

# SOME RESTRICTED PARTITION FUNCTIONS: CONGRUENCES MODULO 3

D. B. LAHIRI

(Received 22 May 1968)

## 1. Introduction

In a previous communication [5] the author has dealt with the congruence properties of some restricted partition functions. The general category of such functions may be denoted by

$$(1) \quad {}_t^r p(n).$$

The restriction merely imposes the condition that no number of the forms  $tn$ ,  $tn \pm r$  where  $t$  and  $r$  are fixed integers shall be a part of the relevant partitions. In other words, in order to determine the value of this function one should count all the unrestricted partitions of  $n$  excepting those which contain a number of any of the above forms as a part. The paper referred to above which deals with congruences modulo 5 for functions with  $t = 75$  made use of a classical identity by Catalan and another by Glaisher. It is possible to derive congruences for other moduli from these identities by more or less similar methods. For example, it can be shown that for  $t = 27$ ,

$$(2) \quad {}_{3}^{27} p(n) \equiv {}_{6}^{27} p(n) \equiv {}_{12}^{27} p(n) \equiv 0 \pmod{3}$$

for 'almost all' values of  $n$ . In this paper we shall be dealing with modulus 3 also, but for a different set of restricted partition functions where  $t = 243$ . There are important points of difference between the method of derivation of the two series of results. It is no longer adequate to depend only upon Catalan and Glaisher, and we shall have to make use of further identities of the same type established by the author [3]. Another point of departure is the fact that the intermediate congruences on which the final results depend relate not to the modulus 3 as in the former series but on (higher) powers of 3. This is also in contrast to the method of derivation which could be used for congruences with other moduli; two such results are given below. For 'almost all' values of  $n$ ,

$$(3) \quad {}_{35}^{147} p(n) \equiv {}_{14}^{147} p(n-7) \pmod{7},$$

$$(4) \quad {}_{176}^{363} p(n) \equiv -{}_{55}^{363} p(n-22) \pmod{11}.$$

These and other related results deserve separate publication.

**THEOREM 1.** *For almost all value of  $n$ , the following congruences modulo 3 hold,*

- (i)  $-\frac{243}{90}p(n) \equiv \frac{243}{72}p(n-3) \equiv \frac{243}{9}p(n-24),$
- (ii)  $-\frac{243}{99}p(n) \equiv \frac{243}{63}p(n-6) \equiv \frac{243}{18}p(n-21),$
- (iii)  $-\frac{243}{117}p(n) \equiv \frac{243}{45}p(n-12) \equiv \frac{243}{56}p(n-15).$

### 2. Definitions and notations

We shall be using the same definitions and notations as in the earlier paper [5] with the exception that we shall use the symbols  $U_i$  and  $P_i(v)$  with new meanings. However in order to make the paper self contained we are not only giving these new meanings but also the interpretations of the other older symbols. Some additional symbols are also introduced.

We shall use  $m$  to denote an integer positive, zero or negative, but  $n$  is reserved for a positive or a non-negative integer only.

We define the function  $u_r(x)$ , or simply  $u_r$ , by

$$(5) \quad u_r = \sum_{n=0}^{\infty} n^r a_n x^n \cdot \sum_{n=0}^{\infty} p(n) x^n,$$

where  $a_n$  is defined by the well known ‘pentagonal number’ theorem of Euler,

$$(6) \quad f(x) = \prod_{n=1}^{\infty} (1-x^n) = \sum_{-\infty}^{+\infty} (-1)^m x^{\frac{1}{2}m(3m+1)} = \sum_{n=0}^{\infty} a_n x^n,$$

and  $p(n)$  is the number of unrestricted partitions of  $n$  given by the expansion,

$$(7) \quad [f(x)]^{-1} = \left[ \prod_{n=1}^{\infty} (1-x^n) \right]^{-1} = \sum_{n=0}^{\infty} p(n) x^n.$$

When  $n = r = 0$ ,  $n^r$  is to be interpreted as unity, so that

$$(8) \quad u_0 = \sum_{n=0}^{\infty} a_n x^n \cdot \sum_{n=0}^{\infty} p(n) x^n = 1.$$

We assume  ${}^t p(n)$  to be unity when  $n = 0$ , and vanishing when  $n$  is negative.

We shall use  $v$  to denote the pentagonal numbers,

$$(9) \quad v = \frac{1}{2}m(3m+1), \quad m = 0, \pm 1, \pm 2, \dots;$$

and with each  $v$  there corresponds an ‘associated’ sign, viz.,  $(-1)^m$ . We shall come across sums of the type

$$(10) \quad \sum_v [\mp V(v)],$$

where it is understood that the sign to be prefixed is the 'associated' one. It is clear that with the above notation,

$$(11) \quad u_r = \sum_v (\mp v^r x^v) / f(x),$$

$$(12) \quad \sum_v (\mp x^v) / f(x) = 1.$$

We shall use the symbols  $U_i$ ,  $i = 1, 2, 3, 4, 5$  to denote the following linear functions of  $u_r$ 's.

$$(13) \quad U_1 = -4u_5 - u_4 - 2u_3 + 4u_2,$$

$$(14) \quad U_2 = -u_5 + u_4 + u_3 - u_2,$$

$$(15) \quad U_3 = -u_5 + 2u_3 - u_1,$$

$$(16) \quad U_4 = 2u_5 - u_4 - 2u_3 + u_2,$$

$$(17) \quad U_5 = -u_5 + 4u_4 + 4u_3 + 2u_2.$$

We also need certain polynomials  $P_i(v)$  in  $v$ ,  $i = 1, 2, 3, 4, 5$  which are obtained by replacing  $U_i$  by  $P_i(v)$ , and  $u_r$  by  $v^r$  in the above relations. For example,

$$P_3(v) = -v^5 + 2v^3 - v.$$

Ramanujan [6] defined

$$(18) \quad \Phi_{r,s}(x) = \sum_{\alpha=1}^{\infty} \sum_{\beta=1}^{\infty} \alpha^r \beta^s x^{\alpha\beta} = \sum_{n=1}^{\infty} n^r \sigma_{s-r}(n) x^n,$$

where  $\sigma_k(n)$  is, as usual, the sum of the  $k$ -th powers of the divisors of  $n$ . The author has found it convenient to simplify the notation to  $\Phi_{r,s}$ , [3], and even to just  $(r, s)$ , [4], so that

$$(19) \quad (r, s) = \sum_{n=1}^{\infty} n^r \sigma_{s-r}(n) x^n.$$

### 3. Lemmas

By putting successively 0, 1, 2, 3, 4, 5, 6, 7, 8 in place of  $v$  in the polynomials  $P_i(v)$  the truth of the following lemma becomes evident.

LEMMA 1. For  $i = 1, 2, 3, 4, 5$

$$\begin{aligned} P_i(v) &\equiv -3 \pmod{3^2}, \text{ if } v \equiv i \pmod{3^2} \\ &\equiv 3 \pmod{3^2}, \text{ if } v \equiv i+3 \pmod{3^2} \\ &\equiv 0 \pmod{3^2} \text{ for other values of } v. \end{aligned}$$

If we replace the  $u_r$ 's appearing in the expressions for  $U_i$  by

$$\sum (\mp v^r x^v) / f(x),$$

which is justified according to (11) we obtain

$$(20) \quad U_i = \sum_v [\mp P_i(v) x^v] / f(x);$$

and then the use of Lemma 1 leads to the lemma given below.

LEMMA 2. For  $i = 1, 2, 3, 4, 5$

$$U_i \equiv 3 \sum_{v \equiv i+3} (\mp x^v) / f(x) - 3 \sum_{v \equiv i} (\mp x^v) / f(x) \pmod{3^2},$$

where the summations extend respectively over all pentagonal numbers

$$v \equiv i+3 \pmod{3^2} \quad \text{and} \quad v \equiv i \pmod{3^2}.$$

The truth of the following lemma can be verified without much difficulty by writing  $9m+j$  with  $j = 0, -1, 1, -3, -4, -2, 3, 2$  and  $4$  respectively in place of  $m$  in the expression  $\frac{1}{2}m(3m+1)$  for the pentagonal numbers and in  $(-1)^m$  its associated sign. It is also to be remembered (when  $j$  is negative, say,  $-j'$ ) that  $\frac{1}{2}(9m-j')(27m-3j'+1)$  and  $\frac{1}{2}(9m+j')(27m+3j'-1)$  represent the same set of numbers.

LEMMA 3. The solutions of

$$v \equiv i \pmod{3^2}$$

are as given below, — the associated signs are also shown,

$i$	solutions	sign
0	$\frac{1}{2}(243m^2 + 9m)$	$(-1)^m$
1	$\frac{1}{2}(243m^2 + 45m) + 1$	$-(-1)^m$
2	$\frac{1}{2}(243m^2 + 63m) + 2$	$-(-1)^m$
3	$\frac{1}{2}(243m^2 + 153m) + 12$	$-(-1)^m$
4	$\frac{1}{2}(243m^2 + 207m) + 22$	$(-1)^m$
5	$\frac{1}{2}(243m^2 + 99m) + 5$	$(-1)^m$
6	$\frac{1}{2}(243m^2 + 171m) + 15$	$-(-1)^m$
7	$\frac{1}{2}(243m^2 + 117m) + 7$	$(-1)^m$
8	$\frac{1}{2}(243m^2 + 225m) + 26$	$(-1)^m$

The identities given in the next lemma are simple applications of a special case of a famous identity of Jacobi [2, p. 283], viz.,

$$(21) \quad \prod_{n=0}^{\infty} [(1-x^{2kn+k-l})(1-x^{2kn+k+l})(1-x^{2kn+2k})] = \sum_{-\infty}^{+\infty} (-1)^m x^{km^2+lm}.$$

In establishing this lemma  $k$  and  $l$  are given values which are in conformity with the quadratic expressions in  $m$  given in Lemma 3.

LEMMA 4. *Writing  $v \equiv i$  simply for  $v \equiv i \pmod{3^2}$ ,*

$$\begin{aligned} \sum_{v \equiv 0} (\mp x^v) &= \prod_{n=0}^{\infty} [(1-x^{243n+117})(1-x^{243n+126})(1-x^{243n+243})], \\ \sum_{v \equiv 1} (\mp x^v) &= -x \prod_{n=0}^{\infty} [(1-x^{243n+99})(1-x^{243n+144})(1-x^{243n+243})], \\ \sum_{v \equiv 2} (\mp x^v) &= -x^2 \prod_{n=0}^{\infty} [(1-x^{243n+90})(1-x^{243n+153})(1-x^{243n+243})], \\ \sum_{v \equiv 3} (\mp x^v) &= -x^{12} \prod_{n=0}^{\infty} [(1-x^{243n+45})(1-x^{243n+198})(1-x^{243n+243})], \\ \sum_{v \equiv 4} (\mp x^v) &= x^{22} \prod_{n=0}^{\infty} [(1-x^{243n+18})(1-x^{243n+225})(1-x^{243n+243})], \\ \sum_{v \equiv 5} (\mp x^v) &= x^5 \prod_{n=0}^{\infty} [(1-x^{243n+72})(1-x^{243n+171})(1-x^{243n+243})], \\ \sum_{v \equiv 6} (\mp x^v) &= -x^{15} \prod_{n=0}^{\infty} [(1-x^{243n+36})(1-x^{243n+207})(1-x^{243n+243})], \\ \sum_{v \equiv 7} (\mp x^v) &= x^7 \prod_{n=0}^{\infty} [(1-x^{243n+63})(1-x^{243n+180})(1-x^{243n+243})], \\ \sum_{v \equiv 8} (\mp x^v) &= x^{26} \prod_{n=0}^{\infty} [(1-x^{243n+9})(1-x^{243n+234})(1-x^{243n+243})]. \end{aligned}$$

For the derivation of the next lemma we have to use the following fact.

(22)

$$\begin{aligned} &\prod_{n=0}^{\infty} [(1-x^{243n+r})(1-x^{243n+243-r})(1-x^{243n+243})]/f(x) \\ &= \prod_{n=0}^{\infty} [(1-x^{243n+r})(1-x^{243n+243-r})(1-x^{243n+243})]/[(1-x)(1-x^2)(1-x^3)\dots] \\ &= \sum_{n=0}^{\infty} {}_{243}p_r(n)x^n. \end{aligned}$$

Using this result and Lemma 4 in Lemma 2 we obtain the following.

LEMMA 5. *With respect to the modulus  $3^2$ ,*

$$\begin{aligned} U_1 &\equiv 3 \sum_{n=0}^{\infty} {}_{18}p(n-22)x^n + 3 \sum_{n=0}^{\infty} {}_{99}p(n-1)x^n, \\ U_2 &\equiv 3 \sum_{n=0}^{\infty} {}_{72}p(n-5)x^n + 3 \sum_{n=0}^{\infty} {}_{90}p(n-2)x^n, \\ U_3 &\equiv -3 \sum_{n=0}^{\infty} {}_{36}p(n-15)x^n + 3 \sum_{n=0}^{\infty} {}_{45}p(n-12)x^n, \end{aligned}$$

$$U_4 \equiv 3 \sum_{n=0}^{\infty} \frac{243}{63} p(n-7)x^n - 3 \sum_{n=0}^{\infty} \frac{243}{18} p(n-22)x^n,$$

$$U_5 \equiv 3 \sum_{n=0}^{\infty} \frac{243}{9} p(n-26)x^n - 3 \sum_{n=0}^{\infty} \frac{243}{72} p(n-5)x^n.$$

We require an important set of congruences which are directly derivable from the identities for  $u_r = u_{r,0}$  given by the author in [3, p. 128] for  $r = 1, 2, 3, 4$  and 5. These identities express  $u_r$ 's as linear functions of  $\Phi_{a,b}$ 's. As an example we have,

$$(23) \quad u_3 = u_{3,0} = -\frac{1}{192}\Phi_{0,1} + \frac{3}{16}\Phi_{1,2} - \frac{15}{8}\Phi_{2,3} - \frac{5}{96}\Phi_{0,3} + \frac{25}{32}\Phi_{1,4} - \frac{7}{192}\Phi_{0,5}.$$

These linear functions have constant coefficients. Obviously by suitable multiplications of both sides of these identities the fractional coefficients can be made integral. We have reduced these integral coefficients with respect to the modulus  $3^6$ , and for the sake of simplicity we have written  $(a, b)$  for  $\Phi_{a,b}$  in the following.

LEMMA 6. *With respect to the modulus  $3^6$  for the congruences we have,*

$$u_1 = - (0, 1);$$

$$3u_2 \equiv -182(0, 1) + 360(1, 2) - 181(0, 3);$$

$$3u_3 \equiv -262(0, 1) - 45(1, 2) - 279(2, 3) + 296(0, 3) - 66(1, 4) + 353(0, 5);$$

$$3^3u_4 \equiv 104(0, 1) + 216(1, 2) + 54(2, 3) + 351(3, 4)$$

$$+ 102(0, 3) - 144(1, 4) + 189(2, 5) - 3(0, 5) + 42(1, 6) - 209(0, 7);$$

$$3^4u_5 \equiv - 5(0, 1) + 360(1, 2) + 135(2, 3) + 297(3, 4)$$

$$- 100(0, 3) + 126(1, 4) + 216(2, 5) + 162(3, 6)$$

$$- 210(0, 5) + 48(1, 6) - 162(2, 7) - 100(0, 7) - 108(1, 8) - 11(0, 9).$$

The next lemma is obtained by the use of the above relations in the expressions for  $U_r$  given in (13)–(17).

LEMMA 7. *With respect to the modulus  $3^6$  we have,*

$$3^4U_i \equiv A_0(0, 9)$$

$$+ B_1(1, 8) + B_0(0, 7)$$

$$+ C_2(2, 7) + C_1(1, 6) + C_0(0, 5)$$

$$+ D_3(3, 6) + D_2(2, 5) + D_1(1, 4) + D_0(0, 3)$$

$$+ E_3(3, 4) + E_2(2, 3) + E_1(1, 2) + E_0(0, 1),$$

*the set of coefficients*

$$(A_0; B_1, B_0; C_2, C_1, C_0; D_3, D_2, D_1, D_0; E_3, E_2, E_1, E_0)$$

*being respectively*

- ( 44; -297, 298; -81, -318, 12;  
81, 27, -153, 283; -54, -216, -144, 32);
- ( 11; 108, 202; 162, 78, 255;  
-162, 351, -153, 163; 27, -216, 288, 344);
- ( 11; 108, 100; 162, -48, 318;  
-162, -216, -45, 46; -297, 108, 126, -211);
- (-22; -216, -302; -324, -30, 210;  
324, -135, -126, -236; 270, -135, -171, 164);
- ( 11; 108, -221; 162, -273, -339;  
-162, -135, -234, 190; 270, 270, 45, 308).

We also require an identical relation among the restricted partition functions appearing in Lemma 5 and the function

$$\frac{243}{117}p(n).$$

To establish such an identity we first note that (12) may be written as

$$(24) \quad \sum_i \sum_{v \equiv i \pmod{3^9}} (\mp x^v)/f(x) = 1,$$

where  $\sum_i$  denotes summation over  $i = 0, 1, 2, \dots, 8$ . We then apply Lemma 4 to expand, - with the help of (22) -

$$(25) \quad \sum_{v \equiv i \pmod{3^9}} (\mp x^v)/f(x)$$

as a power series in  $x$  whose coefficients are expressed in terms of suitable restricted partition functions. Passage to the following lemma is then a simple matter; we just equate the coefficient of  $x^n$  in the expansion of (24) to zero.

LEMMA 8. *The following identity holds for  $n > 0$ ,*

$$\begin{aligned} \frac{243}{117}p(n) = & \frac{243}{99}p(n-1) + \frac{243}{90}p(n-2) - \frac{243}{72}p(n-5) \\ & - \frac{243}{63}p(n-7) + \frac{243}{45}p(n-12) + \frac{243}{36}p(n-15) \\ & - \frac{243}{18}p(n-22) - \frac{243}{9}p(n-26). \end{aligned}$$

### 4. Proof of Theorem 1

Theorem 1 rests upon a basic result which we shall now establish. By comparing the coefficients of the two expressions for  $U_i \pmod{3^6}$  as obtainable from Lemmas 5 and 7 we obtain the following theorem.

THEOREM 0. *With respect to the modulus  $3^6$  we have for  $n > 0$ ,*

- (i)  $3^5 \cdot \frac{243}{18}p(n-22) + 3^5 \cdot \frac{243}{99}p(n-1)$   
 $\equiv 44\sigma_9(n) - (297n - 298)\sigma_7(n) - (81n^2 + 318n - 12)\sigma_5(n)$   
 $+ (81n^3 + 27n^2 - 153n + 283)\sigma_3(n)$   
 $- (54n^3 + 216n^2 + 144n - 32)\sigma(n);$
- (ii)  $3^5 \cdot \frac{243}{72}p(n-5) + 3^5 \cdot \frac{243}{90}p(n-2)$   
 $\equiv 11\sigma_9(n) + (108n + 202)\sigma_7(n) + (162n^2 + 78n + 255)\sigma_5(n)$   
 $- (162n^3 - 351n^2 + 153n - 163)\sigma_3(n)$   
 $+ (27n^3 - 216n^2 + 288n + 344)\sigma(n);$
- (iii)  $-3^5 \cdot \frac{243}{36}p(n-15) + 3^5 \cdot \frac{243}{45}p(n-12)$   
 $\equiv 11\sigma_9(n) + (108n + 100)\sigma_7(n) + (162n^2 - 48n + 318)\sigma_5(n)$   
 $- (162n^3 + 216n^2 + 45n - 46)\sigma_3(n)$   
 $- (297n^3 - 108n^2 - 126n + 211)\sigma(n);$
- (iv)  $3^5 \cdot \frac{243}{63}p(n-7) - 3^5 \cdot \frac{243}{18}p(n-22)$   
 $\equiv -22\sigma_9(n) - (216n + 302)\sigma_7(n) - (324n^2 + 30n - 210)\sigma_5(n)$   
 $+ (324n^3 - 135n^2 - 126n - 236)\sigma_3(n)$   
 $+ (270n^3 - 135n^2 - 171n + 164)\sigma(n);$
- (v)  $3^5 \cdot \frac{243}{9}p(n-26) - 3^5 \cdot \frac{243}{72}p(n-5)$   
 $\equiv 11\sigma_9(n) + (108n - 221)\sigma_7(n) + (162n^2 - 273n - 339)\sigma_5(n)$   
 $- (162n^3 + 135n^2 + 234n - 190)\sigma_3(n)$   
 $+ (270n^3 + 270n^2 + 45n + 308)\sigma(n).$

Theorem 1 is now a simple consequence of the above theorem when one remembers the well known congruence [7, 1 p. 167]

$$(26) \quad \sigma_s(n) \equiv 0 \pmod{k}$$

for ‘almost all’  $n$  for arbitrarily fixed  $k$  and odd  $s$ . We are here concerned with the cases  $s = 1, 3, 5, 7, 9$ , and the modulus  $k = 3^6$ . Now the first congruence of Theorem 1 follows easily from the second and the fifth congruences of Theorem 0; the second congruence of Theorem 1 follows from the first and the fourth congruences of Theorem 0. And from the remaining congruence of the last theorem we find that

$$(27) \quad \frac{243}{45}p(n-12) \equiv \frac{243}{36}p(n-15) \pmod{3}$$

for ‘almost all’ values of  $n$ , which is a part of the last congruence of Theorem 1. In virtue of the identical relation given in Lemma 8 it is not difficult to complete the proof of this last congruence when use is made of all the remaining results of Theorem 1 which we have already established.

Incidentally, it might be of interest to note that each of the five right



hand expressions of Theorem 0 is divisible by  $3^5$  for all positive  $n$ 's. For example from the first relation of the theorem we obtain,

$$(28) \quad \begin{aligned} 44\sigma_9(n) - (297n - 298)\sigma_7(n) - (81n^2 + 318n - 12)\sigma_5(n) \\ + (81n^3 + 27n^2 - 153n + 283)\sigma_3(n) \\ - (54n^3 + 216n^2 + 144n - 32)\sigma(n) \equiv 0 \pmod{3^5}. \end{aligned}$$

Several congruences of the same type are given by the author in [4].

### References

- [1] G. H. Hardy, *Ramanujan* (Cambridge, 1940).
- [2] G. H. Hardy and E. M. Wright, *An introduction to the theory of numbers* (Oxford, 4th ed. 1960).
- [3] D. B. Lahiri, 'On a type of series involving the partition function with applications to certain congruence relations', *Bull. Calcutta Math. Soc.* **38** (1946), 125—132.
- [4] D. B. Lahiri, 'On Ramanujan's function  $\tau(n)$  and the divisor function  $\sigma_k(n)$ , I', *Bull. Calcutta Math. Soc.* **38** (1946), 193—206.
- [5] D. B. Lahiri, 'Some restricted partition function: congruences modulo 5', *Jour. Austr. Math. Soc.* **9** (1969), 424—432.
- [6] S. Ramanujan, *Collected Papers* (Cambridge 1927, and Chelsea Publishing Company, New York, 1962).
- [7] G. N. Watson, 'Über Ramanujansche Kongruenzeigenschaften der Zerfallungszahlen', *Math. Z.* **39** (1935), 712—731.

Indian Statistical Institute  
Calcutta