semi-infinite.

By definition, the length X of any jump of the counter is independent of all other jumps, and we denote its probability distribution by

$$f_j = P(X = j), \qquad j \geqslant 1$$

with the sum of f_j being 1. For example, if jumps are given by a fair cubical die, then

$$f_j = \tfrac{1}{6}, \qquad 1 \leqslant j \leqslant 6.$$

We require the following assumption:

Assumption 1: For some finite $S, f_j = 0$ for $j > S$.

Under Assumption 1, we shall prove by elementary methods, in Theorem 2 below, that

$$\lim_{n \to \infty} u_n = \frac{1}{\mathrm{E}(X)} \tag{1}$$

where $\mathrm{E}(X)$ is the expected value of any jump X. That is to say, for all practical purposes, sufficiently distant points are all equally likely to be visited by the counter with probability $1/\mathrm{E}(X)$. First we give this preliminary result:

Theorem 1.

For all $n \geqslant 1$

$$u_n = f_n u_0 + f_{n-1} u_1 + \dots + f_1 u_{n-1}, \tag{a}$$

and

$$1 = \mathrm{P}(X > n) + \mathrm{P}(X > n - 1)u_1 + \dots + \mathrm{P}(X > 1)u_{n-1} + u_n. \tag{b}$$

Proof:

(a) For $1 \leqslant j \leqslant n$, the counter leaves S_0, lands first on S_{n-j}, and then subsequently lands on S_n with probability $f_j u_{n-j}$. Summing these probabilities yields our result, using the partition theorem (the law of total probability) [1].

(b) There are 3 cases to consider

- The counter 'visits' S_n with probability u_n
- For some $j < n$, the counter first visits S_j and then makes a jump greater than $n - j$ with probability $u_j \, \mathrm{P}(X > n - j)$
- The counter jumps from S_0 directly 'over' S_n, with probability $\mathrm{P}(X > n)$.

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Summing all of these probabilities, the partition theorem [1] is again used to yield our result.

We make the following observations.

1. A positive sequence u_n defined by (a) is called a *renewal sequence* with respect to the distribution f_n.
2. The sum on the right-hand-side of (a) is called the *convolution* of the sequences u_n and f_n [2].
3. Under Assumption 1, there are at most S and $S + 1$ terms in the right-hand-sides of (a) and (b) respectively.

Convergence of Sequence

To justifiably use the limit formula in (1), we need to establish that u_n converges as $n \to \infty$.

Theorem 2.

The sequence u_n has a finite limit as $n \to \infty$, this being

$$\lim_{n \to \infty} u_n = \frac{1}{\mathrm{E}(X)}$$

where $\mathrm{E}(X)$ is the expected value of any jump X.

Proof: From Assumption 1, for some finite $S, f_j = 0$ for $j > S$. For $n > S$. Theorem 1(a) then reduces to

$$u_n = f_S u_{n-S} + f_{S-1} u_{n-S+1} + \dots + f_2 u_{n-2} + f_1 u_{n-1}.$$

Letting the maximum of the terms $u_{n-S}, \dots, u_{n-1}$ be denoted by M_n, all of these terms are clearly less than or equal to M_n. As the sum of f_j is 1, u_n must be less than or equal to M_n. Avoiding the trivial case where u_n is constant, it can be proved by induction that at least one of $u_{n-S}, \dots, u_{n-1}$ is not M_n and the inequality becomes strict,

$$u_n < M_n \tag{2}$$

and so all terms $u_{n-S+1}, \dots, u_n$ are all less than or equal to M_n such that

$$M_{n+1} \leqslant M_n$$

and hence M_n is a monotonically decreasing sequence. Similarly denoting by m_n, the minimum of $u_{n-S}, \dots, u_{n-1}$, with precisely the same logic, m_n is a monotonically increasing sequence. Clearly with the context of probability, both sequences are bounded. By the Monotone Convergence Theorem [3], M_n and m_n both converge to finite limits as $n \to \infty$. We wish to prove that these limits are the same.

Denoting the limits of M_n and m_n as M and m respectively, we assume, for the sake of contradiction, that $m \neq M$. There must then exist a $\delta > 0$ such that

$$M - m = k\delta$$

where $kf_j > 1, 1 \leqslant j \leqslant S$. As m_n is monotonically increasing to m, for all n it is less than or equal to m. We have

$$m_n \leqslant M - k\delta. \tag{3}$$

As M_n is monotonically decreasing to M, there must be some N such that for all $t > N$,

$$M_t - M < \delta. \tag{4}$$

Now consider equation (a) under Assumption 1. One of $u_{t-S}, \dots, u_{t-1}$ is m_t and all others are less than or equal to M_t. Let m_t here have coefficient f', then as the sum of the f_j is 1,

$$u_t \leqslant f'm_t + (1 - f')M_t$$

and we can substitute equations (3) and (4) into this

$$u_t \leqslant M - (kf' + f' - 1)\delta.$$

With all the above constraints, $(kf' + f' - 1)\delta$ is positive such that for all $t > N$,

$$u_t < M.$$

But then all the terms $u_t, \dots, u_{t+S-1}$ are strictly less than M. Setting $n = t + S$ we have

$$M_n < M,$$

a clear contradiction of the proven fact that M_n monotonically decreases to M. The assumption that the sequences have different limits is false, and so M_n and m_n converge to the same limit. From (2),

$$m_n < u_n < M_n.$$

As M_n and m_n have the same limit, the 'Squeeze Theorem' [4] applies and u_n must have this same limit L.

We may now complete the proof of Theorem 2. We have established that u_n converges to some limit L as $n \to \infty$. Allowing $n \to \infty$ in Theorem 1(b) yields the required result immediately, using the fact that the sum is finite, noting that X has a proper distribution [so $\text{P}(X > 0) = 1$], and using the tail sum formula for $\text{E}(X)$. That is, for a positive integer-valued X,

$$\text{E}(X) = \text{P}(X > 0) + \text{P}(X > 1) + \text{P}(X > 2) + \dots .$$

It follows that

$$\lim_{n \to \infty} u_n = \frac{1}{\text{E}(X)}.$$

We note, that in contrast to the Erdős-Feller-Pollard theorem [5] (described below) which proves the more general renewal theorem, we have been able to prove this specific result using only very elementary methods.

Background and History

Theorem 2, as given above, is actually true without Assumption 1, where the limit becomes zero when $\mathrm{E}(X)$ is infinite. This was shown in a famous paper of 1949 by P. Erdős, W. Feller and H. Pollard [5], using the discrete generating function '*dgf*', and proving a key property of power series with positive coefficients to acquire the result. Explicitly, if the sequences u_n and f_n have dgf's $U(s)$ and $F(s)$ respectively, then from Theorem 1(a), we have

$$U(s) = \frac{1}{1 - F(s)}, \tag{5}$$

as used in the Erdős-Feller-Pollard theorem. For a simpler proof of this theorem, a summary of all the standard notation, and its various applications, one can see chapter XIII of [1]. One can also find the original proof here (renyi.hu) [5].

Note that under Assumption 1, $[1 - F(s)]$ is a polynomial, and it follows from (5), and the theory of partial fractions, that

$$\lim_{n \to \infty} u_n = \frac{1}{\mathrm{E}(X)}$$

which is our Theorem 2. However, in the special case of Assumption 1, the elementary proof outlined in our first two sections is sufficient.

Discrete renewal theory is very strongly linked with certain properties of Markov chains. The discrete renewal theorem can be proved using suitable Markov chain convergence theorems, and vice versa [6].

In more general renewal processes, the lengths of jumps X are allowed to have an arbitrary distribution on the positive real line. In such cases, the Laplace transform may be used in place of the dgf, and a suitably modified version of the renewal theorem proved. Even more generally, one may allow counters to jump both forwards and backwards, so that X may also take negative values with a distribution on the entire real line. Here the Fourier transform may be used to acquire appropriate results [7].

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References

1. William Feller, *An introduction to probability theory and its applications*, *Volume I*, (3rd edition), Princeton University (1968).
2. D. R. Cox, Point processes and renewal theory: a brief survey. *Electronic systems effectiveness and life cycle costing*, (1983) pp. 107-112.
3. Albert Adu-Sackey, Francis T. Oduro, G. O. Fosu, Inequalities approach in determination of convergence of recurrence sequences, *Open Journal of*

Mathematical Sciences, **5**(1) (2021) pp. 65-72.
4. William R Fuller, *Sequences*, Springer (1977).
5. Paul Erdős, William Feller and Harry Pollard, A property of power series with positive coefficients, *Bulletin of the American Mathematical Society*, **55**(2) (1949) pp. 201–204.
6. Walter L. Smith, Renewal theory and its ramifications, *Journal of the Royal Statistical Society: Series B (Methodological)*, **20**(2) (1958) pp. 243-284.
7. Sidney I. Resnick, *Adventures in stochastic processes*, Springer Science & Business Media (1992).

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