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A PARTITION OF FINITE T_0 TOPOLOGIES

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The aim of this paper is to study a decomposition of finite T_0 spaces into topological entities called chains and cells. These objects behave like complete units under homeomorphisms and they appear to be useful in investigating certain aspects of finite spaces. As an elementary illustration of how these entities can be used, the concept of an A_2 space is introduced (in the next paragraph) and it is demonstrated that the order of the automorphism group of an A_2 space is expressible as 2^t, for some $t \geqq 0$.

All the spaces in this paper are assumed to have the T_0 separation property and are defined on a finite point set N. Let $\mathscr{T}$ be a topology on N and A a subset of N. Then $A^*(\mathscr{T})$, or more simply A^* when there is no risk of confusion, will denote the minimal open set of $\mathscr{T}$ that contains A. That is, $A^*(\mathscr{T}) = \bigcap \{O|A \subseteq O \in \mathscr{T}\}$. For any set A, $|A|$ will denote the cardinality of A. The single element set $\{\alpha\}$, $\alpha \in N$, will be written simply as α. The union of α with a set A is written $\alpha + A$ and the relative difference of two sets A and B as $A - B$. If $O \in \mathscr{T}$, then an open set Q is a $(*)$ cover of O if $O \subset Q$ and $|Q - O| = 1$. An open set $O \in \mathscr{T}$ is an A_p set (of $\mathscr{T}$) provided there exist p, and only p, $(*)$ covers of O. A topology $\mathscr{T}$ on N is an A_p space if

(1) there exists an A_p set of $\mathscr{T}$, and

(2) $N \neq O \in \mathscr{T}$ implies that, for some $q \leqq p$, O is an A_q set of $\mathscr{T}$.

1.

LEMMA 1. *Let $\mathscr{T}$ be a topology on N.*

(a) *If $\alpha \in N$, then $\alpha^* - \alpha$ is an open set of $\mathscr{T}$.*

(*b*) *$\mathscr{T}$ has isolated points. In other words, there exists at least one open set of $\mathscr{T}$ which contains a single element of N.*

(c) *If $N \neq O \in \mathscr{T}$, then there exists at least one open $Q \in \mathscr{T}$ such that $O \subset Q$ and $|Q - O| = 1$.*

The proofs are omitted as the results follow directly from the T_0 property. Lemma 1 is basic to the development of this paper.

The concept of a *chain* will now be introduced. Let $\mathscr{T}$ be a topology on N. Let

$$\mathbf{C} = [\alpha_1, \ldots, \alpha_i, \ldots, \alpha_m]$$

be a sequence of m distinct elements, $m \geqq 1$, of N. $\mathbf{C}$ is called a *chain* of $\mathscr{T}$ of *length m* provided:

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(1) if $\alpha_1^* - \alpha_1 = \beta^*$ for some $\beta \in N$, then there exists a $\gamma \in N$ such that $\gamma \neq \alpha_1$ and $\gamma^* - \gamma = \beta^*$;

(2) if $\beta^* - \beta = \alpha_m^*$ for some $\beta \in N$, then there exists a $\gamma \in N$ such that $\gamma \neq \beta$ and $\gamma^* - \gamma = \alpha_m^*$, and if $m > 1$ and $1 \leqq i < m$, then

(3) $\alpha_{i+1}^* - \alpha_{i+1} = \alpha_i^*$;

(4) $\beta^* - \beta = \alpha_i^*$ for some $\beta \in N$ implies $\beta = \alpha_{i+1}$.

The length of the chain $\mathbf{C}$ will be denoted by $L(\mathbf{C})$. The *supporting open set* of the chain $\mathbf{C}$, written as $^*\mathbf{C}(\mathscr{T})$, or more simply as $^*\mathbf{C}$ when there is no risk of confusion, is defined to be the open set $\alpha_1^* - \alpha_1$ of $\mathscr{T}$. The notation $\{\mathbf{C} : i\}$ will be used to indicate the subset $\{\alpha_1, \ldots, \alpha_i\}$ consisting of the first i terms of the sequence $\mathbf{C}$. Also $\{\mathbf{C} : 0\} = \emptyset$. Then $\alpha_1^* = {}^*\mathbf{C} + \alpha_1$ and condition (3) becomes $\alpha_i^* = {}^*\mathbf{C} \cup \{\mathbf{C} : i\}$. $\mathbf{C}$ will be used to denote both the sequence $[\alpha_1, \ldots, \alpha_m]$ and the unordered set $\{\alpha_1, \ldots, \alpha_m\}$. The meaning of $\mathbf{C}$ will always be clear from the context in which it will be used.

Example. Let $\mathscr{V}$ be a topology, on the set $\{\alpha | 1 \leqq \alpha \leqq 11\}$, defined by $1^* = 1$, $2^* = \{1, 2\}$, $3^* = \{1, 2, 3\}$, $5^* = 5$, $4^* = \{4, 5\}$, $6^* = 6$, $7^* = \{1, 2, 3, 7\}$, $8^* = \{1, 2, 3, 8\}$, $11^* = \{1, 2, 3, 8, 11\}$, $10^* = \{1, 4, 5, 10\}$ and $9^* = \{1, 4, 5, 9, 10\}$. The chains of $\mathscr{V}$ are: $\mathbf{V}_1 = [1, 2, 3]$, $\mathbf{V}_2 = [5, 4]$, $\mathbf{V}_3 = [6]$, $\mathbf{W}_1 = [7]$, $\mathbf{W}_2 = [8, 11]$ and $\mathbf{X} = [10, 9]$. The supporting open sets are: $^*\mathbf{V}_1 = {}^*\mathbf{V}_2 = {}^*\mathbf{V}_3 = \emptyset$, $^*\mathbf{W}_1 = {}^*\mathbf{W}_2 = \{1, 2, 3\}$ and $^*\mathbf{X} = \{1, 4, 5\}$.

LEMMA 2. *Let $\mathscr{T}$ be a topology on N. If $\alpha \in N$, then α is contained in some chain of $\mathscr{T}$.*

Proof. Let α be specified. A "programme" which generates a chain containing α is described below. The variable (integer) symbols used in this programme are: b, j, m, p and the indexed variable $k(j)$ with the index j running over the $2n + 1$ integer values in the range $-n \leqq j \leqq n$. For the sake of concreteness, it is assumed that N is the set of the first n integers, so that α is itself an integer. At the end of the programme, the chain, of length m, resides in the $k(j)$'s, with $\alpha = k(0)$.

START 1. Set the variable p equal to α, j equal to 0, m equal to 1.

2. Assign the value p to $k(j)$.
3. If there exists a β such that $p^* - p = \beta^*$ then go to 4, otherwise go to 7.
4. If there exists a $\gamma \neq p$ such that $\gamma^* - \gamma = \beta^*$ then go to 7, otherwise go to 5.
5. Increase m by 1, decrease j by 1 and assign the value β to p.
6. Go to 2.
7. Set the variable p equal to α and j equal to 0.
8. If there exists a β such that $\beta^* - \beta = p^*$ then go to 9, otherwise go to 13.
9. If there exists a $\gamma \neq \beta$ such that $\gamma^* - \gamma = p^*$ then go to 13, otherwise go to 10.
10. Increase m by 1, increase j by 1 and set the variable p equal to β.
11. Assign p to $k(j)$.
12. Go to 8.

END 13. $\alpha = k(0)$ in the chain $\mathbf{C}$ of length m, where $\mathbf{C} = [k(1-m), \ldots, k(0)]$ if the current value of $j \leqq 0$ and $\mathbf{C} = [k(j-m+1), \ldots, k(0), \ldots, k(j)]$ if the current value of j is positive.

It is a routine matter to verify that the sequence $\mathbf{C}$ in statement 13 is in fact a chain of $\mathscr{T}$.

LEMMA 3. *Let* $\mathbf{C} = [\alpha_1, \ldots, \alpha_m]$ *be a chain of some topology* $\mathscr{T}$ *on* N. *Then* $^*\mathbf{C}$ *does not contain any element of* $\mathbf{C}$.

Proof. $\alpha_1 \notin {}^*\mathbf{C}$ by the definition of $^*\mathbf{C}$. If $m > 1$, then $1 < j \leqq m$ implies that $\alpha_1 \in \alpha_j^*$ by condition (3) in the definition of a chain. Now if $\alpha_j \in {}^*\mathbf{C}$, then $\alpha_j \in {}^*\mathbf{C} + \alpha_1 = \alpha_1^*$ which contradicts the T_0 property of $\mathscr{T}$.

LEMMA 4. *Let* $\mathbf{C} = [\alpha_1, \ldots, \alpha_p]$ *and* $\mathbf{D} = [\beta_1, \ldots, \beta_q]$ *be two chains of some topology* $\mathscr{T}$ *on* N. *Then* $\mathbf{C} = \mathbf{D}$ *if and only if* $\alpha_1 = \beta_1$.

Proof. If $\alpha_1 = \beta_1$, then $^*\mathbf{C} = \alpha_1^* - \alpha_1 = \beta_1^* - \beta_1 = {}^*\mathbf{D}$. Let $\min(p, q) = p$. First, it is shown that $\alpha_i = \beta_i$, for all $i \leqq p$. This is obvious if $p = 1$. Now suppose that $p > 1$ and assume, as the hypothesis of induction, that $\alpha_i = \beta_i$ if $i \leqq j$ for some $j < p$. The fact that $\mathbf{D}$ is a chain implies that $\beta_{j+1}^* - \beta_{j+1} = \beta_j^*$ and since $\alpha_j = \beta_j$, therefore $\beta_{j+1}^* - \beta_{j+1} = \alpha_j^*$. Since $\mathbf{C}$ is a chain, therefore Condition 4 in the definition of a chain implies that $\beta_{j+1} = \alpha_{j+1}$ and so $\alpha_i = \beta_i$ for all $i \leqq p$. Now if $p = q$, then the proof is complete. However, if $p < q$, then obviously $\beta_{p+1} \notin \mathbf{C}$ and $\beta_{p+1}^* - \beta_{p+1} = \beta_p^* = \alpha_p^*$. Since $p < q$ and $\mathbf{D}$ is a chain, therefore Condition 4 in the chain definition asserts that there does *not* exist a $\gamma \in N$ such that $\gamma^* - \gamma = \beta_p^* = \alpha_p^*$ and $\gamma \neq \beta_{p+1}$. However, because $\mathbf{C}$ is a chain, this result contradicts Condition 2 in the chain definition. This completes the proof.

LEMMA 5. *Let* $\mathbf{C}$ *and* $\mathbf{D}$ *be two distinct chains of a topology* $\mathscr{T}$. *If* $^*\mathbf{C}$ *contains elements of* $\mathbf{D}$, *then* $^*\mathbf{D}$ *does not contain any element of* $\mathbf{C}$.

Proof. Let $\mathbf{C} = [\alpha_1, \ldots, \alpha_j, \ldots]$ and $\mathbf{D} = [\beta_1, \ldots, \beta_i, \ldots]$. By Lemma 4, $\mathbf{C} \neq \mathbf{D}$ implies that $\alpha_1 \neq \beta_1$. Suppose that $\beta_i \in {}^*\mathbf{C}$. Since $\beta_1 \in \beta_i^*$, therefore $\beta_1 \in {}^*\mathbf{C} + \alpha_1 = \alpha_1^*$. Similarly, it can be shown that $\alpha_j \in {}^*\mathbf{D}$ implies that $\alpha_1 \in \beta_1^*$. Therefore if Lemma 5 is not true, then $\mathscr{T}$ is not T_0.

LEMMA 6. *Let* $\mathbf{C}$ *and* $\mathbf{D}$ *be two chains of a topology* $\mathscr{T}$. *Then* $\mathbf{C} \neq \mathbf{D}$ *implies that* $\mathbf{C}$ *and* $\mathbf{D}$ *are disjoint.*

Proof. Let $\mathbf{C} = [\alpha_1, \ldots, \alpha_p]$ and $\mathbf{D} = [\beta_1, \ldots, \beta_q]$. By Lemma 4, $\mathbf{C} \neq \mathbf{D}$ implies that $\alpha_1 \neq \beta_1$. If $p = q = 1$, then Lemma 6 is trivially true. Now suppose that at least one of the chains contains more than one element. If $\mathbf{C}$ and $\mathbf{D}$ are not disjoint, then there exist either (1) an $i \geqq 2$ and a $j \geqq 2$ such that $\alpha_i = \beta_j$ and $\{\mathbf{C} : i - 1\} \cap \{\mathbf{D} : j - 1\} = \emptyset$, or (2) a $j \geqq 2$ such that $\alpha_1 = \beta_j$ and $\alpha_1 \notin \{\mathbf{D} : j - 1\}$, or (3) an $i \geqq 2$ such that $\alpha_i = \beta_1$ and $\beta_1 \notin$

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$\{\mathbf{C} : i - 1\}$. Assume that the conditions of case (1) are satisfied. It is easily demonstrated that this assumption implies that

$$\{\mathbf{D} : j - 1\} \subseteq {}^*\mathbf{C} \text{ and } \{\mathbf{C} : i - 1\} \subseteq {}^*\mathbf{D}$$

so that ${}^*\mathbf{C} \cap \mathbf{D} \neq \emptyset$ and ${}^*\mathbf{D} \cap \mathbf{C} \neq \emptyset$ thus contradicting Lemma 5. Now suppose that there exists a j with the properties indicated for case (2). Then ${}^*\mathbf{C} = \alpha_1{}^* - \alpha_1 = \beta_j{}^* - \beta_j = \beta_{j-1}{}^*$. This last result implies that $\mathbf{C}$ cannot be a chain because condition (1) in the chain definition is violated. For, since $j - 1 < q$ and q is the length of $\mathbf{D}$, Condition 4 asserts that there does *not* exist a $\gamma \neq \beta_j$, that is, a $\gamma \neq \alpha_1$ such that $\gamma^* - \gamma = \beta_{j-1}{}^* = {}^*\mathbf{C}$. Case (3) may be treated similarly. Therefore if $\mathbf{C} \neq \mathbf{D}$, then $\mathbf{C} \cap \mathbf{D} = \emptyset$.

THEOREM 1. *Let $\mathscr{T}$ be a topology on N. Then the collection of chains of $\mathscr{T}$ constitutes a partition of N.*

Proof. This is an immediate consequence of Lemma 2 and Lemma 6.

The next Lemma declares, without proofs, some elementary properties of chains. These will be used in the subsequent discussions, often without any explicit reference being made to the result in question.

LEMMA 7. *Let $\mathscr{T}$ be a topology on N.*

(1) *If $\mathbf{C}$ is a chain of $\mathscr{T}$ and*

(a) *$i \leqq L(\mathbf{C})$, then ${}^*\mathbf{C} \cup \{\mathbf{C} : i\}$ is open.*

(b) *O is an open set of $\mathscr{T}$, and $O \cap \mathbf{C} \neq \emptyset$, then ${}^*\mathbf{C} \subset O$.*

(c) *O is an open set of $\mathscr{T}$, then $\mathbf{C} \cap O = \{\mathbf{C} : i\}$ where $i = |\mathbf{C} \cap O| \geqq 0$.*

(2) *If $O \in \mathscr{T}$ and $\{\mathbf{C}_1, \ldots, \mathbf{C}_i, \ldots, \mathbf{C}_p\}$ is the collection of chains of $\mathscr{T}$ that have non-void intersections with O, then*

$$O = \bigcup_{i=1}^{p} \{\mathbf{C}_i : t_i\}$$

where $t_i = |O \cap \mathbf{C}_i| \geqq 0$.

LEMMA 8. *If $\mathscr{T}$ and $\mathscr{U}$ are topologies on N, then π is a homeomorphism of $\mathscr{T}$ onto $\mathscr{U}$ if and only if $\alpha, \beta \in N$ and $\pi(\alpha) = \beta$ implies that $\pi(\alpha^*(\mathscr{T})) = \beta^*(\mathscr{U})$.*

This elementary result is basic to the study of homeomorphisms between topological spaces.

THEOREM 2. *Let $\mathscr{T}$ and $\mathscr{U}$ be homeomorphic topologies on N and let π be a homeomorphism of $\mathscr{T}$ onto $\mathscr{U}$. Suppose that $\mathbf{C} = [\alpha_1, \ldots, \alpha_i, \ldots, \alpha_u]$ is a chain of $\mathscr{T}$ and $\mathbf{D} = [\beta_1, \ldots, \beta_j, \ldots, \beta_v]$ is a chain of $\mathscr{U}$. Then*

(1) *$\pi(\alpha_1) \in \mathbf{D}$ implies that $\pi(\alpha_1) = \beta_1$;*

(2) *$\pi(\alpha_1) = \beta_1$ implies that $u = v$ and $\pi(\alpha_i) = \beta_i$, $1 \leqq i \leqq u$;*

(3) *$\pi(\mathbf{C}) \cap \mathbf{D} \neq \emptyset$ implies that $u = v$ and $\pi(\alpha_i) = \beta_i$, $1 \leqq i \leqq u$.*

Proof. (1) The result is trivial if $v = 1$. Now suppose that $v > 1$ and that $\pi(\alpha_1) = \beta_j$, where $1 < j \leqq v$. By Condition (1) in the definition of a chain,

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either (a) there exists a $\gamma \neq \alpha_1$ such that $\gamma^*(\mathcal{T}) - \gamma = \alpha_1^*(\mathcal{T}) - \alpha_1$ or (b) $\alpha_1^*(\mathcal{T}) - \alpha_1 \neq \gamma^*(\mathcal{T})$ for any $\gamma \in N$. If (a) is the case, then let $\pi(\gamma) = \mu$. Then it is obvious that

$$\mu^*(\mathcal{U}) - \mu = \beta_j^*(\mathcal{U}) - \beta_j = \beta_{j-1}^*(\mathcal{U})$$

so that, as a consequence of condition (4) in the chain definition, $\mu = \beta_j$. This cannot be true, since $\alpha_1 \neq \gamma$ implies that $\mu = \pi(\gamma) \neq \pi(\alpha_1) = \beta_j$. If (b) is the case, then $\beta_j^*(\mathcal{U}) - \beta_j \neq \mu^*(\mathcal{U})$ for any $\mu \in N$. This contradicts the fact that $\beta_j^*(\mathcal{U}) - \beta_j = \beta_{j-1}^*(\mathcal{U})$. Therefore j cannot be greater than one and so $\pi(\alpha_1) = \beta_1$.

(2) The result is trivial if $u = v = 1$. Now suppose that $1 < v \leqq u$. Suppose further it has already been demonstrated that $\pi(\alpha_j) = \beta_j$ if $j \leqq i$, for some $i < v$. Let $\pi(\alpha_{i+1}) = \gamma$. Then since $\alpha_{i+1}^*(\mathcal{T}) - \alpha_{i+1} = \alpha_i^*(\mathcal{T})$, it follows that $\gamma^*(\mathcal{U}) - \gamma = \beta_i^*(\mathcal{U})$ so that, by Condition 4 in the definition of a chain, $\pi(\alpha_{i+1}) = \beta_{i+1}$. Therefore $\pi(\alpha_i) = \beta_i$ for all $i \leqq v$. Now assume that $v < u$. In that case $\alpha_{v+1}^*(\mathcal{T}) - \alpha_{v+1} = \alpha_v^*(\mathcal{T})$ and if $\pi(\alpha_{v+1}) = \gamma$, then $\gamma^*(\mathcal{U}) - \gamma = \beta_v^*(\mathcal{U})$. Condition (2) of the chain definition now asserts the existence of a $\delta \neq \gamma$ such that $\gamma^*(\mathcal{U}) - \gamma = \delta^*(\mathcal{U}) - \delta$. If $\pi^{-1}(\delta) = \lambda$, then

$$\lambda^*(\mathcal{T}) - \lambda = \alpha_{v+1}^*(\mathcal{T}) - \alpha_{v+1} = \alpha_v^*(\mathcal{T}),$$

so that $\lambda = \alpha_{v+1}$. Since $\delta \neq \gamma$ implies that $\lambda = \pi^{-1}(\delta) \neq \pi^{-1}(\gamma) = \alpha_{v+1}$, this is a contradiction and so $v = u$. Thus the required result has been proved under the assumption that $1 < v \leqq u$. The other possible cases, that is $1 = v \leqq u$, $1 < u \leqq v$ and $1 = u \leqq v$ can be treated similarly.

(3) This is an immediate consequence of (1) and (2).

Thus the chains of a topology behave like complete units under homeomorphisms.

2. Given a topology $\mathcal{T}$ on N, a relation $\approx (\mathcal{T})$ on the set of chains of $\mathcal{T}$ may be defined by requiring that if **C** and **D** are chains of $\mathcal{T}$, then $\mathbf{C} \approx (\mathcal{T})\mathbf{D}$ if and only if $^*\mathbf{C} = {}^*\mathbf{D}$. Clearly, $\approx (\mathcal{T})$ is an equivalence relation and the set of chains of $\mathcal{T}$ is partitioned into pairwise disjoint equivalence classes by $\approx (\mathcal{T})$.

The concept of a *cell* will now be introduced. A collection $\mathcal{C} = \{\mathbf{C}_1, \ldots, \mathbf{C}_r\}$ of r distinct chains of a topology $\mathcal{T}$ is a *r-chain cell*, or more simply a *cell*, of $\mathcal{T}$ provided $\mathcal{C}$ is an equivalence class of the equivalence relation $\approx (\mathcal{T})$. The *supporting open set* of $\mathcal{C}$, denoted by $^*\mathcal{C}(\mathcal{T})$, or more simply by $^*\mathcal{C}$ when there is no risk of confusion, is the (uniquely defined) supporting open set of any chain $C_i \in \mathcal{C}$.

Example. The cells of the previous example space $\mathcal{V}$ are: $\{\mathbf{V}_1, \mathbf{V}_2, \mathbf{V}_3\}$ which is a three chain cell, $\{\mathbf{W}_1, \mathbf{W}_2\}$ which is a two chain cell, and the single chain cell $\{\mathbf{X}\}$. The supporting open sets of these cells are, respectively, $\emptyset$, $\{1, 2, 3\}$, and $\{1, 4, 5\}$.

If $\mathscr{C}$ is a cell, then the symbol $\mathscr{C}$ will be used to denote both the collection of its constituent chains: $\{\mathbf{C}_1, \ldots, \mathbf{C}_r\}$, as well as the subset $\mathbf{C}_1 \cup \ldots \cup \mathbf{C}_r$. The meaning of $\mathscr{C}$ will be clear from the context in which it will be used.

LEMMA 9. *Let $\mathscr{T}$ be a topology on N.*

(1) *The collection of cells of $\mathscr{T}$ constitute a partition of N.*

(2) *Let $\mathscr{C}$ and $\mathscr{D}$ be two distinct cells of $\mathscr{T}$. Then*

(a) $*\mathscr{C} \neq *\mathscr{D}$.

(b) *If $*\mathscr{C}$ contains elements of $\mathscr{D}$, then $*\mathscr{D}$ does not contain any element of $\mathscr{C}$.*

(c) *If $\alpha \in \mathscr{C}$ and $\alpha^* \cap \mathscr{D} \neq \emptyset$ then $\beta \in \mathscr{D}$ implies $\beta^* \cap \mathscr{C} = \emptyset$.*

(3) *For any cell $\mathscr{C}$ of $\mathscr{T}$:*

(a) $*\mathscr{C} \cap \mathscr{C} = \emptyset$ *and $*\mathscr{C} \cup \mathscr{C}$ is an open set of $\mathscr{T}$.*

(b) $*\mathscr{C} = \mathscr{C}^* - \mathscr{C}$.

(c) *If $O \in \mathscr{T}$ and $O \cap \mathscr{C} = A \neq \emptyset$, then $*\mathscr{C} \subset A^* \subseteq O$.*

These results follow almost directly from the demonstrated properties of chains. The proofs are therefore omitted.

LEMMA 10. *For any topology $\mathscr{T}$, there exists a cell $\mathscr{F}$, called the first cell of $\mathscr{T}$, with the property that $*\mathscr{F} = \emptyset$. This first cell is uniquely defined in the sense that if $\mathscr{C}$ is a cell of $\mathscr{T}$ and $*\mathscr{C} = \emptyset$, then $\mathscr{C} = \mathscr{F}$.*

Proof. An immediate consequence of Lemma 1-(b) and Lemma 2 is that there exists a chain $\mathbf{C}$ of $\mathscr{T}$ with $*\mathbf{C} = \emptyset$. $\mathscr{F}$ is the cell having $\mathbf{C}$ as one of its constituent chains. The uniqueness of $\mathscr{F}$ is obvious.

Like chains, the cells of a topology also behave like complete units under homeomorphisms.

THEOREM 3. *Let $\mathscr{T}$ and $\mathscr{U}$ be topologies on N. Then $\mathscr{T}$ and $\mathscr{U}$ are homeomorphic if and only if there exists a single valued map π of N onto itself which satisfies either (and thus both) of the following conditions:*

(1) *If $\mathbf{C} = [\alpha_1, \ldots, \alpha_i, \ldots, \alpha_m]$ is a chain of $\mathscr{T}$, then there exists one, and only one, chain $\mathbf{D}$ of $\mathscr{U}$ such that* (a) $\pi(*\mathbf{C}(\mathscr{T})) = *\mathbf{D}(\mathscr{U})$, (b) $L(\mathbf{C}) = L(\mathbf{D})$ *and* (c) *if $\mathbf{D} = [\beta_1, \ldots, \beta_i, \ldots, \beta_m]$, then $\pi(\alpha_i) = \beta_i$, $1 \leqq i \leqq m$.*

(2) *Let $\mathscr{C}$ be a r chain cell and suppose that $\mathscr{C}$ has p_i chains of length m_i, $1 \leqq i \leqq u$, so that $p_1 + \ldots + p_u = r$. Then there exists one, and only one, r chain cell $\mathscr{D}$ of $\mathscr{U}$ such that* (a) $\pi(*\mathscr{C}(\mathscr{T})) = *\mathscr{D}(\mathscr{U})$, (b) *$\mathscr{D}$ has p_i chains of length m_i, $1 \leqq i \leqq u$ and* (c) if $\mathbf{C} = [\alpha_1, \ldots, \alpha_i, \ldots, \alpha_w] \in \mathscr{C}$, *then there is a chain of equal length $\mathbf{D} = [\beta_1, \ldots, \beta_i, \ldots, \beta_w] \in \mathscr{D}$ such that $\pi(\alpha_i) = \beta_i$, $1 \leqq i \leqq w$.*

COROLLARY. *Suppose that $\mathscr{T}$ and $\mathscr{U}$ are homeomorphic spaces. Then*

(1) *$\mathscr{T}$ and $\mathscr{U}$ possess an equal number of chains and if $\mathscr{T}$ has exactly p chains of length m, then $\mathscr{U}$ also has exactly p chains of length m.*

(2) *$\mathscr{T}$ and $\mathscr{U}$ possess an equal number of cells. In fact, if $\mathscr{T}$ has exactly q cells each of which contains precisely p chains of length m, then $\mathscr{U}$ also has exactly q cells each of which contains precisely p chains of length m.*

(3) *If π is any homeomorphism of $\mathscr{T}$ onto $\mathscr{U}$, $\mathscr{F}$ is the first cell of $\mathscr{T}$ and $\mathscr{G}$ is the first cell of $\mathscr{U}$, then $\pi(\mathscr{F}) = \mathscr{G}$.*

The proofs of the results in Theorem 3 are straightforward and are easily obtained by using Lemma 8 and Theorem 2.

This completes the preliminary survey of the properties of chains and cells. The machinery that has been developed here will be used, in two forthcoming papers (1) to study the structure of finite T_5 spaces (that is, spaces in which every pair of separated sets can be separated by disjoint open sets) and (2) to set up a computer Algorithm for the enumeration of finite, labelled topologies.

The rest of this paper describes an elementary application of the chain concept by studying the automorphisms of an A_2 space. For this purpose a partition of the collection of chains of a topology, other than that into cells, is needed.

3. The set of chains of a topology $\mathscr{T}$ can be partitioned uniquely into the subcollections $E_1(\mathscr{T}), \ldots, E_i(\mathscr{T}), \ldots, E_m(\mathscr{T})$, this partition being defined as follows. A chain $\mathbf{C} \in E_1(\mathscr{T})$ if and only if $^*\mathbf{C} = \emptyset$, and the $E_i(\mathscr{T})$, for $i > 1$, are determined recursively by requiring that a chain $\mathbf{C} \in E_i(\mathscr{T})$ if and only if $^*\mathbf{C} \subseteq D_{i-1}(\mathscr{T})$, and terminating the process at that value m of i for which $D_m = N$. m is the *chain level* of the topology $\mathscr{T}$. Here, and for the rest of this paper, $D_0 = \emptyset$ and $D_i = D_i(\mathscr{T})$, for $i \geqq 1$, denotes the set

$$(\bigcup E_1(\mathscr{T})) \cup \ldots \cup (\bigcup E_i(\mathscr{T})).$$

It is obvious that $D_i(\mathscr{T})$ is an open set of $\mathscr{T}$. A chain $\mathbf{C}$ is defined to be a chain of *level* i if $\mathbf{C} \in E_i(\mathscr{T})$. Clearly, if $\mathbf{C}$ is an i-level chain, then $^*\mathbf{C}$ does not intersect any i-level chain but has a non-void intersection with at least one chain of level j, for all $j < i$.

Before proceeding to study the properties of A_p spaces, some observations regarding their existence are in order. If $\mathscr{T}$ is an A_p space on N and $|N| = n$, then clearly $p \leqq n$. It is obvious that $p = n$ only when $\mathscr{T}$ is the discrete topology. In fact, there exist A_p spaces for all p, in the range $1 \leqq p \leqq n$, on N. To see this, let N be the set of the first n integers. The simplest space is the A_1 space, defined by the collection of open sets: $\{\emptyset, \{1\}, \{1, 2\}, \ldots, \{1, 2, \ldots, n\}\}$. In other words, any A_1 space consists of a single chain. To demonstrate the existence of the other A_p topologies in between the two extremes of A_1 and A_n spaces, consider the space $\mathscr{T}$ whose open sets are: $\emptyset$, all non-void subsets of the first p integers and all subsets of the form: $\{1, 2, \ldots, p\} \cup \{p + 1, \ldots, i\}$ for all i in the range $p + 1 \leqq i \leqq n$. Then $\emptyset$ is an A_p set, if $O \subset \{1, \ldots, p\}$ then O is an A_q set where $q = p - |O|$, and if $N \neq O \supseteq \{1, \ldots, p\}$ then O, if open, is an A_1 set. Therefore $\mathscr{T}$ is an A_p space. Some of the properties of A_p spaces that are described below can be demonstrated more simply without the use of the chain concept. However, the procedure that is used has the advantage of clearly displaying the inherent chain structure of finite spaces.

LEMMA 11. *If* $\mathscr{T}$ *is a* A_p *space, then an* $E_i(\mathscr{T})$ *can contain at most* p *chains.*

Proof. Suppose that $E_i(\mathscr{T})$ is the collection of chains $\mathbf{C}_1, \ldots, \mathbf{C}_q$ where $q > p$. Let $x_1, \ldots, x_q$ be, respectively, the first elements of these chains. Let $Q_\alpha = D_{i-1} + x_\alpha$, $1 \leqq \alpha \leqq q$. Then D_{i-1} and the Q_α are all open sets. Moreover Q_α, $1 \leqq \alpha \leqq q$, is a $(*)$ cover of D_{i-1}. Therefore D_{i-1} is an A_r set of $\mathscr{T}$, for some $r \geqq q$. Since $q > p$, therefore $\mathscr{T}$ is not an A_p space. This contradicts the hypothesis of the Lemma.

LEMMA 12. *Let* $\pi : N \rightarrow N$ *be an automorphism of a topology* $\mathscr{T}$ *and let* $\mathbf{A}$ *be a chain of* $\mathscr{T}$. *Then both the chains* $\mathbf{A}$ *and* $\mathbf{B} = \pi(\mathbf{A})$ *have the same level.*

Proof. An inductive proof is presented. If $\mathbf{A} \in E_1(\mathscr{T})$, then $^*\mathbf{A} = \emptyset$ and so $^*\mathbf{B} = \pi(^*\mathbf{A}) = \emptyset$ which implies that $\mathbf{B} \in E_1(\mathscr{T})$. Now assume, as the hypothesis of induction, that the Lemma is true for all chains with level less than or equal to i and suppose that $\mathbf{A} \in E_{i+1}(\mathscr{T})$. Then, as a consequence of the induction hypothesis, $\pi(D_j) = D_j$ for all $j \leqq i$. Therefore, since $^*\mathbf{A} \subseteq D_i$ and $^*\mathbf{A} \not\subseteq D_{i-1}$, it follows that $^*\mathbf{B} = \pi(^*\mathbf{A}) \subseteq D_i$ and $^*\mathbf{B} \not\subseteq D_{i-1}$. Therefore $\mathbf{B} \in E_{i+1}(\mathscr{T})$.

THEOREM 4. *Let* $\mathscr{T}$ *be an* A_p *space and* $\pi : N \rightarrow N$ *an automorphism of* $\mathscr{T}$. *Then the length of any cycle of* π, *in the disjoint cycle representation of permutations, cannot exceed* p.

Proof. Suppose that $(x_1, \ldots, x_q)$ is a cycle of π of length $q > p$. Then it is a direct consequence of Theorem 2 that there exist chains $\mathbf{C}_1, \ldots, \mathbf{C}_q$ such that $x_1 \in \mathbf{C}_1, \ldots, x_q \in \mathbf{C}_q$, $\pi(\mathbf{C}_i) = \mathbf{C}_{i+1}$ for $1 \leqq i \leqq q - 1$ and $\pi(\mathbf{C}_q) = \mathbf{C}_1$. Lemma 12 now asserts that the chains $\mathbf{C}_1, \ldots, \mathbf{C}_q$ have the same level. This contradicts Lemma 11, and thus the proof is complete.

For the rest of this paper, the discussion will be confined to A_2 spaces. The next result is a trivial corollary to Lemma 11.

LEMMA 13. *In an* A_2 *space, there can exist at most two chains with the same level.*

LEMMA 14. *Let* $\mathscr{T}$ *be an* A_2 *space and let* $\mathbf{A}$, $\mathbf{B}$, *and* $\mathbf{C}$ *be chains of* $\mathscr{T}$. *Suppose that* $\mathbf{A}$ *and* $\mathbf{B}$ *are chains of level* i *and* $\mathbf{C}$ *is of level* $i + 1$. *Then* $^*\mathbf{C}$ *contains at least one of the chains* $\mathbf{A}$ *or* $\mathbf{B}$.

Proof. Without loss of generality, it can be assumed that $^*\mathbf{C} \cap \mathbf{A} \neq \emptyset$. If $\mathbf{A}$ is a single element chain, then the Lemma is trivially true. Now suppose that $\mathbf{A} = [a_1, \ldots, a_u]$ and that $^*\mathbf{C}$ contains the first j elements of $\mathbf{A}$ for some $j < u$. Then it is sufficient to prove that the assumption $\mathbf{B} \not\subseteq {}^*\mathbf{C}$ implies the existence of an A_p set for some $p \geqq 3$. Let $\mathbf{B} = [b_1, \ldots, b_v]$ and $\mathbf{C} = [c_1, \ldots]$. Two situations are now possible. In the first case $^*\mathbf{C} \cap \mathbf{B} = \emptyset$. In this case, let $O = {}^*\mathbf{C} \cup D_{i-1}$, $Q_1 = O + a_{j+1}$, $Q_2 = O + b_1$ and $Q_3 = O + c_1$. In the second case $^*\mathbf{C}$ contains the first k elements of B, for some $k < v$. In this case let $O = {}^*\mathbf{C}$, $Q_1 = O + a_{j+1}$, $Q_2 = O + b_{k+1}$ and $Q_3 = O + c_1$. Then it can be

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easily shown that, in either case, O, Q_1, Q_2 and Q_3 are all open, $O \subset Q_i$ and $|Q_i - O| = 1$, $1 \leqq i \leqq 3$. Therefore O is an A_p set for some $p \geqq 3$.

LEMMA 15. *Let $\mathscr{T}$ be an A_2 space and let* **A**, **B** *and* **C** *be chains of $\mathscr{T}$. If* **A** *is an i-level chain, and* **B**, **C** *are of level $i + 1$ then* **A** *is contained by at least one of* ***B** *or* ***C**.

Proof. Let $\mathbf{A} = [a_1, \ldots]$, $\mathbf{B} = [b_1, \ldots]$, $\mathbf{C} = [c_1, \ldots]$ and suppose that the Lemma is not true. Then two situations are possible. First, neither $^*\mathbf{B}$ nor $^*\mathbf{C}$ contains any element of **A**. In this case, let $O = {}^*\mathbf{B} \cup {}^*\mathbf{C} \cup D_{i-1}$, $Q_1 = O + a_1$, $Q_2 = O + b_1$ and $Q_3 = O + c_1$. In the second case $^*\mathbf{B} \cup {}^*\mathbf{C}$ contains the first j elements of **A**, j being less than the length of **A**. This means that one of $^*\mathbf{B}$ or $^*\mathbf{C}$ contains the first j elements of **A**. Then let $O = {}^*\mathbf{B} \cup {}^*\mathbf{C}$, $Q_1 = O + a_{j+1}$, $Q_2 = O + b_1$ and $Q_3 = O + c_1$. Thus, in either case, O, Q_1, Q_2 and Q_3 are open sets. Moreover Q_1, Q_2 and Q_3 are $(*)$ covers of O. Therefore $\mathscr{T}$ is not an A_2 space, which contradicts the hypothesis of the Lemma.

LEMMA 16. *If* **A** *and* **B** *are chains, both of the same level i, of an A_2 space $\mathscr{T}$, then*
(1) $^*\mathbf{A} = {}^*\mathbf{B}$ *implies that* $^*\mathbf{A} = {}^*\mathbf{B} = D_{i-1}(\mathscr{T})$.
(2) $^*\mathbf{A} \cup {}^*\mathbf{B} = D_{i-1}(\mathscr{T})$.

Proof. (1) Suppose $\mathbf{A} = [a_1, \ldots]$, $\mathbf{B} = [b_1, \ldots]$, $^*\mathbf{A} = {}^*\mathbf{B} = O$ and that the open set O is a proper subset of D_{i-1}. Since $D_{i-1} \in \mathscr{T}$, Lemma 1-(c) ensures the existence of a $Q_1 \in \mathscr{T}$ such that $Q_1 \subseteq D_{i-1}$, $O \subset Q_1$ and $|Q_1 - O| = 1$. Let $Q_2 = O + a_1$ and $Q_3 = O + b_1$. Then the open sets Q_1, Q_2 and Q_3 are $(*)$ covers of O, so that O is an A_p set for some $p \geqq 3$. This contradicts the assumed A_2 nature of $\mathscr{T}$.
(2) Let $O = {}^*\mathbf{A} \cup {}^*\mathbf{B}$ in the proof of (1).

For any A_2 space $\mathscr{T}$, every $E_i(\mathscr{T})$ satisfies one of the following conditions:
(+1) $E_i(\mathscr{T})$ contains two chains **A**, **B** and
(+1a) $^*\mathbf{A} \neq D_{i-1}(\mathscr{T}) \neq {}^*\mathbf{B}$, or
(+1b) one, and only one, of the chains **A**, **B** has $D_{i-1}(\mathscr{T})$ as its supporting open set, or
(+1c) $^*\mathbf{A} = {}^*\mathbf{B} = D_{i-1}(\mathscr{T})$.
(+2) $E_i(\mathscr{T})$ contains only one chain **A** and
(+2a) $^*\mathbf{A} \neq D_{i-1}(\mathscr{T})$, or
(+2b) $^*\mathbf{A} = D_{i-1}(\mathscr{T})$.

Let $\mathscr{T}$ be an A_2 space in which $E_1(\mathscr{T})$ contains two chains. Then $\mathscr{T}$ is a *type 1* space provided either the chain level of $\mathscr{T}$ is 1, or every $E_i(\mathscr{T})$, for $i > 1$, satisfies condition (+1a). $\mathscr{T}$ is a *type 2* space provided the chain level of $\mathscr{T}$ is at least 2 and every $E_i(\mathscr{T})$, for $i > 1$, satisfies one of the conditions (+1a), (+1b) or (+2a) and there exists at least one i, $i > 1$, for which $E_i(\mathscr{T})$ does not satisfy (+1a).

LEMMA 17. *Let $\mathscr{T}$ be a type 1 A_2 space. Then there can exist at most one non-*

identity automorphism of $\mathscr{T}$. In the event that it exists, this automorphism is defined by the requirement that it interchanges the two chains of every $E_i(\mathscr{T})$.

Proof. The proof is by induction over the chain level. If $\mathscr{T}$ is a type 1 A_2 space with chain level 1, then it contains only two chains and the result in question is a trivial consequence of Theorem 2. Now assume, as the hypothesis of induction, that the Lemma is true for all topologies with chain level m. Now let $\mathscr{T}$ be a type 1 topology with chain level $m + 1$, π an automorphism of $\mathscr{T}$ and π_m the restriction of π to the points of $D_m(\mathscr{T})$. Clearly, the topology induced by $\mathscr{T}$ on $D_m(\mathscr{T})$ is a type 1 topology. Therefore, by the induction hypothesis, π_m is either the identity mapping or it interchanges the two chains of every $E_i(\mathscr{T})$ for $i \leqq m$. Hence, supposing $\mathbf{A}$ and $\mathbf{B}$ to be the two $m + 1$ level chains of $\mathscr{T}$, it will be sufficient to show that if $(1)\pi_m$ is the identity mapping, then $\pi(\mathbf{A}) = \mathbf{A}$ and $\pi(\mathbf{B}) = \mathbf{B}$ and if $(2)\pi_m$ is not the identity mapping, then $\pi(\mathbf{A}) = \mathbf{B}$, and $\pi(\mathbf{B}) = \mathbf{A}$. Lemma 16-(1) and the definition of a type 1 space ensure that $^*\mathbf{A} \neq {}^*\mathbf{B}$. Now in case (1) since $^*\mathbf{A}, {}^*\mathbf{B} \subseteq D_m$, therefore

$$\pi(^*\mathbf{A}) = \pi_m(^*\mathbf{A}) = {}^*\mathbf{A} \neq {}^*\mathbf{B} = \pi_m(^*\mathbf{B}) = \pi(^*\mathbf{B})$$

and so, as a consequence of Theorem 3-(1), $\pi(\mathbf{A}) = \mathbf{A}$ and $\pi(\mathbf{B}) = \mathbf{B}$. Now suppose that $\mathbf{C}$ and $\mathbf{F}$ are the two m level chains of $\mathscr{T}$. Hence it can be assumed, as a consequence of Lemmas 14, 15, 16-(2), and without loss of generality, that $\mathbf{C} \subseteq {}^*\mathbf{A}$, $\mathbf{F} \not\subseteq {}^*\mathbf{A}$, $\mathbf{F} \subseteq {}^*\mathbf{B}$, and $\mathbf{C} \not\subseteq {}^*\mathbf{B}$. Then in case (2), as a consequence of the induction hypothesis, $\pi(\mathbf{C}) = \pi_m(\mathbf{C}) = \mathbf{F}$, $\pi(\mathbf{F}) = \mathbf{C}$ and so $\mathbf{C} = \pi(\mathbf{F}) \not\subseteq \pi(^*\mathbf{A})$. Therefore $\pi(^*\mathbf{A}) \neq {}^*\mathbf{A}$, and similarly $\pi(^*\mathbf{B}) \neq {}^*\mathbf{B}$ and so $\pi(\mathbf{A}) = \mathbf{B}$ and $\pi(\mathbf{B}) = \mathbf{A}$. This completes the proof.

LEMMA 18. *Let $\mathscr{T}$ be a type 2 space. Then the identity map is the only automorphism of $\mathscr{T}$.*

Proof. The proof is by induction over the chain level. However, to avoid needless repetition, the demonstration is omitted for the case when the chain level is 2, as this becomes evident from the general inductive proof. Therefore assume, as the hypothesis of induction, that the Lemma is true for all type 2 spaces with chain level m, for some $m \geqq 2$. Now suppose that $\mathscr{T}$ is a type 2 space with chain level $m + 1$, $\mathscr{T}_m$ is the topology induced by $\mathscr{T}$ on $D_m(\mathscr{T})$, π is an automorphism of $\mathscr{T}$ and π_m the restriction of π to the points of $D_m(\mathscr{T})$. First, it will be shown that π_m is an identity mapping. For, suppose the contrary. Then the induction hypothesis implies that $\mathscr{T}_m$ is not a type 2 space. The definitions of type 1 and type 2 spaces now ensure the $\mathscr{T}_m$ is a type 1 space with two m level chains and either (a) there exists only one $m + 1$ level chain $\mathbf{A}$ of $\mathscr{T}$ and $^*\mathbf{A} \neq D_m(\mathscr{T})$ or (b) there exist two $m + 1$ level chains $\mathbf{A}$, $\mathbf{B}$ of $\mathscr{T}$ and $^*\mathbf{A} \neq D_m(\mathscr{T})$, $^*\mathbf{B} = D_m(\mathscr{T})$. Now, following an exactly similar reasoning as the one used in the proof of case (2) in Lemma 18, it becomes evident that $\pi(^*\mathbf{A}) \neq {}^*\mathbf{A}$. However, this leads to a contradiction. For, in case (a) since $\mathbf{A}$ is the only $m + 1$ level chain, therefore $\pi(\mathbf{A}) = \mathbf{A}$ so that $\pi(^*\mathbf{A}) = {}^*\mathbf{A}$, and in case (b) since it is clearly impossible that $\pi(^*\mathbf{A}) = D_m(\mathscr{T})$, therefore $\pi(\mathbf{A}) = \mathbf{A}$ which again implies that $\pi({}^*\mathbf{A}) = {}^*\mathbf{A}$. Hence the demonstration now proceeds with the assumption that π_m is an identity mapping, irrespective of whether $\mathscr{T}_m$ is type 1 or 2. Therefore, if $\mathscr{T}$ has only one $m + 1$ level chain $\mathbf{A}$, then clearly $\pi(\mathbf{A}) = \mathbf{A}$ and so π is the identity mapping. In case $\mathscr{T}$ has two $m + 1$ level chains $\mathbf{A}$ and $\mathbf{B}$, then following a reasoning similar to the one used in the proof of case (1) in Lemma 18, it becomes evident that $\pi(\mathbf{A}) = \mathbf{A}$, $\pi(\mathbf{B}) = \mathbf{B}$ and so again π is the identity mapping.

THEOREM 5. *Let $\mathscr{T}$ be an A_2 topology. Then the number of automorphisms of $\mathscr{T}$ can be expressed as 2^t, for some $t \geqq 0$.*

Proof. Let α be the number of automorphisms of $\mathscr{T}$. In case $\mathscr{T}$ consists of a single chain, clearly $\alpha = 1$. In case $\mathscr{T}$ consists of two chains $\mathbf{A}$ and $\mathbf{B}$, of unequal length, then the only automorphism is the identity mapping so that $\alpha = 1$. In case $\mathbf{A}$ and $\mathbf{B}$ have the same length, then the automorphisms are the identity and the interchange of $\mathbf{A}$ and $\mathbf{B}$ so that $\alpha = 2$. Now suppose that the chain level of $\mathscr{T}$ is m, for some $m \geqq 2$. Consider the partition $N_1, \ldots, N_r, \ldots, N_s$ of N constructed as follows. If no $E_i(\mathscr{T})$, for $i > 1$, satisfies $(+1c)$ or $(+2b)$ then let $N_1 = N$. Otherwise let $j_1 = 1, j_{s+1} = m + 1$ and suppose that $j_i, \ldots, j_r, \ldots, j_s$ are the values of i, in ascending order of magnitude, for which $E_i(\mathscr{T})$ satisfies either $(+1c)$ or $(+2b)$. Then, for $r = 1$ to s, let

$$N_r = \bigcup_i (\bigcup E_i(\mathscr{T})), \quad j_r \leqq i \leqq j_{r+1} - 1.$$

Then it follows directly from the definitions that the topology $\mathscr{T}_r$ induced on N_r by $\mathscr{T}$ is either type 1, type 2 or is a single chain space. Moreover N_1 contains the isolated points of $\mathscr{T}$ and if $\beta \in N_r$, $r > 1$, then $(N_1 \cup \ldots \cup N_{r-1}) \subseteq \beta^*(\mathscr{T})$. Therefore, as a consequence of Lemmas 17 and 18, t of these subspaces, for some $t \geqq 0$, will have exactly 2 automorphisms and the rest will have a single automorphism. The result now becomes evident on observing that the restriction of an automorphism of $\mathscr{T}$ to N_r is an automorphism of $\mathscr{T}_r$ and, conversely, if π_r is an automorphism of $\mathscr{T}_r$ for $1 \leqq r \leqq s$, then $\pi_1 \ldots \pi_r \ldots \pi_s$ is an automorphism of $\mathscr{T}$.

A detailed study of A_2 and A_3 spaces, including the solution to the enumeration problem, will appear elsewhere.

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