Solution of a Geometrical Problem.

By R. F. Muirhead.

"To draw through a given point a transversal of a given triangle so that the segments of the transversal may be in a given ratio."

Figure 25.

Analysis. ABC is a triangle and DEF a transversal and K is the point of concurrence of the four circles circumscribed about the four triangles formed by the transversal and the sides of the triangle. H is a point on the circumference of the circumcircle of ABC, such that AK and AH are equally inclined to the bisector of the angle BAC.

Thus $\angle HBC = \angle KCB = \angle KED$

and $\angle HCB = \angle KBC = \angle KFE$.

Thus the triangles HBC, KCB and KEF are mutually similar.

Again if AH meet BC in X, we have

$$\angle XHC = \angle ABC = \angle FKD.$$

Hence the figures HBCX and KEFD are similar;

hence $CX : XB = FD : DE$.

Let O be any point in EF, and OLDK a circle meeting KB in L.

Then $\angle OLK = \angle ODK = \angle FBK$.

$\therefore$ OL is parallel to FB.

Construction. Given ABC and O, and the ratio $p:q$ to which FD : DE is to be equal. Make $CX : XB = p:q$. Draw AXH meeting the circumcircle of ABC in H. Make arc BK equal and opposite to arc CH. Draw OL parallel to BA to meet KB in L. and let the circle through OLK cut BC in D and D′. Then the lines OD and OD′ give the two solutions of the problem.

https://doi.org/10.1017/S0013091500001991 Published online by Cambridge University Press

Cor. If we make O go off to infinity in a given direction, the arc KDL becomes a straight line through K making ∠BKD equal to the angle between BA and the given direction.

This enables us to solve the problem : *To draw a transversal of a triangle in a given direction so that the segments of the transversal may be in a given ratio.*

Remark. We may note that K is the focus of the parabola which touches the three sides of ABC and the transversal, and that all other transversals (not passing through O) which have their segments in the same ratio, will touch the same parabola. Hence we see that if three fixed tangents be drawn to a parabola, the ratio of the segments they intercept on any variable tangent is a fixed one. This is a known property of tangents to a parabola. (See Professor Gibson's paper in Vol. IX. of our *Proceedings.*)

From this it follows that we could reduce our problem to that of drawing through a given point a fifth tangent to a parabola when four are given. For, ABC being given, and the ratio $p:q$, we can by a very easy construction draw a transversal (say through an arbitrary point in AB) having its segments in the ratio $p:q$. We have then four given tangents.

On the other hand our construction affords a new solution of the problem : to draw through a given point a tangent to the parabola which touches four given lines.

The only previously published solution I have met with of the problem of this paper is that given in Thomas Simpson's *Elements of Geometry* (Problem XXXVII. of the section on the Construction of Geometrical Problems). The solution there given is not at all direct or elegant : a footnote gives a reference to David Gregory's *Astronomy*, B.V. Prob. 8.

I ought to add that the idea of the solution here given was suggested to me by the solution of the special case of the problem when $p:q=1$ which was communicated to me by a friend.