BULL. AUSTRAL. MATH. SOC. 26A54, 26A57
VOL. 17 (1977), 161-175. (53B15)

New proofs of some theorems on infinitely differentiable functions

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New proofs are given for the splitting lemma and two of Mather's theorems in the theory of C^∞-functions. They use a Taylor series approach rather than the usual map-germ theory and are more elementary than previous proofs. A geometric characterisation of a class of degenerate functions is proved to hold in this account but not in the usual framework.

Introduction

It is well-known that C^∞-functions (those possessing derivatives of all orders), possess properties that make their study difficult. For example, the Cauchy function

(1) $$C(x) = \begin{cases} \exp\left(-x^{-2}\right) , & x \neq 0 , \\ 0 , & x = 0 , \end{cases}$$

has a Taylor series expansion about the origin that converges to zero rather than to $C(x)$.

This has led mathematicians to study equivalence classes of such functions. The usual equivalence classes studied nowadays are map-germs (or simply germs). Two C^∞-functions are equivalent if they agree over some neighbourhood of the point under study, taken to be the origin. The germ approach is developed in (*inter alia*) [2], [14]. Another approach uses Taylor expansions as the relevant equivalence classes. This leads to a slightly coarser structure. For example, $C(x)$ possesses the same

Received 3 March 1977.

Taylor expansion as the zero function, but lies in a different map-germ.

The theorems leading to Thom's *Catastrophe Theory* ([2], [13], [14], [15], *inter alia*) are developed in terms of map-germs. The purpose of the present paper is to exhibit proofs of three basic theorems in the Taylor series context. It is believed that the proofs are simplified by this and that this makes them more readily accessible to the non-specialist. In keeping with this view, and to emphasize the difference of approach, we employ a notation different from that of [2], [14], and more in keeping with the classical sources from which the present account derives.

The theorems to be proved are two results due to Mather ([8], [9], but see first [2], [14]) and the so-called splitting lemma ([2], [9], [13], [14]), for which a completely elementary proof is supplied. We further provide a simple characterisation of a particular class of degenerate functions and show the differences between the Taylor series and map-germ approaches.

1. Notation and basic notions

Let $f(\mathrm{x}) = f(x_1, x_2, \ldots, x_n)$ be a real-valued function of the n real variables x_i. We suppose $f(\mathrm{x})$ to be C^∞ - that is all possible derivatives exist at some point, without loss of generality the origin, near which we wish to study the behaviour of f. We write, as is usual,

(2) $$|\mathrm{x}|^2 = \sum_{i=1}^{n} x_i^2 .$$

As $f(\mathrm{x})$ is C^∞, it possesses a Taylor expansion about 0. This is a sum of monomials of the form

$$\prod_{i=1}^{n} \beta x_i^{\alpha_i} ,$$

each of which will be referred to as a *term*. For any given term, the *order* is defined as

$$\sum_{i=1}^{n} \alpha_i .$$

If $\phi(\mathrm{x})$ is a formal expression for a sum of terms, we write

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(3) $$\phi(x) = O\left(|x|^k\right)$$

if $\phi(x)$ contains no terms of order less than k.

Our starting point shall be the theorem that every power series determines an equivalence class of real functions $f(x)$ containing a member which, *par abus de langage*, we also term $f(x)$, which is C^∞, whose Taylor expansion about 0 is the given series, and which is the sum of that series in the event that this converges.

This theorem follows readily from a result on asymptotic expansions due to Borel and van der Corput. See [5, pp. 22-24].

As all the results that follow are directed to the study of singularities, we assume that the constant and all linear terms in the series vanish:

(4) $$f(x) = O\left(|x|^2\right) .$$

2. Coordinate transformations and right-equivalence

An essential object of the theorems that follow is the use of coordinate transformations to simplify the form of $f(x)$. Let $\xi_i(x)$ be new coordinates which preserve the origin and map a neighbourhood of 0 one-to-one onto itself. We require each $\xi_i(x)$ to be a C^∞-function. This leads to the equation

(5) $$\xi = A.x + O\left(|x|^2\right) ,$$

where A is a non-singular square matrix.

Equation (5) may be thought of as the composition of the affine transformation A and a "*quasi-identity*" - that is, a transformation of the form:

(6) $$\xi = x + O\left(|x|^2\right) ,$$

the composition being taken in either order, although with different forms for the non-linear terms in the quasi-identities involved.

According to a classical result (see [6, pp. 388-406]), the transformation (5) is invertible, the inverse being of the same form; two

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such transformations compose to give a third. The same holds for equation (6). If the series involved converge, so do the inverses and the compositions.

A more modern (group theoretic) account of these results is given by Bochner and Martin [*1*, pp. 3-10].

Following a transformation (5), the function $f(\mathrm{x})$ will become a function $g(\boldsymbol{\xi})$ of the new coordinates. Then

$$f(\mathrm{x}) = g(\boldsymbol{\xi}) \; , \tag{7}$$

and we shall say that the functions f, g are *right-equivalent*. A useful notation is

$$f(\mathrm{x}) \overset{R}{\sim} g(\mathrm{x}) \; . \tag{8}$$

In the special case $\mathrm{A} = \mathrm{I}$ (the quasi-identity), we shall write

$$f(\mathrm{x}) \overset{Q}{\sim} g(\mathrm{x}) \; . \tag{9}$$

3. k-jets, k-completeness, and k-determination

Our aim shall be to seek particularly simple forms for the function $f(\mathrm{x})$. One such shall be a suitably truncated version of the power series.

DEFINITION 1. The *k-jet*, ${}^{k}f(\mathrm{x})$, of $f(\mathrm{x})$ is the polynomial consisting of all terms of order less than or equal to k in the Taylor expansion of $f(\mathrm{x})$.

We shall give conditions under which $f(\mathrm{x})$, and indeed other functions as well, are right-equivalent (indeed right-equivalent under quasi-identity) to their k-jets. Heuristic considerations based on formal expansion of $f\left(\mathrm{x}+O\left(|\mathrm{x}|^{2}\right)\right)$ suggest a study of forms with the structure

$$\phi(\mathrm{x}) = \sum_{i=1}^{n} \phi_{i}(\mathrm{x}) f_{,i}(\mathrm{x}) \; , \tag{10}$$

where $f_{,i}(\mathrm{x})$ are the n partial derivatives of $f(\mathrm{x})$ and $\phi_{i}(\mathrm{x})$ are n C^{∞}-functions. If a given C^{∞}-function $\phi(\mathrm{x})$ is expressible in this form, we shall say that it is *generated by* the $f_{,i}(\mathrm{x})$.

A further definition is also required.

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DEFINITION 2. If all $\phi(x)$ such that $\phi(x) = O\left(|x|^k\right)$ are generated by the $f_{,i}(x)$ and the coefficient functions $\phi_i(x)$ obey $\phi_i(x) = O(|x|)$, then $f(x)$ is said to be *k-complete*.

(For example, a non-degenerate quadratic form is 2-complete; $x_1^4 + x_2^4$ is 5-complete, but not 4-complete; $x_1^2 x_2$ is not k-complete for any k.)

It is clear that if $f(x)$ is k-complete, it is also $(k+l)$-complete for any natural number l. It follows that to prove the k-completeness of a function $f(x)$, one need only show that all $\phi(x)$ of order k satisfy

$$\phi(x) = \sum_{i=1}^{n} \phi_i(x) f_{,i}(x) + O\left(|x|^{k+1}\right) \tag{11}$$

for some suitable C^∞-functions $\phi_i(x)$.

Of the possible power series, there will be infinitely many sharing the same k-jet for any given k. If progress is to be possible via a consideration of k-jets alone, the properties of the corresponding C^∞-functions must be determined in some sense, by those of the k-jets. To this end, a further definition is introduced.

DEFINITION 3. Let $f(x)$ be a C^∞-function and let $g(x)$ be any other C^∞-function such that ${}^k f(x) = {}^k g(x)$ for some k. If for all such functions $g(x)$,

$$g(x) \overset{R}{\sim} f(x) ,$$

then $f(x)$ will be said to be *k-determined*.

We now proceed to demonstrate the relationship between k-completeness and k-determination.

4. Mather's first theorem

Clearly k-completeness is an easier property to demonstrate than k-determination, as the first property involves only finitely many calculations for its proof whereas the second does not. However, we have

THEOREM 1 (Mather). *If $f(x)$ is k-complete, then it is k-determined.*

Proof. The proof is in two parts. Define

$$F(x, t) = (1-t)f(x) + tg(x) , \tag{12}$$

where ${}^k f(x) = {}^k g(x)$. Then, writing $F_{,i}(x, t)$ $(1 \le i \le n)$ for the partial derivative functions with respect to the variables x_i, and $F_{,n+1}(x, t)$ for the partial derivative function with respect to t, we have

$$f(x) - g(x) = O\left(|x|^{k+1}\right) , \tag{13}$$

$$F_{,n+1}(x, t) = g(x) - f(x) , \tag{14}$$

and

$$F_{,i}(x, t) = f_{,i}(x) + t\left(g_{,i}(x) - f_{,i}(x)\right) , \tag{15}$$

for $1 \le i \le n$.

1. It is first shown that $F_{,n+1}(x, t)$ is generated by the other n functions $F_{,i}(x, t)$. To this end, seek functions $-\eta_i(x, t)$ such that

$$F_{,n+1}(x, t) = g(x) - f(x) = -\sum_{i=1}^{n} \eta_i(x, t)F_{,i}(x, t) . \tag{16}$$

Put

$$\eta_i(x, t) = \sum_{s=0}^{\infty} \eta_i^{(s)}(x)t^s , \tag{17}$$

and substitute into equation (16).

Equating powers of t now yields

$$\sum_{i=1}^{n} \eta_i^{(0)}(x)f_{,i}(x) = f(x) - g(x) \tag{18}$$

and

$$\sum_{i=1}^{n} \eta_i^{(s+1)}(x)f_{,i}(x) = \sum_{i=1}^{n} \eta_i^{(s)}(x)\left(f_{,i}(x) - g_{,i}(x)\right) , \tag{19}$$

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where use has been made of equation (15).

Since $f(\mathrm{x})$ is k-complete, it is $(k+1)$-complete, and the $\eta_i^{(0)}(\mathrm{x})$ exist. Furthermore $\eta_i^{(0)}(\mathrm{x}) = O(|\mathrm{x}|)$. Thus

$$\sum_{i=1}^{n} \eta_i^{(0)}(\mathrm{x})\left(f_{,i}(\mathrm{x})-g_{,i}(\mathrm{x})\right) = O\left(|\mathrm{x}|^{k+1}\right) , \tag{20}$$

so that $\eta_i^{(1)}(\mathrm{x})$ exists and $\eta_i^{(1)}(\mathrm{x}) = O(|\mathrm{x}|)$.

The argument now proceeds inductively.

2. We now solve for $\mathrm{H}(\mathrm{x}, t)$ the system of differential equations

$$\frac{\partial H_i}{\partial t} = \eta_i\left(\mathrm{H}(\mathrm{x}, t), t\right) \tag{21}$$

with

$$H_i(\mathrm{x}, 0) = x_i . \tag{22}$$

This system possesses a solution by a well-known existence theorem in the theory of differential equations. (See, for example, [3, pp. 57 *et seq*.].) The solution is C^∞ .

It follows that equation (16) may be written

$$F_{,n+1}\left(\mathrm{H}(\mathrm{x}, t), t\right) + \sum_{i=1}^{n} F_{,i}\left(\mathrm{H}(\mathrm{x}, t), t\right) \frac{\partial}{\partial t} H_i(\mathrm{x}, t) = 0 , \tag{23}$$

that is

$$\frac{d}{dt} F\left(\mathrm{H}(\mathrm{x}, t), t\right) = 0 ,$$

from which it follows that $F\left(\mathrm{H}(\mathrm{x}, t), t\right)$ is independent of t .

Thus, for a range of values of t about some t_0 ,

$F\left(\mathrm{H}(\mathrm{x}, t), t\right) = F\left(\mathrm{H}\left(\mathrm{x}, t_0\right), t_0\right)$ and $F(\mathrm{x}, t) \overset{R}{\sim} F\left(\mathrm{x}, t_0\right)$. This process may be continued, using equation (16) from $t = 0$ to $t = 1$ if for all $h(\mathrm{x})$ of order $k + 1$, the set of functions generated by the $f_{,i}(\mathrm{x})$ and the set of functions generated by $f_{,i}(\mathrm{x}) + h_{,i}(\mathrm{x})$ are identical. As

$f(x)$ is k-complete, this follows immediately.

We now have

$$F(H(x, t), t) = F(H(x, 0), 0) = F(x, 0) = f(x) \tag{24}$$

by equations (22), (12).

But also,

$$F(H(x, 1), 1) = g(H(x, 1)) \ , \tag{25}$$

so that, setting

$$H(x, 1) = \xi(x) \ , \tag{26}$$

we reach

$$f(x) = g(\xi) \ , \tag{27}$$

or

$$f(x) \overset{R}{\sim} g(x) \ , \tag{28}$$

the desired result.

As a matter of fact, elementary order considerations or use of equation (22) show that the stronger result

$$f(x) \overset{Q}{\sim} g(x) \tag{29}$$

has been proved.

We have in fact proved a version of the theorem due to Stefan [11]. $f(x)$ is k-determined if it is $(k+1)$-complete and if for all $h(x)$ of order $k+1$, $f_{,i}(x)$ and $f_{,i}(x) + h_{,i}(x)$ generate the same set of functions. Siersma's theorem [10] that $f(x)$ is k-determined if it is $(k+1)$-complete and if $f(x)$ is homogeneous of order k is a ready corollary.

Proofs of Theorem 1 are given in [2], [8], [9], and [14] as well as by Stefan [11]. The above account differs in Part 1 but essentially repeats the others in Part 2, which is included here for completeness.

We may also note that if $f(x)$ is $(k+1)$-complete so is ${}^{k}f(x)$ and conversely. Also, $f(x)$ is k-determined if and only if ${}^{k}f(x)$ is. The proofs of these statements are quite trivial.

5. Mather's second theorem

Mather's second theorem may usefully be thought of as a "semi-converse" of his first. It states:

THEOREM 2 (Mather). *If* $f(x)$ *is* k*-determined, then it is* $(k+1)$*-complete.*

Proof. It will suffice to show that every homogeneous polynomial of order $(k+1)$ is generated by the $f_{,i}(x)$. Let $\phi(x)$ be such a polynomial. Form $f(x) + \phi(tx) = f(x) + t^{k+1}\phi(x)$, where t is a real variable.

By hypothesis

$$f(x) + t^{k+1}\phi(x) = f\{\xi(x,\ t)\} \ , \tag{30}$$

where $\xi(x,\ t)$ is C^∞ in x . As $f(\xi)$ is C^∞ in ξ and $f(x) + t^{k+1}\phi(x)$ is C^∞ in $x,\ t$, we may, without loss of generality, assume $\xi(x,\ t)$ to be C^∞ in t . In fact, the parameter t appears on the left as t^{k+1} and so we may assume that ξ depends only on x and t^{k+1} . Thus

$$\xi(x,\ t) = \xi^{(0)} + t^{k+1}\xi^{(k+1)} + t^{2k+2}\xi^{(2k+2)} + \ldots\ , \tag{31}$$

where $\xi^{(l)}$ is a function of x only for each l . By continuity of $f(x) + t^{k+1}\phi(x)$ at $t = 0$, we may, without loss of generality, set $\xi^{(0)} = x$. Then

$$\begin{aligned} f(x) + t^{k+1}\phi(x) &= f\left(x + t^{k+1}\xi^{(k+1)} + \ldots\right) \\ &= f(x) + t^{k+1}\sum_{i=1}^{n} f_{,i}(x)\xi_i^{(k+1)} + \ldots\ . \end{aligned}$$

If we now differentiate $k + 1$ times with respect to t , and then set $t = 0$, there results

$$\phi(x) = \sum_{i=1}^{n} f_{,i}(x)\xi_i^{(k+1)}\ . \tag{32}$$

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It remains for us to show that $\xi_i^{(k+1)} = O(|\mathrm{x}|)$. Suppose otherwise and recall that by the remark at the end of the previous section $f(\mathrm{x})$ may without loss of generality be taken to be a polynomial of degree k . We then have

$$\sum a_i f_{,i}(\mathrm{x}) = 0$$

for some non-zero a_i , and the solution of this partial differential equation involves us in non-independence of the x_i . Thus $a_i = 0$, for all i , and the theorem is proved.

This result was first proved by Mather [*8*], [*9*], and accounts are also given by Bröcker [*2*, p. 100] and Wassermann [*14*, pp. 41-43]. These differ considerably from that given above.

The proof of Theorem 2 allows us also to deduce the

COROLLARY. *If*

$$f(\mathrm{x}) \overset{R}{\sim} {}^k f(\mathrm{x}) ,$$

then

$$f(\mathrm{x}) \overset{Q}{\sim} {}^k f(\mathrm{x}) ,$$

for some suitably chosen quasi-identity.

6. The splitting lemma

The splitting lemma enables further simplification of a function $f(\mathrm{x})$ under right-equivalence. It states

THEOREM 3.

$$(33)\quad f(x_1, x_2, \ldots, x_n) \overset{R}{\sim} g(x_1, x_2, \ldots, x_r) + q(x_{r+1}, x_{r+2}, \ldots, x_n) ,$$

where $g(x_1, x_2, \ldots, x_r) = O(|\mathrm{x}|^3)$ *and* $q(x_{r+1}, x_{r+2}, \ldots, x_n)$ *is a non-degenerate quadratic form.*

Proof. Since, by equation (4), $f(\mathrm{x})$ is of second order, ${}^2f(\mathrm{x})$ is a quadratic form. This may be reduced by a linear transformation to the canonical form

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$$\pm x_{r+1}^2 \pm x_{r+2}^2 \pm \ldots \pm x_n^2 ,$$

where $n - r$ is the rank of the associated matrix (r is the corank).

In these new coordinates

(34) $$f(x) = g\left(x_1, x_2, \ldots, x_r\right) + \phi(x) + \psi(x) + q\left(x_{r+1}, x_{r+2}, \ldots, x_n\right) ,$$

where q is the form given above and g, ϕ, ψ are of third order; $\phi(x)$ is the sum of all terms divisible by $x_{r+1}^2, x_{r+2}^2, \ldots, x_n^2$; $\psi(x)$ is the sum of all terms involving $x_{r+1}, x_{r+2}, \ldots, x_n$ only once.

Now

(35) $$\psi(x) = x_{r+1}\psi_1(x) + x_{r+2}\psi_2(x) + \ldots + x_n\psi_{n-r}(x) ,$$

where $\psi_1(x)$ does not contain x_{r+1} , $\psi_2(x)$ contains neither x_{r+1} nor x_{r+2} , and so on.

We now form

$$\pm\left(x_{r+1}^2 \mp x_{r+1}\psi_1(x) + \tfrac{1}{4}\psi_1(x)^2\right) \mp \tfrac{1}{4}\left(\psi_1(x)\right)^2 ;$$

that is,

$$\pm\left(x_{r+1} \mp \tfrac{1}{2}\psi_1(x)\right)^2 \mp \tfrac{1}{4}\left(\psi_1(x)\right)^2 ,$$

and set

(36) $$\xi_{r+1} = x_{r+1} \mp \tfrac{1}{2}\psi_1(x) .$$

Proceeding in this way, we absorb the whole of $\psi(x)$ into the quadratic form. g, ϕ are altered in the process, but we reach in any case the form (34) except that the term $\psi(x)$ is absent. For notational convenience, replace ξ_i by x_i .

Now

(37) $$\phi(x) = x_{r+1}^2\phi_1(x) + x_{r+2}^2\phi_2(x) + \ldots + x_n^2\phi_{n-r}(x) ,$$

for suitably chosen $\phi_i(x)$.

We now form

$$\pm x_{r+1}^2\left(1 \mp \phi_1(\mathrm{x})\right)$$

and set

$$\xi_{r+1} = x_{r+1}\left(1 \mp \phi_1(\mathrm{x})\right)^{\frac{1}{2}} , \tag{38}$$

and so, by proceeding in this way, absorb the whole of $\phi(\mathrm{x})$ into the quadratic term. We thus reach the required form. Transformations (37), (38) are, in fact, quasi-identities.

Thom sketches [*13*, p. 59] a proof of this result and Mather [*9*], Wassermann [*14*, pp. 120-123] and Bröcker [*2*, p. 125] provide proofs. Gromoll and Meyer [*7*] prove a Hilbert space analogue.

The above proof proceeds from a more elementary point of view than these others.

7. A geometric characterisation

There exist $f(\mathrm{x})$ which are not finitely determined - that is they are not k-determined for any finite k. Such functions are also described as being of infinite codimension. This section gives a new characterisation of such functions.

Our starting point is a lemma due to Thom [*12*].

LEMMA (Thom). *$f(\mathrm{x})$ has infinite codimension if and only if there exists an i such that x_i^k is not generated by the $f_{,i}(\mathrm{x})$ for any k.*

In order to give a geometric characterisation of this result, we introduce the concept of an ε-neighbourhood.

DEFINITION 4. By the ε-neighbourhood of the singular point 0 of $f(\mathrm{x})$, we shall mean that connected neighbourhd of 0 for which

$$|f_{,i}(\mathrm{x})| < \varepsilon \tag{39}$$

for all i.

We now have

THEOREM 4. *$f(\mathrm{x})$ has infinite codimension if and only if the*

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ε-neighbourhood of 0 *is unbounded.*

Proof. First suppose $f(x)$ to have infinite codimension. Then, by the lemma, there exists an x_i such that x_i^k is not generated by the $f_{,i}(x)$ for any k. In this case, inequalities (39) do not suffice to bound x_i.

Now suppose $f(x)$ to be finitely determined. Then for some suitable k, x_i^k is generated by the $f_{,i}(x)$ for all i, and hence is bounded by the inequalities (39).

The result follows.

In Theorem 4, ε may be arbitrarily small and so there exists as an intersection of all ε-neighbourhoods a continuum of singular points, so that 0 is not an isolated singularity. Conversely if 0 is isolated, its ε-neighbourhood must be bounded for suitable small ε. We thus have the

COROLLARY. *$f(x)$ is finitely determined if and only if 0 is an isolated singularity of $f(x)$.*

It should be noted that Theorem 4 and this corollary are not true on the usual map-germ accounts. $C(x)$, as defined in equation (1), provides a ready counter-example.

8. A conjecture

The following conjecture would allow an elementary proof of the uniqueness of unfoldings [2, Chapter 16], would allow a very ready proof of the splitting lemma and would somewhat simplify existing proofs of Thom's classification theorem. It appeared first in [4, p. 193].

CONJECTURE. Let $f(x)$ be a finitely determined polynomial and suppose that $g(x) = \sum_{i=1}^{n} g_i(x) f_{,i}(x)$, where $g_i(x) = O(|x|^2)$. Then

$$f(x) \overset{Q}{\sim} f(x) + g(x) . \tag{40}$$

This may be regarded as an extension of Theorem 1.

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