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ARTICLE

Equilibrium dynamics in a model of growth and spatial agglomeration

Shota Fujishima[1] and Daisuke Oyama[2,*]

[1]Graduate School of Economics, Hitotsubashi University, 2-1 Naka, Kunitachi, Tokyo 186-8601, Japan and [2]Faculty of Economics, University of Tokyo, Hongo, Bunkyo-ku, Tokyo 113-0033, Japan
*Corresponding author. Email: oyama@e.u-tokyo.ac.jp

Abstract

We present a multiregional endogenous growth model in which forward-looking agents choose their regions to live in, in addition to consumption and capital accumulation paths. The spatial distribution of economic activity is determined by the interplay between production spillover effects and urban congestion effects. We characterize the global stability of the spatial equilibrium states in terms of economic primitives such as agents' time preference and intra- and interregional spillovers. We also study how macroeconomic variables at the stable equilibrium state behave according to the structure of the spillover network.

Keywords: Growth, agglomeration, continuous-time overlapping generations, perfect foresight dynamics, potential function, spillover network

JEL Codes: C62, C73, D62, O41, R13

1. Introduction

Empirical evidence suggests that the growth process of one economy is not independent of those of other economies (Ertur and Koch (2007)). Spatial interdependence matters for economic growth through, at least, two channels: first, there are technological externalities across space given the spatial distribution of economic agents, where the degree of externalities varies according to the physical, economic, and social interconnections among regions; second, the agents themselves, at least in the medium/long run, move across regions. The location choice of the agents is determined by the trade-off between agglomeration benefits and congestion costs, which in turn are affected by the future growth in each location, to the extent that location decisions are irreversible, investment decisions. In understanding the relationship between growth and agglomeration, therefore, it is important to fully incorporate intertemporal optimization by forward-looking agents, with respect to location decisions as well as saving/capital accumulation decisions.

In this paper, we develop a tractable multiregional endogenous growth model with overlapping generations à la Yaari (1965) and Blanchard (1985), in which we study the equilibrium spatial dynamics in the spirit of Matsuyama (1991). The world consists of n regions. Production technology is that of the AK-type, where spatial interdependence is expressed by the intra- and interregional spillover effects on the factor productivity in each region. We incorporate agents' mobility with complete irreversibility in location choice as in Matsuyama's (1991) model of sectoral choice: each agent chooses which region to locate in only upon birth, and once the choice is made, he stays in that region throughout his life. The value of each region is determined by the expected lifetime utility of the optimal consumption-saving path that realizes in the region, where we assume congestion effects in a form of consumption costs (e.g., intraregional transport costs) that are increasing in the regional population. Thus, agents' location decisions are based on the positive externalities from spillovers and the negative externalities from congestion, given their expectations for the future evolution of the aggregate population distribution over the regions. A *spatial equilibrium path*, or equilibrium path in short, is a path of the population distribution along which every agent chooses a location that maximizes his expected lifetime utility against the expectation of the path itself.

In our equilibrium dynamics, there generally exist multiple stationary equilibrium states, in particular when the positive externalities are sufficiently strong. We are thus interested in the stability of equilibrium states, which offers a criterion for equilibrium selection among the multiple equilibrium states. Moreover, for a given initial population state, there may exist multiple equilibrium paths, approaching different equilibrium states. Hence, local analysis is not sufficient and global stability analysis is necessary, as emphasized by, for example, Matsuyama (1991) among others. Formally, we would like to characterize an equilibrium state that is *absorbing* and *globally accessible*, a state $\bar{x}$ such that (i) any equilibrium path starting in a neighborhood of $\bar{x}$ converges to $\bar{x}$ and (ii) for any initial state, there exists an equilibrium path that converges to $\bar{x}$.

That said, global stability analysis is generally a difficult task for nonlinear equilibrium dynamics such as ours, especially when the state variable is of high dimension (i.e., when there are more than two regions in our model). To maintain our many-region setting, we focus on environments in which a *potential function* (Monderer and Shapley (1996), Sandholm (2001)) exists. A potential function, which is defined on (a neighborhood of) the set of population distributions (i.e., the unit simplex of $\mathbb{R}^n$), is a function such that the change in any agent's stationary utility from relocation is always equal to the marginal change in the value of this function. A necessary and sufficient condition for the existence of a potential function here is that the matrix of coefficients of technological spillovers, or *spillover matrix*, satisfies some form of symmetry that we call triangular integrability, which is satisfied in particular when the spillover matrix is symmetric. Our main technical result shows that, under certain regularity conditions, the equilibrium state that uniquely maximizes the potential function is absorbing and globally accessible when the discount rate is sufficiently close to zero.

Given the above result, our task is to inspect the shape of the potential function, which embodies the agents' incentives in location decisions. In particular, we consider sufficient conditions under which the potential function becomes convex or concave on the state space. First, if intraregional spillover effects net of congestion effects are sufficiently large relative to interregional spillover effects, then the potential function is strictly convex when the discount rate is sufficiently close to zero. In this case, the global maximizer of the potential function is attained at a vertex of the state space, which implies that all agents agglomerate in one region at the stable equilibrium state. If people are farsighted, they attach greater importance to agglomeration economies, which affect the future gains through accelerating economic growth, than to congestion, which affects the current consumption. This results in the full agglomeration that achieves the largest growth rate. Second, the potential function is strictly concave when the discount rate is sufficiently large. In this case, there is a unique equilibrium state, at which the potential function is maximized, and every equilibrium path converges to this state. The unique equilibrium state lies in the interior of the state space, which means that agents are dispersed over multiple regions at the stable equilibrium state. When agents are nearly myopic, they care about current congestion more than agglomeration economies which lead to future growth, resulting in a dispersed population distribution.

Our model with multiple regions enables us to study network effects through technological spillovers across the regions. We thus examine how the outcomes at the stable equilibrium state are shaped by the structure of the spillover network. First, we consider a pair of networks, each of which comprises two clusters of regions, but which differ in the strength of connections within each cluster and across the clusters. We demonstrate how the welfare-maximizing network

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among the two is determined in a subtle way by the congestion effects. Specifically, we find that connections within each cluster are more significant than those across the clusters for the equilibrium utility level when the congestion effects are weak, and vice versa when they are strong. Second, we look at the long-run spatial distribution of economic variables such as capital and income, besides the population distribution. In particular, we discuss a spatial inequality issue called *σ-convergence*, which means that the income difference among regions diminishes over time (Barro and Sala-i-Martin (1992)). In our model, the *σ-divergence*, which is the opposite of the σ-convergence, can also occur at the stable equilibrium state. We present an example in which at the stable equilibrium state, there coexist regions that will ride on balanced growth paths where income perpetually grows and regions that will stop growing. Finally, we show that, under certain conditions, the stable equilibrium state is represented as the vector of each region's *Katz-Bonacich centrality*, a centrality concept in network theory (Zenou (2016)).[1] This is an analogue of the result of Ballester et al. (2006) for a finite-player game with linear best responses.

Our spatial growth model is characterized by the three important ingredients: spatial externalities, agents' mobility, and agents' forward-looking expectations. Most closely related to our model is that by Eaton and Eckstein (1997). They build a multiregional endogenous growth model with continuous-time OLG of Blanchard-Yaari type, irreversible location choice, and intra- and interregional externalities, where growth is driven by human capital acquisition à la Lucas (1988). Rather than analyzing the dynamic properties of their model, they focus on stationary states in the limit case where the discount rate is equal to the negative of the birth–death rate and compute the coefficients of spatial externalities for which the observed population distribution across French cities is supported as a stationary state.[2] Brock et al. (2014) consider a dynamic model of competitive firms within an industry with finitely many locations where, in each region, firms accumulate capital subject to adjustment costs and intra- and interregional externalities. They study conditions on the externality coefficients and technology parameters, among others, under which a stationary equilibrium that has a spatially uniform capital distribution is locally unstable.

Whereas our paper and those by Eaton and Eckstein (1997) and Brock et al. (2014) study models with a finite discrete spatial domain, there is parallel literature on economic growth in continuous space.[3] Boucekkine et al. (2009) consider a continuous-space growth model with neoclassical production technology and capital mobility, which they call the spatial Ramsey model. They consider the planner's optimal control problem subject to a partial differential equation that describes the law of motion of capital in equilibrium. However, it entails technical difficulties known as ill-posedness.[4] Moreover, this class of models do not consider spatial spillovers, which are an important focus here. Desmet and Rossi-Hansberg (2014) take a strategy to abstract away from forward-looking expectations while keeping the rich structure of the model to allow calibration/estimation of the model to match the data. They impose assumptions under which all the decisions by consumers and firms are in fact static decisions; in particular, each agent is able to migrate at any (discrete) period and each firm's investment in innovation does not affect its future productivity. Aiming at providing theoretical results, we instead pursue a full dynamic model with forward-looking agents for spatial growth, by imposing simplifying assumptions such as irreversible location decisions.

In methodological aspects, our study contributes to the literature on perfect foresight dynamics in population games (Matsui and Matsuyama (1995), Hofbauer and Sorger (1999), Oyama et al. (2008), among others). In the previous studies in this literature, a static game is repeatedly played over time, so that the stationary equilibrium states are solely determined by the static game, independent of the discount rate.[5] In contrast, our model involves a stock variable, that is, capital stock, through which the future growth benefits affect the agents' location decisions, relative to the congestion costs. The relative importance between the benefits and the costs is governed by the discount rate, and thus, the equilibrium states as well as the potential function depend on the discount rate. Accordingly, in our stability analysis, which follows the turnpike-theoretic approach by Hofbauer and Sorger (1999), this feature requires us an extra care in studying the trajectory of the critical points of the potential function as the discount rate varies. We resolve this issue by introducing a certain regularity condition on the parameters, analogous to the conditions of regular Nash equilibrium (van Damme (1983)) and regular evolutionarily stable strategy (Taylor and Jonker (1978)), that allows an argument based on the Implicit Function Theorem which guarantees the existence of a neighborhood that isolates the potential maximizer from other critical points uniformly over all sufficiently small values of the discount rate; refer to the discussion below Theorem 3.1 in Section 3.3 for details.

The rest of this paper proceeds as follows. In Section 2, we state the basic structure of the model and define our spatial equilibrium dynamics. In Section 3, we study the stability of stationary equilibrium states. In Section 4, we study the relationship between the stable equilibrium states and the structure of technological spillover networks. Section 5 is a conclusion and discusses subjects for future research. Proofs omitted from the main text are provided in the Appendix.

2. Model

2.1 A continuous-time OLG model with multiple regions

We consider a continuous-time overlapping generations model with $n \geq 2$ regions. Let $S = \{1, 2, ..., n\}$ be the set of regions. A mass one of the initial agents at time 0 are distributed across the regions according to the exogenously given distribution $x_0 \in \Delta$, where $\Delta = \left\{x \in \mathbb{R}^n_+ : \sum_{i \in S} x_i = 1\right\}$, each of whom is endowed with an amount $k_0 > 0$ of capital. Each agent is replaced by a newborn, also endowed with an initial capital amount k_0, according to a Poisson process with parameter $\lambda > 0$. We normalize the time unit in such a way that $\lambda = 1$. These processes are assumed to be independent, so that during each short time interval $[t, t + \mathrm{d}t)$, a mass $\mathrm{d}t$ of agents are replaced by the same mass of entrants, where the total mass of the population is fixed to one over time. We call the agents born at time τ generation τ.

Upon birth, an individual makes a once-and-for-all location decision, that is, he chooses which region to live in and settles in that region throughout his life. Let $\alpha_i(\tau)$ denote the population share of generation τ in region i. The time-t mass of generation τ locating in region i is $e^{-(t-\tau)}\alpha_i(\tau)$, and therefore, the total population distribution $x(t) = (x_1(t), \ldots, x_n(t)) \in \Delta$ at time t is given by

$$x_i(t) = e^{-t}x_{0i} + \int_0^t e^{-(t-\tau)}\alpha_i(\tau)\mathrm{d}\tau \tag{1}$$

for each region i. We denote the time path of $x(t)$, to be determined in equilibrium, by $x(\,\cdot\,) = (x(t))_{t\geq 0}$.

2.1.1 Production

In each region, firms produce the consumption good with capital as the only input under AK technology, where both the consumption and capital goods are assumed to be non-transferable to other regions.[6] Production is subject to externalities in the spirit of Romer (1986); specifically, it benefits from spillover effects within the region as well as from other regions. Intraregional spillovers will work as agglomeration forces, while interregional spillovers tend to mitigate the former. To simplify the argument, we assume that the capital productivity depends directly on the population distribution $x \in \Delta$ and, in particular, it is increasing in the population of the region where the production takes place. The aggregate production function in region i takes the AK form,

$$Y_i = Z_i(x)K_i, \tag{2}$$

where K_i is the aggregate capital input, and the productivity factor $Z_i(x)$ depends on the populations, and hence the levels of production, of the own region as well as the other regions

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through

$$Z_i(x) = \sum_{j \in S} z_{ij} x_j \tag{3}$$

with $z_{ii} > 0$ and $z_{ij} \geq 0$ for all $j \neq i$. Intra- and interregional spillover effects are captured by z_{ii} and z_{ij}, respectively, and in fact define a network over the regions, where links are weighted according to the *spillover matrix* $Z = (z_{ij})$.

The term $Z_i(x)$ captures the externalities arising from interactions of individuals with knowledge spillovers within and across regions, which are typically decreasing in the physical and/or social distance between regions. Our assumption of its dependence on the population is for tractability: in particular, it allows us to solve for the market equilibrium capital stocks within each region (given a path of population distribution $x(\cdot)$ fixed), as to be done below.[7][8]

2.1.2 Consumption

Each agent of generation τ located in region i decides on the path of consumption $(c_i(\tau, t))_{t \geq \tau}$ to maximize his expected lifetime utility, where we assume that the instantaneous utility is given by $\ln c_i(\tau, t)$. Since the lifespan is exponentially distributed with mean 1, the expected lifetime utility is

$$\int_0^\infty \int_\tau^{\tau+s} e^{-\rho(t-\tau)} \ln c_i(\tau, t) \mathrm{d}t e^{-s} \mathrm{d}s = \int_\tau^\infty e^{-(1+\rho)(t-\tau)} \ln c_i(\tau, t) \mathrm{d}t, \tag{4}$$

where $\rho > 0$ is the common rate of time preference, while $1 + \rho$ is interpreted as the effective discount rate. We will be interested mainly in the case where agents are farsighted, that is, ρ is close to zero. Note that the model is well defined whenever $\rho > -1$.

Agents in region i earn the returns to their capital by renting it to the firms within region i. For consumption, we assume that there are congestion externalities in the form of iceberg-type intraregional transport costs. Specifically, in order to consume one unit of the good, an individual in region i has to purchase $\phi_i(x_i(t))$ units, where $\phi_i(x_i) \geq 1$ is continuously differentiable on $[0, 1]$,[9] and satisfies $\phi_i'(x_i) > 0$ for all $x_i \in [0, 1]$.[10] The intertemporal budget constraint of generation τ is then given by

$$\dot{k}_i(\tau, t) = r_i(t) k_i(\tau, t) - \phi_i(x_i(t)) c_i(\tau, t), \quad k_i(\tau, \tau) = k_0, \tag{5}$$

where $k(\tau, t)$ denotes the capital holding with $\dot{k}_i(\tau, t) = \partial k_i(\tau, t)/\partial t$, $r_i(t)$ denotes the rental rate of capital (to be determined in equilibrium), and the depreciation rate is assumed to be zero.

Given $x(\cdot)$, $r_i(\cdot)$, and k_0, an agent of generation τ in region i maximizes (4) subject to (5). With the (current-value) Hamiltonian

$$H_i(k_i, c_i, \eta_i, t) = \ln c_i + \eta_i \big[r_i(t) k_i - \phi_i(x_i(t)) c_i \big],$$

the necessary conditions for optimality are

$$\frac{\partial H_i}{\partial c_i}(k_i(\tau, t), c_i(\tau, t), \eta_i(\tau, t), t) = \frac{1}{c_i(\tau, t)} - \eta_i(\tau, t) \phi_i(x_i(t)) = 0, \tag{6}$$

$$\begin{aligned} \dot{\eta}_i(\tau, t) &= (1 + \rho) \eta_i(\tau, t) - \frac{\partial H_i}{\partial k_i}(k_i(\tau, t), c_i(\tau, t), \eta_i(\tau, t), t) \\ &= -\big[r_i(t) - (1 + \rho) \big] \eta_i(\tau, t), \end{aligned} \tag{7}$$

$$\lim_{t \to \infty} e^{-(1+\rho)(t-\tau)} \eta_i(\tau, t) k_i(\tau, t) = 0, \tag{8}$$

where η_i is the adjoint variable, and (8) is the transversality condition (see, e.g., Kamihigashi (2001), Section 4.2). By (7), we have

$$\eta_i(\tau, t) = \eta_i(\tau, \tau) e^{-R_i(\tau, t)}, \tag{9}$$

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where

$$R_i(\tau, t) = \int_\tau^t \left[r_i(s) - (1+\rho)\right]\mathrm{d}s. \tag{10}$$

We also have

$$\begin{aligned}\frac{\partial}{\partial t}\big(\eta_i(\tau,t)k_i(\tau,t)\big) &= \dot{\eta}_i(\tau,t)k_i(\tau,t) + \eta_i(\tau,t)\dot{k}_i(\tau,t)\\ &= -\left[r_i(t) - (1+\rho)\right]\eta_i(\tau,t)k_i(\tau,t) + \eta_i(\tau,t)\big(r_i(t)k_i(\tau,t) - \eta_i(\tau,t)^{-1}\big)\\ &= (1+\rho)\left(\eta_i(\tau,t)k_i(\tau,t) - \frac{1}{1+\rho}\right),\end{aligned}$$

where the second equality follows from (5), (6), and (7), so that

$$\eta_i(\tau,t)k_i(\tau,t) - \frac{1}{1+\rho} = e^{(1+\rho)(t-\tau)}\left(\eta_i(\tau,\tau)k_0 - \frac{1}{1+\rho}\right). \tag{11}$$

But the transversality condition (8) holds only if $\eta_i(\tau,\tau)k_0 - \frac{1}{1+\rho} = 0$, and therefore by (11) we have $\eta_i(\tau,t)k_i(\tau,t) = \frac{1}{1+\rho}$ for all $t \geq \tau$. It follows that the optimal consumption and capital holding are given as

$$c_i(\tau,t) = \frac{1+\rho}{\phi_i(x_i(t))}k_i(\tau,t) \tag{12}$$

and

$$k_i(\tau,t) = k_0 e^{R_i(\tau,t)} \tag{13}$$

by (6) and (9), respectively.

Therefore, by (4), (12), and (13), we obtain the expected lifetime utility of agents born at time τ from locating in region i as

$$\int_\tau^\infty e^{-(1+\rho)(s-\tau)}\left\{R_i(\tau,s) + \ln\frac{(1+\rho)k_0}{\phi_i(x_i(s))}\right\}\mathrm{d}s. \tag{14}$$

2.1.3 Market equilibrium

The capital market clearing condition at each time t is

$$K_i(t) = e^{-t}x_{0i}k_i(0,t) + \int_0^t e^{-(t-\tau)}\alpha_i(\tau)k_i(\tau,t)\mathrm{d}\tau, \tag{15}$$

where the right hand side is the aggregate supply obtained by aggregating the capital holdings of the agents in region i.

The equilibrium rental rate $r_i(t)$ is generally not equal to the marginal productivity $Z_i(x)$, depending on how the capital holdings of the exiting agents are handled. Here we assume that only a fraction $\mu \in [0,1]$ of each individual's holding of capital is transferable within a region upon exit, due to transaction costs which amount $1-\mu$ per unit (Heijdra and Mierau (2012)), and that there is a perfectly competitive insurance market, so that

$$r_i(t) = Z_i(x(t)) + \mu \tag{16}$$

holds under free entry.[11] The value of μ depends on the extent of intraregional mobility of capital. In the limiting case where $\mu = 0$, the capital is considered as completely sunk, while in the other polar case where $\mu = 1$ as in the standard continuous-time OLG models (Yaari (1965)), capital may be interpreted as general assets which are tradable (within the region).

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2.2 Spatial equilibrium dynamics

We have characterized the optimal paths of consumption and capital holding *given* a path of population distributions $x(\,\cdot\,)$. Thus, the next task is to define the equilibrium condition regarding the location choice. When the agents choose their locations, they compare the expected lifetime payoff from each location. By (10), (14), and (16), the expected lifetime payoff from region i is given by

$$V_i(x(\,\cdot\,),\tau) = (1+\rho)\int_\tau^\infty e^{-(1+\rho)(s-\tau)}\left\{\int_\tau^s [Z_i(x(\nu)) + \mu - (1+\rho)]\,\mathrm{d}\nu + \ln\frac{(1+\rho)k_0}{\phi_i(x_i(s))}\right\}\mathrm{d}s$$
$$= (1+\rho)\int_\tau^\infty e^{-(1+\rho)(s-\tau)}\left(\frac{Z_i(x(s))}{1+\rho} - \ln\phi_i(x_i(s))\right)\mathrm{d}s + C, \tag{17}$$

where the second equality obtains by integration by parts, and $C = \frac{\mu-(1+\rho)}{1+\rho} + \ln(1+\rho)k_0$. The value $V_i(x(\,\cdot\,),\tau)$ of locating in region i is determined by the path $x(\,\cdot\,)$ of population distributions through the spillover effects $Z_i(x(\,\cdot\,))$, which depend on the position of region i in the spillover network defined by Z, as well as the congestion effects $\phi_i(x_i(\,\cdot\,))$. Note that the value is normalized by the effective discount rate $1+\rho$.

Our equilibrium dynamics falls in the class of *perfect foresight dynamics* studied in the context of sectoral choice by Matsuyama (1991) as well as in game theory by Matsui and Matsuyama (1995), Hofbauer and Sorger (1999), and Oyama et al. (2008), among others. A spatial equilibrium path is a path $x^*(\,\cdot\,)$ along which agents optimally choose locations under the expectation of $x^*(\,\cdot\,)$ itself.

Definition 2.1. $x^*\colon [0,\infty) \to \Delta$ *is a* spatial equilibrium path, *or* equilibrium path *in short, from* $x_0 \in \Delta$ *if it is Lipschitz continuous and satisfies* $x^*(0) = x_0$*, and for all* $i \in S$*,*

$$\dot{x}_i^*(t) > -x_i^*(t) \Rightarrow i \in \arg\max_{j\in S} V_j(x^*(\,\cdot\,),t) \tag{18}$$

for almost all $t \geq 0$.

Recall from (1) that we have $\dot{x}_i(t) = \alpha_i(t) - x_i(t)$. Thus, $\dot{x}_i^*(t) > -x_i^*(t)$ (i.e., $\alpha_i(t) > 0$) implies that some positive fraction of new entrants choose to locate in region i during short time interval $[t, t+\mathrm{d}t)$. The condition says that region i must maximize $V_j(x^*(\,\cdot\,),t)$ with respect to j given $x^*(\,\cdot\,)$ itself.

The continuity of the integrand in (17), $\frac{Z_i(x)}{1+\rho} - \ln\phi_i(x_i)$, in $x \in \Delta$ guarantees the existence of an equilibrium path for each initial state x_0 (see, e.g., Oyama et al. (2008, Subsection 2.3)).

2.3 Stationary spatial equilibrium states

We say that a path $x(\,\cdot\,)$ is *stationary* at $\bar{x} \in \Delta$ if $x(t) = \bar{x}$ for all $t \geq 0$ (i.e., $x(t)$ is constant at $\bar{x}$ over time). If $x(\,\cdot\,)$ is stationary at $\bar{x}$, the payoff from locating in region i is given by

$$V_i(x(\,\cdot\,),\tau) = \frac{Z_i(\bar{x})}{1+\rho} - \ln\phi_i(\bar{x}_i)$$

for all $i \in S$ and all $\tau \geq 0$. We define the function $Q\colon \Delta \to \mathbb{R}^n$ by the right-hand side:

$$Q_i(x) = \frac{Z_i(x)}{1+\rho} - \ln\phi_i(x_i) \tag{19}$$

for each $i \in S$. Thus, the function V in (17) is written as

$$V_i(x(\,\cdot\,),\tau) = (1+\rho)\int_\tau^\infty e^{-(1+\rho)(s-\tau)} Q_i(x(s))\mathrm{d}s + C.$$

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Definition 2.2. $\bar{x} \in \Delta$ *is a* stationary spatial equilibrium state, *or* equilibrium state *in short, if for all* $i \in S$,

$$\bar{x}_i > 0 \Rightarrow i \in \arg\max_{j \in S} Q_j(\bar{x}). \tag{20}$$

By construction, $\bar{x}$ is an equilibrium state if and only if the stationary path $x(\cdot)$ at $\bar{x}$ is an equilibrium path from $\bar{x}$. By the continuity of $Q(x)$ in $x \in \Delta$, the existence of an equilibrium state follows from a standard fixed point argument. We say that $\bar{x}$ is a *strict* equilibrium state if for all $i \in S$, $\bar{x}_i > 0 \Rightarrow \{i\} = \arg\max_{j \in S} Q_j(\bar{x})$. Equivalently, a strict equilibrium state is a full agglomeration state e_{i^*}, a state in which all the agents are located in one region i^*, such that $Q_{i^*}(e_{i^*}) > Q_j(e_{i^*})$ for all $j \neq i^*$.

Formally, the stationary payoff function $Q = (Q_i)_{i \in S}$ can be interpreted as a static *population game* (Sandholm (2010)), a game in which a continuum of homogeneous players choose among actions in $S = \{1, \ldots, n\}$ and their payoff function $Q(x)$ depends only on the action distribution $x \in \Delta$ (rather than action profile) as well as one's own action. Our (strict) equilibrium states are precisely the (strict) Nash equilibrium action distributions of the population game.

An important departure of our model from the existing literature of perfect foresight dynamics in population games (such as Matsui and Matsuyama (1995) and Hofbauer and Sorger (1999) in random matching settings and Matsuyama (1991) and Oyama (2009) in economic contexts) is that the stationary payoff function Q and hence the equilibrium states depend on the discount rate ρ. Our model involves a stock variable, that is, capital stock, the returns to which constitute the first term in the expression (19), where the population distribution x affects the payoffs through the rate of return as spillovers. The second term in (19) represents the congestion costs, which by assumption are directly affected by the local population x_i. The relative importance between these benefits and costs is governed by the discount rate ρ.

As a preliminary, we discuss some intuitive properties of the equilibrium states of our model. First, if intraregional spillover effects net of congestion effects dominate interregional spillover effects for region i, then the full agglomeration state in region i is a strict equilibrium state, provided that the discounting of the future benefits from spillovers is sufficiently small.

Observation 2.1. *For* $i \in S$, *suppose that*

$$z_{ii} - \ln \phi_i(1) > \max_{j \neq i} z_{ji}. \tag{21}$$

Then there exists $\rho_0(i) > 0$ *such that if* $\rho < \rho_0(i)$, *then the full agglomeration in* i *is a strict equilibrium state.*

This immediately follows from the observation that the full agglomeration state in region i is a strict equilibrium state if and only if $\frac{z_{ii}}{1+\rho} - \ln \phi_i(1) > \frac{z_{ji}}{1+\rho}$ for all $j \neq i$ (recall that $\phi_j(0) = 1$), that is, the *effective* intraregional net spillovers $\frac{z_{ii}}{1+\rho} - \ln \phi_i(1)$ are greater than the effective interregional spillovers $\frac{z_{ji}}{1+\rho}$; then let $\rho_0(i) = \min_{j \neq i} (z_{ii} - z_{ji} - \ln \phi_i(1)) / \ln \phi_i(1) > 0$.

Second, if the time discounting is sufficiently large, so that the congestion effects become dominant relative to the effective spillover terms, then there is a unique equilibrium state, and it is a full dispersion state, a state in which every region hosts a positive mass of agents. Formally, for sufficiently large ρ, first, the payoff function Q satisfies the *strict contractivity* condition (Sandholm (2015)),[12]

$$(y - x)'(Q(y) - Q(x)) < 0 \text{ for all } x, y \in \Delta \text{ with } x \neq y, \tag{22}$$

in which case Q necessarily has a unique equilibrium state, and second, the unique equilibrium state lies in the interior of Δ.

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Observation 2.2. *There exists $\rho_1 > 0$ such that if $\rho > \rho_1$, then the strict contractivity condition* (22) *holds, and $\bar{x}_i > 0$ for all i in the unique equilibrium state $\bar{x}$.*

Recall that, by (10), (13), and (16), at the full agglomeration state in region i, the growth rate of individual capital holding in each region j is $\frac{\dot{k}_j(\tau,t)}{k_j(\tau,t)} = z_{ji} + \mu - (1+\rho)$, which, under the condition (21), is maximized at $j = i$. Thus, agglomeration makes the capital grow faster, which tends to increase the future benefits in i. On the other hand, by (12), the initial consumption is $\frac{(1+\rho)k_0}{\phi_i(1)}$ in i and $(1+\rho)k_0$ in $j \neq i$ (where $\phi_j(0) = 1$), and it is minimized at region i. Thus, agglomeration causes congestion, which tends to reduce the current consumption. When the discount rate ρ is small, or people are farsighted, they attach greater importance to the agglomeration benefits on the capital growth than to the congestion costs on the current consumption. As a result, agglomeration in region i is likely to attain. In particular, there are multiple strict equilibrium states if the condition in (21) holds for multiple regions and the discount rate is small accordingly.[13] If the discounting is sufficiently large, or people are sufficiently myopic, in contrast, they care about congestion more than economic growth, and therefore agglomeration is less likely to attain and in fact, in a unique equilibrium state, the population is fully dispersed across the regions.

2.4 Long-run capital and income at equilibrium states

Before closing this section, we turn our attention to the long-run levels of capital and income at equilibrium states. Consider the equilibrium path that is stationary at $x \in \Delta$ and any region i such that $x_i > 0$. Then by (16), the rental rate $r_i(t)$ in region i is equal to

$$r_i(t) = Z_i(x) + \mu \tag{23}$$

for all $t \geq 0$. Thus, by (10) and (13), we have $k_i(\tau, t) = k_0 e^{[r_i - (1+\rho)](t-\tau)}$, and therefore, by (15) with $\alpha_i(\tau) = x_i$,

$$K_i(t) = \begin{cases} \dfrac{(r_i - \rho - 1)e^{(r_i-\rho-2)t} - 1}{r_i - \rho - 2} K_i(0) & \text{if } r_i - \rho - 2 \neq 0, \\ (t+1)K_i(0) & \text{otherwise,} \end{cases} \tag{24}$$

where $K_i(0) = x_i k_0$. Let $k_i(t) = K_i(t)/x_i$ and $y_i(t) = Y_i(t)/x_i = Z_i(x)k_i(t)$, which are the per-capita capital and per-capita income in region i, respectively. If $r_i - \rho - 2 < 0$, or $Z_i(x) < -\mu + \rho + 2$, then $k_i(t) \to \frac{k_0}{-Z_i(x) - \mu + \rho + 2}$ and $y_i(t) \to \frac{Z_i(x)k_0}{-Z_i(x) - \mu + \rho + 2}$ as $t \to \infty$, so that the region i tends to a steady state. On the other hand, if $r_i - \rho - 2 \geq 0$, or $Z_i(x) \geq -\mu + \rho + 2$, the per-capita capital grows without bound, where $\frac{\dot{k}_i(t)}{k_i(t)}, \frac{\dot{y}_i(t)}{y_i(t)} \to Z_i(x) + \mu - \rho - 2$ as $t \to \infty$, so that the region rides on a balanced growth path asymptotically. Hence, we obtain the following result.

Proposition 2.3. *Suppose that the equilibrium path is stationary at $x \in \Delta$. Then,*

$$\begin{cases} (k_i(t), y_i(t)) \to \left(\frac{k_0}{-Z_i(x) - \mu + \rho + 2}, \frac{Z_i(x)k_0}{-Z_i(x) - \mu + \rho + 2} \right) & \text{if } Z_i(x) < -\mu + \rho + 2 \\ \frac{\dot{k}_i(t)}{k_i(t)}, \frac{\dot{y}_i(t)}{y_i(t)} \to Z_i(x) + \mu - \rho - 2 & \text{if } Z_i(x) \geq -\mu + \rho + 2 \end{cases} \quad \text{as } t \to \infty.$$

Since the productivity at an equilibrium state differs across regions in general, the proposition above implies that the σ*-convergence* (Barro and Sala-i-Martin (1992)), which is the diminishing spatial variation of per-capita incomes over time, will not always occur. In our model, the σ*-divergence* can also happen. In particular, it is possible that, while some regions will succeed in riding on balanced growth paths, some other regions will be stuck with fixed income levels.[14] In fact, we present a numerical example for such a case in Subsection 4.2.

3. Global stability analysis

3.1 Stability concepts

As we discussed in the previous section, our dynamic model may generally have multiple equilibrium states, and it is indeed the case when spillover effects dominate congestion effects. Furthermore, from an equilibrium state $\bar{x}$, there may exist an equilibrium path, other than the stationary path at $\bar{x}$, that departs away from $\bar{x}$ and converges to another equilibrium state. We regard such a state $\bar{x}$ as unstable or fragile. We are interested in equilibrium states that are globally stable in the following sense (Matsui and Matsuyama (1995)). For $x \in \Delta$ and $\varepsilon > 0$, we let $B_\varepsilon(x) = \{y \in \Delta \mid |y - x| < \varepsilon\}$ denote the ε-neighborhood of x in Δ, where $|z| = \max_{i \in S}|z_i|$ is the max-norm of $z \in \mathbb{R}^n$.

Definition 3.1.

(i) *An equilibrium state* $\bar{x} \in \Delta$ *is* absorbing *if there exists* $\varepsilon > 0$ *such that any equilibrium path from any* $x_0 \in B_\varepsilon(\bar{x})$ *converges to* $\bar{x}$*;* $\bar{x}$ *is* fragile *if it is not absorbing.*

(ii) *An equilibrium state* $\bar{x} \in \Delta$ *is* accessible *from* $x_0 \in \Delta$ *if there exists an equilibrium path from* x_0 *that converges to* $\bar{x}$*;* $\bar{x}$ *is* globally accessible *if* $\bar{x}$ *is accessible from any* $x_0 \in \Delta$.

In what follows, we aim to characterize an equilibrium state $\bar{x}$ that is absorbing and globally accessible for sufficiently small discount rates $\rho > 0$. Recall that the equilibrium path is not necessarily unique for an initial state. Nevertheless, the absorption of $\bar{x}$ requires that any equilibrium path from a neighborhood of $\bar{x}$ converge to $\bar{x}$. On the other hand, for the global accessibility, we require that from any initial state, there exist at least one equilibrium path that converges to $\bar{x}$. By definition, if a state is absorbing (globally accessible, resp.), then no other state can be globally accessible (absorbing, resp.), and thus an absorbing and globally accessible state is unique if it exists.

Our dynamic equilibrium model is highly nonlinear and infinite dimensional and thus is difficult to analyze in general, in particular when there are more than two regions. Accordingly, to maintain our n-region setup, the analysis in this paper will be conducted under a certain symmetry assumption on the spillover matrix Z, as introduced in the next subsection.

3.2 Potential

In our analysis, the concept of *potential* will play an important role. Let $\overline{\Delta}$ be an open neighborhood of Δ in $\mathbb{R}^n$.

Definition 3.2. *A function* $W: \overline{\Delta} \to \mathbb{R}$ *is a* potential function *of* Q *if it is differentiable and satisfies*

$$\frac{\partial W}{\partial x_i}(x) - \frac{\partial W}{\partial x_j}(x) = Q_i(x) - Q_j(x) \text{ for all } x \in \Delta \text{ and all } i, j \in S. \tag{25}$$

The function W is defined on $\overline{\Delta}$ only for its derivatives to be well defined on Δ; otherwise, it is innocuous. By definition, W is a potential function of Q if and only if $z'\nabla W(x) = z'Q(x)$ for all $x \in \Delta$ and all $z \in T\Delta$, where $\nabla W(x) = \left(\frac{\partial W}{\partial x_1}(x), \ldots, \frac{\partial W}{\partial x_n}(x)\right)'$ denotes the gradient vector of W at x, and $T\Delta = \{z \in \mathbb{R}^n \mid \sum_{i \in S} z_i = 0\}$ the tangent space of Δ.

This concept is a natural extension of that by Monderer and Shapley (1996), defined for finite-player normal form games, to population games (Sandholm (2001, 2009, 2010), Oyama (2009)). Observe that if Q admits a potential function W, the set of equilibrium states of Q coincides

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with the set of Karush–Kuhn–Tucker (KKT) points for the optimization problem max $W(x)$ subject to $x \in \Delta$, that is, those points $x \in \Delta$ for which there exist $\nu \in \mathbb{R}$ and $\eta \in \mathbb{R}^n_+$ such that for all $i \in S$,

$$\frac{\partial W}{\partial x_i}(x) = \nu - \eta_i \tag{26a}$$

$$x_i > 0 \Rightarrow \eta_i = 0. \tag{26b}$$

It is clear from the definition that a potential function always exists when there are only two regions. For three or more regions, the existence of a potential function requires a nontrivial restriction on the parameters. In our model, Q admits a potential function if and only if the spillover matrix Z satisfies the *triangular integrability condition*:

$$z_{ij} + z_{jk} + z_{ki} = z_{ik} + z_{kj} + z_{ji} \text{ for all } i, j, k \in S. \tag{27}$$

See, for example, Oyama (2009, Appendix A) or Sandholm (2010, Section 3.2) for more details. Assume that the triangular integrability condition holds. Let $\widetilde{Z} = (\tilde{z}_{ij})$ be the $n \times n$ matrix defined by

$$\tilde{z}_{ij} = z_{ij} + z_{j1} - z_{1j},$$

which is symmetric (i.e., $\tilde{z}_{ij} = \tilde{z}_{ji}$ for all $i, j \in S$) by the triangular integrability. Then, a potential function of Q is given by

$$W(x) = \frac{1}{2(1+\rho)} x' \widetilde{Z} x - \sum_{i \in S} \int_0^{x_i} \ln \phi_i(y) \mathrm{d}y, \tag{28}$$

for which, for all $x \in \Delta$, we have

$$\begin{aligned} \frac{\partial W}{\partial x_i}(x) &= \frac{1}{1+\rho} (\widetilde{Z}x)_i - \ln \phi_i(x_i) \\ &= \frac{1}{1+\rho} (Zx)_i - \ln \phi_i(x_i) + \sum_{k=1}^{n} (z_{k1} - z_{1k}) x_k \end{aligned}$$

and hence $\frac{\partial W}{\partial x_i}(x) - \frac{\partial W}{\partial x_j}(x) = Q_i(x) - Q_j(x)$ for all $i, j \in S$. In a special case where Z is symmetric, we have $\widetilde{Z} = Z$ and $\nabla W(x) = Q(x)$ for all $x \in \Delta$. Note that a potential function is unique on Δ up to a constant.

In our model, the stationary payoff function Q depends on the discount rate ρ. We will denote it as well as its potential function by $Q(\cdot, \rho)$ and $W(\cdot, \rho)$, respectively, when we want to make the dependence on ρ explicit. Note that $Q(\cdot, \rho)$ and $W(\cdot, \rho)$ as expressed in (19) and (28) are well defined for all $\rho \in (-1, \infty)$.

3.3 Stability theorem

In this subsection, we present our global stability result, which holds under the following assumptions. For $x \in \Delta$, we let supp $(x) = \{i \in S \mid x_i > 0\}$ denote the support of x.

A1. The triangular integrability condition (27) holds, so that the function $W(\cdot, \rho)$ defined by (28) is a potential function of $Q(\cdot, \rho)$ for each $\rho \in (-1, \infty)$.

A2. $W(\cdot, 0)$ has a unique maximizer $\bar{x}^0$ on Δ.

A3. $\bar{x}^0$ is a *regular equilibrium state* of $Q(\cdot, 0)$ in the following sense, where we denote $C =$ supp $(\bar{x}^0)$:

(1) $\bar{x}^0$ is a *quasi-strict equilibrium state* of $Q(\cdot, 0)$, that is, it is an equilibrium state of $Q(\cdot, 0)$ such that $Q_i(\bar{x}^0, 0) > Q_j(\bar{x}^0, 0)$ for all $i \in C$ and all $j \notin C$.

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(2) The matrix

$$\begin{pmatrix} D_x Q(\bar{x}^0, 0)_C & \mathbf{1}_C \\ \mathbf{1}'_C & 0 \end{pmatrix} \in \mathbb{R}^{(|C|+1)\times(|C|+1)}$$

is nonsingular, where $D_x Q(\cdot, 0)_C \in \mathbb{R}^{|C|\times|C|}$ is the submatrix of the Jacobian matrix $D_x Q(\cdot, 0)$ of $Q(\cdot, 0)$ restricted to C and $\mathbf{1}_C \in \mathbb{R}^{|C|}$ is the $|C|$-dimensional column vector of ones.

The regularity concept in A3 is analogous to the concepts of regular Nash equilibrium in normal form games (van Damme (1983)) and regular evolutionarily stable strategy in population games (Taylor and Jonker (1978)).

The Jacobian $D_x Q(\cdot, \rho)$ is written as

$$D_x Q(x, \rho) = \frac{1}{1+\rho} Z - \Psi(x),$$

where $\Psi(x)$ is the matrix such that $\Psi_{ii}(x) = \phi'_i(x)/\phi_i(x)$ and $\Psi_{ij}(x) = 0$ for all i and $j \neq i$. With the existence of a potential function $W(\cdot, \rho)$, the inequality in A3(1) is written as $\frac{\partial W}{\partial x_i}(\bar{x}^0, 0) > \frac{\partial W}{\partial x_j}(\bar{x}^0, 0)$, which implies that $\bar{x}^0$ satisfies the KKT conditions (26) with strict complementary slackness (i.e., $\eta_i = 0$ if and only if $x_i > 0$ in place of (26b)), while the nonsingularity of the matrix in A3(2) is equivalent to the nonsingularity of the bordered Hessian of $W(x, 0)$ at $\bar{x}^0$ on $\{x \in \Delta \mid \sum_{i\in C} x_i = 1\}$,

$$\begin{pmatrix} D_x^2 W(\bar{x}^0, 0)_C & \mathbf{1}_C \\ \mathbf{1}'_C & 0 \end{pmatrix}.$$

In particular, a strict equilibrium state is necessarily a regular equilibrium state. Assumption A2 is a genericity condition which excludes, for example, perfect symmetry among regions. From A3 it follows in particular that, for ρ sufficiently close to 0, $W(\cdot, \rho)$ has a unique maximizer on Δ (as shown in Lemma A.6 in the Appendix).

Under these assumptions A1–A3, we have the main theorem on the stability of equilibrium states under our equilibrium dynamics.

Theorem 3.1. *Assume A1–A3. Then there exists $\bar{\rho} > 0$ such that the equilibrium state that uniquely maximizes $W(\cdot, \rho)$ on Δ is absorbing and globally accessible for any $\rho \in (0, \bar{\rho}]$.*

Recall from Observation 2.1 that when the discount rate ρ is small, that is, agents are sufficiently farsighted or patient, there tend to exist multiple agglomeration equilibrium states provided that intraregional net spillover effects dominate interregional spillover effects. As agents become more farsighted, expectations that the economy will move from one equilibrium state to another become more likely to be self-fulfilling, making some of the equilibrium states fragile. Our theorem shows that for sufficiently small ρ, all the equilibrium states but one, the state $\bar{x}^\rho$ that maximizes the potential function, become fragile. Whatever initial condition x_0 the history picks, there is some form of self-fulfilling expectations that leads the economy to $\bar{x}^\rho$, that is, $\bar{x}^\rho$ is globally accessible. On the other hand, history also matters, in that if history picks the initial condition in a neighborhood of $\bar{x}^\rho$, any form of self-fulfilling expectations cannot prevent the economy from converging to $\bar{x}^\rho$, that is, $\bar{x}^\rho$ is absorbing. Thus, our result offers a natural criterion to select among multiple equilibrium states, and in fact the selected equilibrium is characterized by the maximizer of the potential function of $Q(\cdot, \rho)$ on Δ. Hence, our task is, then, to inspect the shape of the potential function. In the next subsection (Subsection 3.4), we discuss the globally stable equilibrium state for some simple cases, while in the next section (Section 4), we study stability in relation to the spillover network structure.

In the remainder of this subsection, we briefly discuss the proof of our theorem; the full proof is provided in Appendix A.1.

Suppose that $W(\cdot, \rho)$ is a potential function of $Q(\cdot, \rho)$ with a unique maximizer $\bar{x}^\rho$ on Δ. We utilize two results from the previous literature (Hofbauer and Sorger (1999), Oyama (2009)) which apply to our model for a *fixed* discount rate ρ. (i) First, for a given initial condition $x_0 \in \Delta$, consider the dynamic optimization problem,

$$\text{maximize } \mathcal{W}(x(\,\cdot\,), \rho) = (1+\rho)\int_0^\infty e^{-\rho t} W(x(t), \rho)\mathrm{d}t \tag{29a}$$

$$\text{subject to } \dot{x}(t) = \alpha(t) - x(t),\ \alpha(t) \in \Delta,\ x(0) = x_0. \tag{29b}$$

Then, any solution to this problem is an equilibrium path from x_0 (Hofbauer and Sorger (1999, Theorem 2) or Oyama (2009, Lemma C.2)). This may be seen as a dynamic analogue of the property that a solution to the static optimization problem max $W(x, \rho)$ subject to $x \in \Delta$ is an equilibrium state of $Q(\cdot, \rho)$. (ii) To state the second result, call a state $x^c \in \Delta$ a *critical* point of $W(\cdot, \rho)$ if $\frac{\partial W}{\partial x_i}(x^c, \rho) = \frac{\partial W}{\partial x_j}(x^c, \rho)$ for all $i, j \in \text{supp}\,(x^c)$, and denote the set of critical points of $W(\cdot, \rho)$ by $\mathcal{C}(\rho)$. Then, for any equilibrium path $x(\,\cdot\,)$, if there exists $t \geq 0$ such that $W(x(t), \rho) > W(x^c, \rho)$ for all $x^c \in \mathcal{C}(\rho) \setminus \{\bar{x}^\rho\}$, then $x(\,\cdot\,)$ necessarily converges to $\bar{x}^\rho$ (Hofbauer and Sorger (1999, Lemma 4) or Oyama (2009, Lemma C.6)). The task is then to guarantee the existence of an $\varepsilon > 0$ such that for any sufficiently small $\rho > 0$, the ε-neighborhood $B_\varepsilon(\bar{x}^\rho)$ of $\bar{x}^\rho$ in Δ isolates $\bar{x}^\rho$ from other critical points (in that $W(x, \rho) > W(x^c, \rho)$ for all $x \in B_\varepsilon(\bar{x}^\rho)$ and all $x^c \in \mathcal{C}(\rho) \setminus \{\bar{x}^\rho\}$) and for any $x_0 \in \Delta$, any solution to the problem (29) visits $B_\varepsilon(\bar{x}^\rho)$ at least once.

Here, we have to notice the difference between the previous models and ours. In the previous models, where a static game is repeatedly played over time, the payoff function Q and hence the potential function W and the unique potential maximizer $\bar{x}$ are independent of the discount rate ρ, so that, under an assumption that $\bar{x}$ is isolated from other critical states,[15] one can first fix an $\hat{\varepsilon} > 0$, again independent of ρ, such that $W(x) > W(x^c)$ for all $x \in B_{\hat{\varepsilon}}(\bar{x})$ and all $x^c \in \mathcal{C}(\rho) \setminus \{\bar{x}\}$. Given this $\hat{\varepsilon} > 0$, a version of the so-called Visit Lemma (Oyama (2009, Lemma C.3)) from turnpike theory shows that there exists $\hat{\rho} > 0$ such that for any $\rho \in (0, \hat{\rho}]$ and any $x_0 \in \Delta$, if $x(\,\cdot\,)$ is an equilibrium path for ρ and x_0, then there exists $t \geq 0$ such that $x(t) \in B_{\hat{\varepsilon}}(\bar{x})$. Combined with the results (i)–(ii) stated above, these imply that $\bar{x}$ is absorbing and globally accessible for $\rho \in (0, \hat{\rho}]$ in the previous setting.

Our model, in contrast, involves stock variables, and as a consequence, the potential function does depend on the discount rate ρ, so that in general there is no guarantee that we can take an isolating ε as above *uniformly* for all (sufficiently small) values of ρ. It would not be possible, in particular, if the trajectory of critical points bifurcates as ρ changes from $\rho = 0$ to $\rho > 0$. This is where our regularity condition A3 comes in: By A3(1), for ρ's sufficiently close to zero and in a neighborhood of $\bar{x}^0$, the critical states have the same support as $\bar{x}^0$ and hence are precisely the solutions to the system of equations $\frac{\partial W}{\partial x_i}(x, \rho) = \frac{\partial W}{\partial x_j}(x, \rho)$ for all $i, j \in C = \text{supp}\,(\bar{x}^0)$, $\sum_{i \in S} x_i = 1$, and $x_i = 0$ for all $i \notin C$, and A3(2) then allows us to apply the Implicit Function Theorem to this system to show that, under A2, there exist a continuous (in fact C^1) function $\phi(\rho)$ defined on a neighborhood J of $\rho = 0$ and $\bar{\varepsilon} > 0$ such that for all $\rho \in J$, $\phi(\rho)$ is a unique maximizer of $W(\cdot, \rho)$ on Δ, and $W(x, \rho) > W(x^c, \rho)$ for all $x \in B_{\bar{\varepsilon}}(\phi(\rho))$ and all $x^c \in \mathcal{C}(\rho) \setminus \{\phi(\rho)\}$ (Lemmas A.6–A.7 in the Appendix). Then, this implies, first, by result (ii) above that the potential maximizer $\phi(\rho)$ is absorbing for any $\rho > 0$ contained in J. Second, for $\bar{\varepsilon} > 0$ as obtained, our version of the Visit Lemma (Lemma A.9) shows that there exists $\bar{\rho} > 0$ in J such that for any $\rho \in (0, \bar{\rho}]$ and any initial condition $x_0 \in \Delta$, any solution to the optimization problem (29), which is an equilibrium path from x_0 by result (i), must visit the $\bar{\varepsilon}$-neighborhood of $\phi(\rho)$, so that it converges to $\phi(\rho)$ by the choice of $\bar{\varepsilon}$ and result (ii). This means that $\phi(\rho)$ is globally accessible for $\rho \in (0, \bar{\rho}]$.

3.4 Agglomeration and dispersion as stable equilibrium states

In this subsection, under the triangular integrability assumption (27), we study the shape of the potential function as given in (28) to characterize the stable equilibrium state of our model.

Specifically, we consider sufficient conditions under which the potential function becomes convex or concave.

The potential function $W(\cdot,\rho)$ is strictly convex (concave, resp.) on Δ if and only if $(y-x)'(\nabla W(y,\rho)-\nabla W(x,\rho))>0$ (<0, resp.) holds for all $x,y\in\Delta$ with $x\neq y$, where by the definition of a potential function, we have $(y-x)'(\nabla W(y,\rho)-\nabla W(x,\rho))=(y-x)'(Q(y,\rho)-Q(x,\rho))$.

First, the following observation gives us a sufficient condition for the potential function to be strictly convex on Δ for $\rho=0$.

Observation 3.2. *Assume A1. If*

$$z_{ii}-\max_{y\in[0,1]}\frac{\phi_i'(y)}{\phi_i(y)}>\frac{1}{2}\sum_{j\neq i}(z_{ij}+z_{ji})\ \text{for all } i\in S, \tag{30}$$

then $W(\cdot,0)$ is strictly convex on Δ.

The condition (30) says that $D_xQ(x,0)+D_xQ(x,0)'$ has a positive dominant diagonal for all $x\in\Delta$. It implies that $D_xQ(x,0)$ is positive definite for all $x\in\Delta$, which in turn implies that $(y-x)'(\nabla W(y,0)-\nabla W(x,0))=(y-x)'(Q(y,0)-Q(x,0))>0$ for all $x,y\in\Delta$, $x\neq y$, and hence, $W(\cdot,0)$ is strictly convex on Δ.

Thus, the potential function is strictly convex when $\rho=0$ if, for each region, the agglomeration economy z_{ii} net of the congestion force $\phi_i'(x_i)/\phi_i(x_i)$ is sufficiently large and/or the spillover effect from the other regions is sufficiently small.

The maximizer of a strictly convex potential function on Δ is a full agglomeration state (i.e., a vertex of Δ) and is a strict equilibrium state, which automatically satisfies the regularity condition A3. Thus, by Theorem 3.1, we obtain the following result for the case of strict convexity.

Proposition 3.3. *Assume A1–A2, and suppose that $W(\cdot,0)$ is strictly convex on Δ and uniquely maximized at a full agglomeration state e_{i^*}. Then there exists $\bar{\rho}>0$ such that e_{i^*} is absorbing and globally accessible for all $\rho\in(0,\bar{\rho}]$.*

Being farsighted, people agglomerate in a single region if the positive effect of agglomeration on the economic growth is sufficiently large relative to its congestion effect on the current consumption (so that the potential function is strictly convex). In particular, if the spillover matrix Z is symmetric (i.e., $z_{ij}=z_{ji}$ for all i,j) and if $z_{11}=\cdots=z_{nn}$, the global maximizer is attained when people agglomerate in region i^* where $\int_0^1\ln\phi_{i^*}(y)\mathrm{d}y<\int_0^1\ln\phi_j(y)\mathrm{d}y$ for all $j\neq i^*$. Thus, the stable population distribution is the full agglomeration in the region with the smallest congestion force. On the other hand, again under the symmetry of Z, if $\int_0^1\ln\phi_1(y)\mathrm{d}y=\cdots=\int_0^1\ln\phi_n(y)\mathrm{d}y$, which trivially holds when $\phi_1=\cdots=\phi_n$, the global maximizer is attained when people agglomerate in region i^* where $z_{i^*i^*}>z_{jj}$ for all $j\neq i^*$. Thus, the stable population distribution is the full agglomeration in the region with the largest agglomeration benefit.[16]

Next, we consider the case where people are nearly myopic, or ρ is very large. Suppose that ρ is sufficiently large as in Observation 2.2. Then the strict contractivity condition (22) implies that the potential function $W(\cdot,\rho)$ is strictly concave on Δ. In this case, the unique maximizer $\bar{x}^\rho$ of W on Δ, which is a full dispersion state (i.e., $\bar{x}_i^\rho>0$ for all i) by Observation 2.2, in fact attracts all the equilibrium paths, as shown in Proposition A.10 in Appendix A.1.4.

Proposition 3.4. *Assume A1, and let $\rho_1>0$ be as in Observation 2.2. If $\rho>\rho_1$ so that $W(\cdot,\rho)$ is strictly concave and uniquely maximized at a dispersed state $\bar{x}^\rho$, then every equilibrium path converges to $\bar{x}^\rho$.*

By definition, the unique equilibrium state is also absorbing and globally accessible in this case. We have already discussed the intuitions behind the fact that a fully dispersed population distribution is attained at a unique equilibrium state when ρ is large, that is, myopic people care about

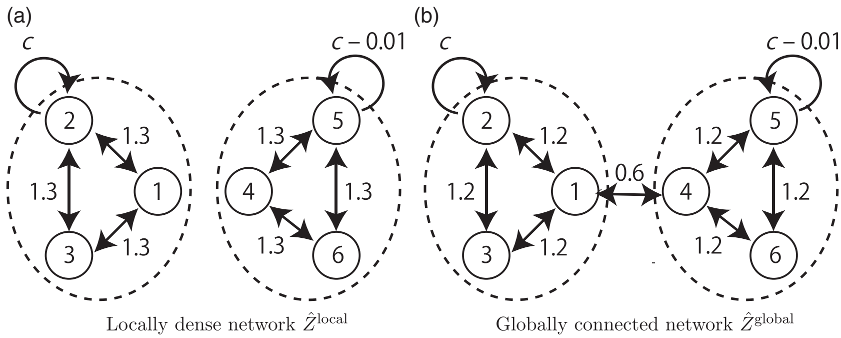

Figure 1. Clustering versus reach.

congestion more than agglomeration economies. The proposition above confirms that this unique equilibrium state is globally stable.

4. Stable equilibrium states and network structure

In this section, we see how the network structure of externalities affects the stable equilibrium. We make an assumption analogous to that in the network game literature, where the best response function is usually assumed to be linear (see, e.g., Bramoullé and Kranton (2016)). Specifically, we assume $\phi_i(x_i) = \exp(\kappa_i x_i)$ where $\kappa_i > 0$ for all $i \in S$ so that the stationary payoff function is linear in $x \in \Delta$, that is,

$$Q_i(x) = \frac{1}{1+\rho} \sum_{j \in S} z_{ij} x_j - \kappa_i x_i \tag{31}$$

for $i \in S$. We set $\rho = 0$ to approximate the condition "ρ is sufficiently small" as in Theorem 3.1. To further simplify expositions, we focus on the case where Z is symmetric. When Q is linear as in (31), z_{ii} and κ_i are not distinguishable. Hence, in the following, we consider the *net*-spillover matrix $\hat{Z} = (\hat{z}_{ij})$ defined by

$$\hat{z}_{ij} = \begin{cases} z_{ij} - \kappa_i & \text{if } i = j, \\ z_{ij} & \text{otherwise.} \end{cases} \tag{32}$$

Note that the potential function is then given by

$$W(x) = \frac{1}{2} x' \hat{Z} x. \tag{33}$$

In general, a weighted directed network, or simply network, is represented by an *adjacency matrix* M, where $m_{ij} \neq 0$ if there is a directed link from node i to node j, and the value of m_{ij} represents the weight attached to that link. We call the network with adjacency matrix M simply network M. We may regard our spatial economy as a network with adjacency matrix $\hat{Z}$.

4.1 Clustering versus reach

In this subsection, we compare two different kinds of networks: in one network, connections are strong locally but weak globally, while in the other, connections are weak locally but strong globally. The networks are given in Figure 1, where c is a constant to be specified below. That is, their net-spillover matrices $\hat{Z}^{\text{local}}$ and $\hat{Z}^{\text{global}}$ are given by $\hat{Z}^{\text{local}}_{ii} = c$ if $i \in \{1, 2, 3\}$, $\hat{Z}^{\text{local}}_{ii} = c - 0.01$ if $i \in \{4, 5, 6\}$, $\hat{Z}^{\text{local}}_{ij} = 1.3$ if $i \neq j$ and $[i, j \in \{1, 2, 3\}$ or $i, j \in \{4, 5, 6\}]$, and $\hat{Z}^{\text{local}}_{ij} = 0$ otherwise; and

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Table 1. Stable equilibrium states and utility levels for $c = -0.5$ and $c = -3$

	x_1	x_2	x_3	x_4	x_5	x_6	u^*
Local	0.3333	0.3333	0.3333	0.	0.	0.	0.7
Global	0.3333	0.3333	0.3333	0.	0.	0.	0.6333
(a) $c = -0.5$							
	x_1	x_2	x_3	x_4	x_5	x_6	u^*
Local	0.1687	0.1687	0.1687	0.1646	0.1646	0.1646	−0.0675
Global	0.1852	0.159	0.159	0.1832	0.1568	0.1568	−0.064
(b) $c = -3.0$							

$\hat{Z}^{\text{global}}_{ii} = c$ if $i \in \{1, 2, 3\}$, $\hat{Z}^{\text{global}}_{ii} = c - 0.01$ if $i \in \{4, 5, 6\}$, $\hat{Z}^{\text{global}}_{ij} = 1.2$ if $i \neq j$ and $[i, j \in \{1, 2, 3\}$ or $i, j \in \{4, 5, 6\}]$, $\hat{Z}^{\text{global}}_{14} = \hat{Z}^{\text{global}}_{41} = 0.6$ and $\hat{Z}^{\text{global}}_{ij} = 0$ otherwise. In these networks, we think of regions 1, 2, 3 and regions 4, 5, 6 as clusters, respectively. In network $\hat{Z}^{\text{local}}$, the two clusters are completely isolated, but connections within each of the clusters are strong. In network $\hat{Z}^{\text{global}}$, on the other hand, connections within each of the clusters are weaker than in the left network, but the two clusters are connected through regions 1 and 4. Therefore, it is possible to reach farther regions in $\hat{Z}^{\text{global}}$ than in $\hat{Z}^{\text{local}}$. We call network $\hat{Z}^{\text{local}}$ a *locally dense network* while network $\hat{Z}^{\text{global}}$ a *globally connected network*.

In this example, we illustrate how which of the two networks achieves the higher steady-state lifetime utility, as given in (19), is affected by the intraregional net spillover effects through the population distribution at the stable equilibrium state. To this end, we assume $\hat{z}_{ii} = c$ for $i = 1, 2, 3$ while $\hat{z}_{ii} = c - 0.01$ for $i = 4, 5, 6$ in both networks and compute the stable equilibrium states via the optimization of the potential function and compare the utility levels under the two networks for different values of c. We slightly differentiate intraregional net spillover effects for the two clusters to guarantee that the potential function has a unique maximizer. Note that the total strength of interregional spillover effects (i.e., the sum of the off-diagonal entries of the spillover matrix) is the same between the two networks so that there is no scale effect.

We consider two values for c: $c = -0.5$ and $c = -3$.[17] First, let $c = -0.5$.[18] The maximizers of the potential function are summarized in Table 1.[19] The equilibrium utility u^* is 0.7 in $\hat{Z}^{\text{local}}$ whereas 0.6333 in $\hat{Z}^{\text{global}}$. In this case, as the intraregional spillover effects are strong and/or the congestion effects are weak (relative to the case of $c = -3$ that follows), only one of the two clusters is populated at the stable equilibrium states. Thus, the locally dense network gives a higher utility level, where the agents benefit from the larger magnitude of within-cluster spillover coefficients than in the globally connected network.

Next, let $c = -3$. The maximizers of the potential function, which is strictly concave for each network in this case, are summarized in Table 1, where, as opposed to the previous case, $\hat{Z}^{\text{global}}$ achieves the higher equilibrium utility ($u^* = -0.064$) than $\hat{Z}^{\text{local}}$ ($u^* = -0.0675$). With weak intraregional spillover effects and/or strong congestion effects, the population is dispersed across the regions in both networks. While the three regions in each cluster have an equal fraction of agents in the locally dense network, the “hub” regions 1 and 4 in the globally connected network attract larger fractions than the others. As a consequence, these hubs generate larger spillover benefits, compensating the smaller magnitude of within-cluster spillover coefficients, in the latter network than in the former.[20]

4.2 σ-Divergence

In Section 2.4, we show the possibility that some regions perpetually grow whereas the others eventually stop growing at an equilibrium state (Proposition 2.3). In this subsection, we provide an

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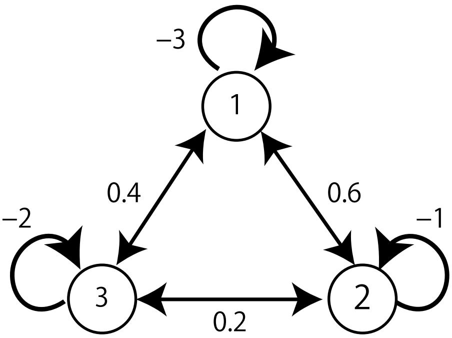

Figure 2. σ-Divergence.

example where such a situation actually takes place in the stable equilibrium state. Let us consider the net-spillover network $(\hat{z}_{ij})$ in Figure 2, where we let $z_{11} = 3$, $z_{22} = 2$, and $z_{33} = 2$, and $\kappa_1 = 6$, $\kappa_2 = 3$, and $\kappa_3 = 4$. We also set $\mu = 1$. The unique potential maximizer, hence the stable equilibrium state, is computed as $(\bar{x}_1, \bar{x}_2, \bar{x}_3) = (0.2392, 0.5064, 0.2545)$, where all regions are populated, and the production coefficients are $Z_1(\bar{x}) = 1.1232, Z_2(\bar{x}) = 1.2071$, and $Z_3(\bar{x}) = 0.7059$. Hence, $Z_1(\bar{x})$ and $Z_2(\bar{x})$ are greater than 1 $(= -\mu + \rho + 2)$, whereas $Z_3(\bar{x})$ is less than 1. Therefore, by Proposition 2.3, at the stable equilibrium state, regions 1 and 2 will ride on balanced growth paths with the growth rate $Z_i(\bar{x}) - 1$, $i = 1, 2$, in the limit as $t \to \infty$, while region 3 will converge to a steady state in the long run with the per-capita capital stock $k_3 = \frac{k_0}{1 - Z_3(\bar{x})}$. Moreover, notice that $\bar{x}_1 < \bar{x}_3$. That is, a stagnating region has a larger population than a growing region. This implies that spatial agglomeration and economic growth is not always positively correlated.

4.3 Katz–Bonacich centrality

In network theory, various measures have been proposed to measure the centrality of each node. Among others, what we consider here is the *Katz–Bonacich centrality* (see, e.g., Zenou (2016)). Let us consider network M, and denote its spectral radius by $r(M)$. Let $\delta > 0$ be such that $\delta r(M) < 1$, so that $\sum_{k=0}^{\infty} \delta^k M^k$ is well defined (and is equal to $(I - \delta M)^{-1}$). Then, the vector of Katz–Bonacich centralities in network M with decay factor δ is defined as

$$b(M, \delta) = \sum_{k=0}^{\infty} \delta^k M^k \mathbf{1} \; \left(= (I - \delta M)^{-1} \mathbf{1}\right), \tag{34}$$

where $\mathbf{1}$ is the vector of ones with the same dimension as M. The ith element of $\delta^k M^k \mathbf{1}$ is the weighted sum of walks of length k of network M that start from node i.[21] Hence, the Katz–Bonacich centrality of node i, which is denoted by $b_i(M, \delta)$, is the weighted sum of all walks that start from node i.[22]

Ballester et al. (2006) consider a finite-player game with linear best responses and derive a sufficient condition under which its Nash equilibrium is unique and is represented as a (normalized) vector of Katz–Bonacich centralities of a network, whose adjacency matrix is obtained by appropriately transforming the matrix of the coefficients in the payoff function to express the interdependencies among the players as a nonnegative matrix. In this subsection, we conduct an analogous exercise for our population game, that is, we derive a sufficient condition under which an equilibrium state is represented as a (normalized) vector of Katz–Bonacich centralities through transforming the net-spillover matrix $\hat{Z}$.

To simplify our argument, we impose the following assumptions.

A4. (0) $z_{ij} \geq 0$ for all $i \neq j$.
(1) $z_{in} = z_{ni} = 0$ for all $i \neq n$.
(2) $\hat{z}_{nn} < 0$.

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Assumption A4(0) is our standing assumption that the spillover coefficients are nonnegative, which is included for reference. A4(1) makes region n, completely isolated from other regions, serve as an outside option. Under A4(1), A4(2) prevents the full agglomeration in region n from being an equilibrium state.

In the following, we focus on a fully dispersed equilibrium state (i.e., an equilibrium state whose support equals $S = \{1, \ldots, n\}$). Assuming the existence of such an equilibrium state amounts to imposing some structural assumption on the net-spillover matrix $\hat{Z}$, as summarized in the proposition below. We denote by $\hat{Z}_{S\setminus\{n\}}$ the $(n-1) \times (n-1)$ submatrix of $\hat{Z}$ restricted to $S \setminus \{n\}$, by I_{n-1} the $(n-1) \times (n-1)$ identity matrix, and by $\mathbf{1}_{n-1}$ the $(n-1)$ dimensional vector of ones.

Proposition 4.1. *Assume A4. Then the following conditions are equivalent:*

(a) *There exists an equilibrium state $\bar{x} \in \Delta$ such that $\bar{x}_n > 0$.*
(b) *There exists an equilibrium state $\bar{x} \in \Delta$ such that $\bar{x}_i > 0$ for all $i \in S$.*
(c) *There exists $\xi \in \mathbb{R}^{n-1}_{++}$ such that $(-\hat{Z}_{S\setminus\{n\}})\xi = (-\hat{z}_{nn})\mathbf{1}_{n-1}$.*
(d) *All the leading principal minors of $-\hat{Z}_{S\setminus\{n\}}$ are positive.*
(e) *$-\hat{Z}_{S\setminus\{n\}}$ is nonsingular, and $(-\hat{Z}_{S\setminus\{n\}})^{-1}$ exists and is nonnegative.*
(f) *$-\hat{Z}_{S\setminus\{n\}}$ is a nonsingular M-matrix, that is, it is a matrix written as $sI_{n-1} - B$ with $s > 0$ and $B \geq O$ for which $s > r(B)$.*

Proof. (a) $\Rightarrow$ (b): Let $\bar{x}$ be an equilibrium state such that $\bar{x}_n > 0$. We show that $\bar{x}_i > 0$ for all $i \in S$. Fix any $i \neq n$. By the equilibrium condition, we have $(\hat{Z}\bar{x})_i \leq (\hat{Z}\bar{x})_n$, where $(\hat{Z}\bar{x})_n = \hat{z}_{nn}\bar{x}_n < 0$ by A4(1) and A4(2). On the other hand, we have $(\hat{Z}\bar{x})_i = \hat{z}_{ii}\bar{x}_i + \sum_{j\neq i} z_{ij}\bar{x}_j \geq \hat{z}_{ii}\bar{x}_i$ (by A4(0)). Hence, we must have $\bar{x}_i > 0$ (and $\hat{z}_{ii} < 0$).

(b) $\Rightarrow$ (a): Obvious.

(b) $\Rightarrow$ (c): Given an equilibrium state $\bar{x} \gg 0$, let $\xi = \frac{1}{\bar{x}_n}\bar{x}_{S\setminus\{n\}} \gg 0$. Then we have $\hat{Z}_{S\setminus\{n\}}\xi = \frac{1}{\bar{x}_n}\hat{Z}_{S\setminus\{n\}}\bar{x}_{S\setminus\{n\}} = \frac{1}{\bar{x}_n}\hat{z}_{nn}\bar{x}_n\mathbf{1}_{n-1} = \hat{z}_{nn}\mathbf{1}_{n-1}$ by the equilibrium condition.

(c) $\Rightarrow$ (b): Given $\xi \gg 0$ such that $\hat{Z}_{S\setminus\{n\}}\xi = \hat{z}_{nn}\mathbf{1}_{n-1}$, let $\bar{x} = \frac{1}{1+\mathbf{1}'_{n-1}\xi}(\xi, 1) \gg 0$. Then we have $\hat{Z}_{S\setminus\{n\}}\bar{x}_{S\setminus\{n\}} = \frac{1}{1+\mathbf{1}'_{n-1}\xi}\hat{z}_{nn}\mathbf{1}_{n-1} = \hat{z}_{nn}\bar{x}_n\mathbf{1}_{n-1}$, and hence $\bar{x}$ is an equilibrium state.

Given that $-\hat{Z}_{S\setminus\{n\}}$ is a Z-matrix[23] (A4(0)), the equivalence among conditions (c)–(f) is well known from the theory of M-matrices (see, e.g., Berman and Plemmons (1979, Theorem 6.2.3)). □

Suppose that there exists an equilibrium state $\bar{x}$ such that $\bar{x}_n > 0$, or equivalently, assume any (hence all) of the conditions in Proposition 4.1. As is clear from the proof of the proposition, it is in fact a unique equilibrium state, uniquely determined by $\bar{x}_{S\setminus\{n\}} = \frac{1}{1+\mathbf{1}'_{n-1}\xi}\xi$ (and $\bar{x}_n = 1 - \mathbf{1}'_{n-1}\bar{x}_{S\setminus\{n\}}$), where $\xi = (-\hat{Z}_{S\setminus\{n\}})^{-1}(-\hat{z}_{nn})\mathbf{1}_{n-1}$. We also have $\hat{z}_{ii} < 0$ for all i.

Now let

$$\beta = \max_{i\in S\setminus\{n\}} (-\hat{z}_{ii}) > 0, \tag{35}$$

which can be interpreted as the maximum net congestion effect among the regions in $S \setminus \{n\}$, and define the $(n-1) \times (n-1)$ nonnegative matrix, or network, G by

$$G = \hat{Z}_{S\setminus\{n\}} + \beta I_{n-1}, \tag{36}$$

so that $-\hat{Z}_{S\setminus\{n\}} = \beta^{-1}(I_{n-1} - \beta^{-1}G)$. By condition (e) in Proposition 4.1, $(I_{n-1} - \beta^{-1}G)^{-1}$ exists and is nonnegative, which implies that $r(\beta^{-1}G) < 1$, or $r(G) < \beta$, by the Perron–Frobenius

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theorem (or by condition (f)), where $r(M)$ denotes the spectral radius of a matrix M. Thus, ξ is written as

$$\xi = \beta^{-1}(I_{n-1} - \beta^{-1}G)^{-1}(-\hat{z}_{nn})\mathbf{1}_{n-1} = \beta^{-1}(-\hat{z}_{nn})b(G, \beta^{-1}),$$

through which the equilibrium state is represented as (a multiple of) the vector $b(G, \beta^{-1})$ of Katz–Bonacich centralities of the network G with decay factor β^{-1}, as defined in (34). To summarize:

Proposition 4.2. *Assume A4. Suppose that there exists an equilibrium state with region n in its support and denote it by $\bar{x}$. Let β be defined by (35) and G be the $(n-1)\times(n-1)$ nonnegative matrix defined by (36). Then we have*

$$\bar{x}_{S\setminus\{n\}} = \frac{-\hat{z}_{nn}}{\beta + \mathbf{1}'_{n-1}b(G, \beta^{-1})}b(G, \beta^{-1}).$$

In particular, the share of the population of regions $i \in S \setminus \{n\}$ is equal to the share of the Katz–Bonacich centrality of that region.

Finally, since $\hat{Z}$ is symmetric as assumed, condition (d) in Proposition 4.1, under Assumption A4, implies that $\hat{Z}$ is negative definite, and hence the potential function $W(x) = \frac{1}{2}x'\hat{Z}x$ is strictly concave. Therefore, by Proposition A.10, the unique equilibrium state $\bar{x}$ has a strong global stability property, that from any state in Δ, any equilibrium path converges to $\bar{x}$.

5. Conclusion

In this paper, we have presented a continuous-time overlapping generations model of endogenous growth with many regions in which forward-looking agents make irreversible location decisions upon birth, as well as saving/capital accumulation decisions. By invoking techniques utilizing potential functions from population game theory, we characterized the global stability of the spatial equilibrium states under our equilibrium dynamics for sufficiently small discount rates. We thereby studied how the population distribution and macroeconomic variables are determined at the stable equilibrium state according to the structure of the network defined by intra- and interregional spillovers relative to congestion costs.

Our geographical model is kept very simple, in order to fully incorporate intertemporal optimization of the agents in a tractable multiregional setting. It is desirable to consider trade across regions and other types of engines of growth,[24] as well as other types of dispersion forces than intraregional transport costs, such as commuting and land costs. We have also been extreme in location decisions by assuming perfect irreversibility. The model would be equivalent under an alternative setting that agents are in fact infinitely lived but receive relocation opportunities only randomly according to a Poisson process, *if* we assume that the capital holding is initialized to the fixed amount of k_0 every time an agent receives a relocation opportunity. If we want to assume more naturally that the agent moves with the capital accumulated in the previous region, we would need to keep track also of the capital distribution in a tractable way. Finally, we followed the perpetual youth modeling by assuming Poisson birth–death processes. It would be interesting to consider more realistic processes while keeping the model simple and tractable,[25] to study, for example, the correlation between the geographic and demographic structures. These are subjects for future research.

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Notes

1 In our context, the Katz–Bonacich centrality of a region is interpreted as the total (discounted) sum of indirect spillover effects that the region receives.
2 Grafeneder-Weissteiner and Prettner (2013) consider a two-region growth model with continuous-time OLG where the total population changes over time. They study the effects of the demographic structure on per capita expenditures and capital stocks at the stationary state.
3 See Desmet and Rossi-Hansberg (2010) for a survey of this literature.
4 Boucekkine et al. (2009) study the optimal capital diffusion with linear utility. Note that the current paper focuses on the decentralized outcome. Boucekkine et al. (2013) overcome ill-posedness by considering AK-type technology and a circular location space, while Camacho and Pérez-Barahona (*forthcoming*) consider a finite horizon setting.
5 Oyama (2009) applies this class of dynamics to a multi-regional New Economic Geography model with forward-looking agents. Absent saving and capital accumulation, this model reduces to the same type of dynamics with a time-invariant payoff function.
6 We thus think of K in a broad sense to include human capital (Barro and Sala-i-Martin (1995)).
7 The linear dependence on x is mainly for simplification of the exposition; the stability theorem in Section 3.3 does not rely on it (see Appendix A.1).
8 An alternative assumption would be that the magnitude of externality in a region depends on the levels of aggregate capital stocks of that and other regions as in the standard (i.e., aspatial) endogenous growth models (e.g., Romer (1986)), but with multiple regions, it would require simultaneous determination of capital stock paths of all the regions, making the model intractable.
9 That is, there is a continuously differentiable extension of ϕ on an open neighborhood of $[0, 1]$.
10 This assumption captures negative externalities from agglomeration, while the specific, iceberg form is for tractability of the model (see, e.g., Martin and Rogers (1995), Alonso-Villar (2001) for similar formulations). Incorporating congestion costs directly as disutilities in an additive manner (i.e., assuming the instantaneous utility function to be $\ln c_i - \psi_i(x_i)$ for some increasing function ψ_i) will not change the qualitative properties of the equilibrium dynamics (and, as can be seen below, will lead to exactly the same dynamics if $\psi_i(x_i) = \ln \phi_i(x_i)$). Behind our "reduced form" of congestion costs, there would be other factors such as land rents and commuting costs (as, e.g., in Krugman and Livas Elizondo (1996), see also Gaspar (2018, Section 3.2) for later contributions), but we abstain from incorporating those factors for the sake of simplicity.
11 At each time interval $[t, t + \mathrm{d}t)$, a fraction $\mathrm{d}t$ of agents die leaving a total amount $\mu K_i(t)\mathrm{d}t$ of the transferable value of the capital good, which must be distributed among the remaining agents under the zero-profit condition for the insurance companies, where each agent receives an amount proportional to his current capital holding, so that he receives $\mu \mathrm{d}t \times k_i(\tau, t)$ in addition to the return $Z_i(x(t))\mathrm{d}t \times k_i(\tau, t)$ to the capital rental.
12 A population game with strict contractivity (22), or a strictly contractive game, is called a *strictly stable game* in Hofbauer and Sandholm (2009).
13 In fact, it follows from the Poincaré-Hopf theorem (see, e.g., Hofbauer and Sigmund (1998, Chapter 13)) that if there are m full agglomeration equilibrium states and they are all strict, then there are at least $m - 1$ dispersion states (states in which more than one regions host a positive mass of agents).
14 The σ-divergence is also empirically observed. For example, Young et al. (2008) find the σ-divergence in US county-level data.
15 A sufficient condition for this is that the potential function W has finitely many critical values (Oyama (2009, Assumption C.1)), which is satisfied if W is analytic, and in particular, if the payoff function is linear as in Hofbauer and Sorger (1999).
16 In Appendix A.2, we discuss the relevance of capital accumulation in supporting agglomeration in a stable state by studying the case where capital accumulation is prohibited. There we formulate a sense in which, and identify conditions under which, agglomeration forces are stronger with capital accumulation than without.
17 If c is positive and sufficiently large, or the intraregional net spillover effects are sufficiently strong, then the potential function is strictly convex and is maximized at the vertices e_i, $i = 1, 2, 3$, of the state space Δ (i.e., the full agglomeration states at regions $i = 1, 2, 3$) in both networks (where unique maximization would be guaranteed by differentiating the value of c region-wise). At each full agglomeration state e_i, utility levels are the same $u^* = c$ between the networks.
18 Inspecting the eigenvalues of $\hat{Z}$ reveals that the potential function is neither concave nor convex on Δ in either network in this case.
19 Since the potential function W is quadratic in x, its maximizer can be computed by a simple support enumeration algorithm, where one solves finitely many systems of linear equations, one for each possible support, for the critical points of W and compares the values of W at the critical points.

20 Hosting a positive amount of agents in the equilibrium state in each network, all regions (not only hub regions) benefit from the hub effects in the globally connected network over the locally dense network.
21 A *walk* of length k in network M from node i to node j is a sequence of links $\{(i_0, i_1), (i_1, i_2), \ldots, (i_{k-1}, i_k)\}$ such that $i_1 = i$, and $i_k = j$. Hence, if network M is unweighted, the i-th element of $M^k\mathbf{1}$ is the total number of walks of length k that start from node i.
22 The row sums of the Leontief inverse in the input-output analysis are also represented as Katz-Bonacich centralities when we consider a network where the input-output matrix is taken as an adjacency matrix. In that case, the Katz-Bonacich centrality of node i is interpreted as the total indirect effects through input-output linkages that node i receives.
23 A Z-matrix is a square matrix whose off-diagonal entries are all nonpositive. This terminology should not be confused with our notation for the spillover matrix.
24 De la Roca and Puga (2017) empirically find that agglomeration has a positive effect on human capital accumulation. Some theoretical studies in the New Economic Geography literature consider interregional trade and other engines of growth such as variety expansion with a two-regional growth model (See, e.g., Walz (1996), Peng et al. (2006), Fujita and Thisse (2013)).
25 See, for example, d'Albis and Augeraud-Véron (2011).

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APPENDIX

A.1 STABILITY THEOREM

In this section, we prove the stability theorem (Theorem 3.1) for general payoff functions Q (under certain regularity conditions). In Sections A.1.1 and A.1.2, the set-up with general payoff functions is introduced, and potential functions and perfect foresight dynamics are defined. In Section A.1.3, the stability theorem is proved in a general form (Theorem A.3). In Section A.1.4, a stronger, global absorption result is proved for the case of strictly concave potential (Proposition A.10).

A.1.1 GENERAL PAYOFF FUNCTIONS

We are given a function $Q\colon \Delta \times (-1, \infty) \to \mathbb{R}^n$, a family of payoff functions parameterized by the discount rate ρ, where $Q(\cdot, \rho)$ is defined for all $\rho \in (-1, \infty)$, and assume that Q is C^1 in the sense that it has a C^1 extension (again denoted by Q) on $\overline{\Delta} \times (-1, \infty)$ for some open neighborhood $\overline{\Delta}$ of Δ in $\mathbb{R}^n$.

A state $\bar{x} \in \Delta$ is a *critical state* of $Q(\cdot, \rho)$ if $Q_i(\bar{x}, \rho) = Q_j(\bar{x}, \rho)$ for all $i, j \in \operatorname{supp}(\bar{x})$, where $\operatorname{supp}(x) = \{i \in S \mid x_i > 0\}$. $\bar{x} \in \Delta$ is a *stationary equilibrium state*, or *equilibrium state* in short, of $Q(\cdot, \rho)$ if for all $i \in \operatorname{supp}(\bar{x})$, $Q_i(\bar{x}, \rho) \geq Q_j(\bar{x}, \rho)$ for all $j \in S$, or equivalently, if it is a critical state of $Q(\cdot, \rho)$ such that $Q_i(\bar{x}, \rho) \geq Q_j(\bar{x}, \rho)$ for all $i \in \operatorname{supp}(\bar{x})$ and all $j \notin \operatorname{supp}(\bar{x})$.

We denote the set of critical states of $Q(\cdot, \rho)$ by $\mathcal{C}(\rho)$. A critical state $x^c \in \mathcal{C}(\rho)$ is *isolated* in $\mathcal{C}(\rho)$ if there exists a neighborhood $U \subset \Delta$ of x^c such that $U \cap \mathcal{C}(\rho) = \{x^c\}$.

A function $W\colon \overline{\Delta} \to \mathbb{R}$ is a *potential function* of $Q(\cdot, \rho)$ if it is differentiable and for all $i, j \in S$,

$$\frac{\partial W}{\partial x_i}(x) - \frac{\partial W}{\partial x_j}(x) = Q_i(x, \rho) - Q_j(x, \rho)$$

for all $x \in \Delta$. It is well known that $Q(\cdot, \rho)$ admits a potential function if and only if

$$\frac{\partial Q_i}{\partial x_j}(x) + \frac{\partial Q_j}{\partial x_k}(x) + \frac{\partial Q_k}{\partial x_i}(x) = \frac{\partial Q_i}{\partial x_k}(x) + \frac{\partial Q_k}{\partial x_j}(x) + \frac{\partial Q_j}{\partial x_i}(x)$$

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holds for all $i, j, k \in S$ and all $x \in \Delta$. As discussed in the main text, any (local or global) solution to the maximization problem,

$$\begin{aligned} &\text{maximize} \;\; W(x) \\ &\text{subject to} \;\; x \in \Delta, \end{aligned}$$

is an equilibrium state, and thus a critical point, of $Q(\cdot, \rho)$.

$x^c \in \Delta$ is a *critical point* of W if $\frac{\partial W}{\partial x_i}(x^c) = \frac{\partial W}{\partial x_j}(x^c)$ for all $i, j \in \operatorname{supp}(x^c)$; by definition, the set of critical points of W equals $\mathcal{C}(\rho)$, the set of critical states of $Q(\cdot, \rho)$.

A.1.2 PERFECT FORESIGHT EQUILIBRIUM PATHS

Given a payoff function $(Q(\cdot, \rho))_\rho$, the perfect foresight dynamics is defined as in the main text. A path $x\colon [0, \infty) \to \Delta$ is a *feasible path* if it is Lipschitz continuous and for almost all $t \geq 0$, there exists $\alpha(t) \in \Delta$ such that $\dot{x}(t) = \alpha(t) - x(t)$. For a feasible path $x(\,\cdot\,)$, define

$$V(x(\,\cdot\,), t, \rho) = (1+\rho) \int_t^\infty e^{-(1+\rho)(s-t)} Q(x(s), \rho) \mathrm{d}s.$$

A feasible path $x(\,\cdot\,)$ is an *equilibrium path* from $x_0 \in \Delta$ for ρ if $x(0) = x_0$, and for all $i \in S$ and almost all $t \geq 0$,

$$\dot{x}_i(t) > -x_i(t) \Rightarrow i \in \arg\max_{j \in S} V_j(x(\,\cdot\,), t, \rho).$$

The existence of an equilibrium path is guaranteed by the continuity of $Q(\cdot, \rho)$; see Oyama et al. (2008, Subsection 2.3).

Proposition A.1. *For all $\rho > -1$ and for all $x_0 \in \Delta$, there exists an equilibrium path from x_0.*

The equilibrium states of $Q(\cdot, \rho)$ are precisely the stationary states of the perfect foresight dynamics.

Proposition A.2.
(1) *$\bar{x} \in \Delta$ is an equilibrium state if and only if the stationary path at $\bar{x}$ is an equilibrium path.*
(2) *If an equilibrium path converges to $\bar{x} \in \Delta$, then $\bar{x}$ is an equilibrium state.*

Part (1) is by construction; part (2) is due to Oyama et al. (2008, Proposition 2.1).

Finally, the concepts of absorption and global accessibility are defined as in Definition 3.1.

A.1.3 PROOF OF THE STABILITY THEOREM

For general payoff functions $(Q(\cdot, \rho))_\rho$, Assumption A1 in the main text is replaced with the following, while the other assumptions are maintained.

A1'. There exist an open interval $I \subset (-1, \infty)$ with $0 \in I$ and a C^2 function $W\colon \overline{\Delta} \times I \to \mathbb{R}$ such that $W(\cdot, \rho)$ is a potential function of $Q(\cdot, \rho)$ for each $\rho \in I$.

The stability theorem is restated as follows.

Theorem A.3. *Assume A1', A2, and A3. Then there exists $\bar{\rho} > 0$, $\bar{\rho} \in I$, such that the unique maximizer of $W(\cdot, \rho)$ on Δ is absorbing and globally accessible for any $\rho \in (0, \bar{\rho}]$.*

In what follows, we prove the theorem with a series of lemmas. The main obstacle due to the dependence of the potential function on ρ, as discussed in the main text below the statement of Theorem 3.1, is resolved under our regularity assumptions in Lemmas A.6–A.7, where we obtain an ε-neighborhood that isolates the unique potential maximizer from other critical states uniformly for all values of ρ in a neighborhood of $\rho = 0$. The absorption part will then follow from Lemma A.5 which in turn follows from Lemma A.4 (Hofbauer and Sorger (1999, Lemma 4) or Oyama (2009, Lemma C.6)). The global accessibility part will follow from Lemmas A.8–A.9.

For a path $x\colon [0, \infty) \to \Delta$, let $\omega(x(\,\cdot\,))$ denote the set of accumulation points of $x(\,\cdot\,)$, that is, $\hat{x} \in \omega(x(\,\cdot\,))$ if and only if $\hat{x} = \lim_{k\to\infty} x(t_k)$ for some sequence $(t_k)_{k=1}^\infty$ with $\lim_{k\to\infty} t_k = \infty$.

Lemma A.4. *Fix any $\rho > 0$, and suppose that $W(\cdot, \rho)$ is a potential function of $Q(\cdot, \rho)$. For any equilibrium path $x(\,\cdot\,)$,*
(1) *$W(\hat{x}, \rho) \geq W(x(t), \rho)$ for all $\hat{x} \in \omega(x(\,\cdot\,))$ and all $t \geq 0$, and*
(2) *$\omega(x(\,\cdot\,)) \subset \mathcal{C}(\rho)$.*

Lemma A.5. *Fix any $\rho > 0$, and suppose that $W(\cdot, \rho)$ is a potential function of $Q(\cdot, \rho)$ and has a unique maximizer $\bar{x}^\rho$ on Δ. Then for any equilibrium path $x(\,\cdot\,)$, if there exists $t \geq 0$ such that $W(x(t), \rho) > W(x^c, \rho)$ for all $x^c \in \mathcal{C}(\rho) \setminus \{\bar{x}^\rho\}$, then $\lim_{t'\to\infty} x(t') = \bar{x}^\rho$.*

There is a subtlety in proving the global accessibility in our framework: Whereas in Hofbauer and Sorger (1999) and Oyama (2009), the instantaneous payoff function (the function Q in our model) is independent of the discount rate ρ, in ours

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it does vary depending on ρ. Thus, the result of Hofbauer and Sorger (1999) and Oyama (2009) is not directly applicable, and this is where we invoke our regularity condition A3, which allows us to use the Implicit Function Theorem in the proof of the lemma below.

Lemma A.6. *Assume A1′, A2, and A3. Then there exist an open neighborhood U of $\bar{x}^0$ and a compact interval $J \subset I$ with $0 \in \operatorname{int} J$, and a continuous (in fact C^1) function $\phi\colon J \to U$ such that for all $\rho \in J$, $\phi(\rho)$ is a unique maximizer of $W(\cdot, \rho)$ and $\mathcal{C}(\rho) \cap U = \{\phi(\rho)\}$.*

Proof. Let $X^*(\rho) = \arg\max_{x\in\Delta} W(x, \rho)$, and let $\{\bar{x}^0\} = X^*(0)$ as in A2. As noted, $X^*(\rho) \subset \mathcal{C}(\rho)$. By A3(1) and the continuity of $\frac{\partial W}{\partial x_i}$, $i \in S$, we can take an open neighborhood $U^0 \subset \overline{\Delta}$ of $\bar{x}^0$ and an open interval $J^0 \subset I$ with $0 \in J^0$ such that (i) for all $x \in U^0$, $x_i > 0$ for all $i \in \operatorname{supp}(\bar{x}^0)$, and (ii) for all $x \in U^0$ and all $\rho \in J^0$, $\frac{\partial W}{\partial x_i}(x, \rho) > \frac{\partial W}{\partial x_j}(x, \rho)$ for all $i \in \operatorname{supp}(\bar{x}^0)$ and all $j \notin \operatorname{supp}(\bar{x}^0)$.

By the continuity of W in (x, ρ) and the compactness of Δ, the correspondence X^* is nonempty-valued and upper semi-continuous by the Maximum Theorem. Therefore, there exists an open interval $J^1 \subset J^0$ with $0 \in J^1$ such that $X^*(\rho) \subset U^0$ (and $X^*(\rho) \neq \emptyset$) for all $\rho \in J^1$.

Assume without loss of generality that $\operatorname{supp}(\bar{x}^0) = \{1, \ldots, m\}$, $m \leq n$, and define the C^1 function $F\colon U^0 \times J^1 \to \mathbb{R}^n$ by

$$F_i(x, \rho) = \frac{\partial W}{\partial x_i}(x, \rho) - \frac{\partial W}{\partial x_m}(x, \rho), \quad i = 1, \ldots, m-1,$$

$$F_m(x, \rho) = \sum_{i\in S} x_i - 1,$$

$$F_i(x, \rho) = x_i, \quad i = m+1, \ldots, n.$$

By the choice of U^0 and J^1, for $x \in U^0$ and $\rho \in J^1$, $F(x, \rho) = 0$ if and only if $x \in \mathcal{C}(\rho)$.

Then consider $F(x, \rho) = 0$ as an equation in x. We have $F(\bar{x}^0, 0) = 0$, and

$$|D_x F(\bar{x}^0, 0)| = \begin{vmatrix} D_x Q(\bar{x}^0, 0)_C & \mathbf{1}_C \\ \mathbf{1}'_C & 0 \end{vmatrix} \neq 0$$

by A3, where $C = \operatorname{supp}(\bar{x}^0) = \{1, \ldots, m\}$. Therefore, we can apply the Implicit Function Theorem so that we have a compact interval $J \subset J^1$ with $0 \in \operatorname{int} J$, an open neighborhood $U \subset U^0$ of $\bar{x}^0$, and a C^1 function $\phi\colon J \to U$ such that $\phi(0) = \bar{x}^0$, and for all $\rho \in J$, $\phi(\rho)$ is the unique solution to $F(x, \rho) = 0$ in U. Thus, for all $\rho \in J$, we have $\mathcal{C}(\rho) \cap U = \{\phi(\rho)\}$ and hence $X^*(\rho) = \{\phi(\rho)\}$, as desired. □

Lemma A.7. *Assume A1′, A2, and A3, and let J and $\phi(\cdot)$ be as in Lemma A.6. Then there exists $\bar{\varepsilon} > 0$ such that for any $\rho \in J$, if $|x - \phi(\rho)| < \bar{\varepsilon}$, then $W(x, \rho) > W(x^c, \rho)$ for all $x^c \in \mathcal{C}(\rho) \setminus \{\phi(\rho)\}$.*

Proof. Let U be an open set as in Lemma A.6, where we assume that $\Delta \not\subset U$; otherwise, the claim trivially holds. Let $w(\rho) = \max_{x\in\Delta\setminus U} W(x, \rho)$ and $X(\rho) = \{x \in \Delta \mid W(x, \rho) \leq w(\rho)\}$. By the continuity of W and w, X is (nonempty- and) compact-valued and upper semi-continuous. Let $D(\rho) = \min_{x\in X(\rho)} |x - \phi(\rho)|$, which is lower semi-continuous. Let $\bar{\varepsilon} = \min_{\rho\in J} D(\rho)$. Since for every ρ, $\phi(\rho) \notin X(\rho)$ and hence $D(\rho) > 0$, we have $\bar{\varepsilon} > 0$. This $\bar{\varepsilon}$ satisfies the condition in the statement. Indeed, fix $\rho \in J$, and let $|x - \phi(\rho)| < \bar{\varepsilon}$. Then $x \notin X(\rho)$, and therefore $W(x, \rho) > w(\rho) \geq W(x', \rho)$ for all $x' \in \Delta \setminus U$; in particular, $W(x, \rho) > W(x^c, \rho)$ for all $x^c \in \mathcal{C}(\rho) \setminus \{\phi(\rho)\} \subset \Delta \setminus U$ as claimed. □

Now suppose that $W(\cdot, \rho)$ is a potential function of $Q(\cdot, \rho)$ for $\rho > 0$ and consider the following optimization problem:

$$\text{maximize} \ \ \mathcal{W}(x(\cdot), \rho) = (1 + \rho) \int_0^\infty e^{-\rho t} W(x(t), \rho) dt \tag{A1a}$$

$$\text{subject to} \ \ x(\cdot) \in \mathcal{X}(x_0), \tag{A1b}$$

where $\mathcal{X}(x_0)$ is the set of feasible paths from a given initial state $x_0 \in \Delta$, which is compact with respect to the topology of uniform convergence on compact intervals. The continuity of $W(\cdot, \rho)$ implies the continuity of $\mathcal{W}(\cdot, \rho)$, which in turn guarantees the existence of an optimal solution to the problem.

The maximizers of $\mathcal{W}(\cdot, \rho)$ are equilibrium paths of the perfect foresight dynamics; see Hofbauer and Sorger (1999, Theorem 2) or Oyama (2009, Lemma C.2).

Lemma A.8. *Fix any $\rho > 0$, and suppose that $W(\cdot, \rho)$ is a potential function of $Q(\cdot, \rho)$. Then every optimal solution to the problem (A1) is an equilibrium path from x_0.*

The following corresponds to the so-called "Visit Lemma" in the turnpike theory literature (e.g., Scheinkman (1976)).

Lemma A.9. *Assume A1′, A2, and A3, and let J and $\phi(\cdot)$ be as in Lemma A.6. For any $\varepsilon > 0$, there exists $\bar{\rho}(\varepsilon) > 0$, $\bar{\rho}(\varepsilon) \in J$, such that for any $\rho \in (0, \bar{\rho}(\varepsilon)]$ and any $x_0 \in \Delta$, if $x(\cdot)$ is an optimal solution to (A1) for ρ and x_0, then there exists $t \geq 0$ such that $|x(t) - \phi(\rho)| < \varepsilon$.*

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Proof. Assume the contrary: there exists $\varepsilon > 0$ such that for all $\bar{\rho} > 0$, $\bar{\rho} \in J$, there exists an optimal solution $x(\cdot)$ for some $\rho \in (0, \bar{\rho}]$ and some $x_0 \in \Delta$ such that $|x(t) - \bar{x}^0| \geq \varepsilon$ for all $t \geq 0$. Given such an $\varepsilon > 0$, for $\rho \in J$ define $c(\rho)$ by

$$c(\rho) = W(\phi(\rho), \rho) - \max_{x \in \Delta}\{W(x, \rho) \mid |x - \phi(\rho)| \geq \varepsilon\} > 0,$$

which is lower semi-continuous in ρ. Write $m(x, \rho) = W(x, \rho) - (W(\phi(\rho), \rho) - c(\rho)/2)$. Note that $m(\bar{x}^0, 0) > 0$. Since $m(x, \rho)$ is lower semi-continuous in (x, ρ), there exist $d > 0$ and $\rho_0 > 0$, $\rho_0 \in J$, such that $m(x, \rho) > 0$ whenever $|x - \bar{x}^0| \leq d$ and $\rho \in [0, \rho_0]$. Let $T \geq 0$ be such that $e^{-T} \leq d$, and let $\rho_1 \in (0, \rho_0]$ be such that

$$(1 - e^{-\rho_1 T})2M < e^{-\rho_1 T} \min_{\rho' \in [0, \rho_0]} c(\rho')/2,$$

where $M > 0$ is a constant such that $|W(x, \rho)| \leq M$ for all $x \in \Delta$ and $\rho \in [0, \rho_0]$. Given such a ρ_1, let $x(\cdot)$ be an optimal solution for $\rho \in (0, \rho_1]$ and $x_0 \in \Delta$ such that $|x(t) - \phi(\rho)| \geq \varepsilon$ for all $t \geq 0$, as assumed. Note that by the definition of $c(\rho)$, we have $W(\phi(\rho), \rho) - W(x(t), \rho) \geq c(\rho)$ for all $t \geq 0$.

Let $y(\cdot)$ be the feasible path from x_0 given by $y(t) = e^{-t}x_0 + (1 - e^{-t})\bar{x}^0$. Then by the choice of T, for all $t \geq T$ we have $|y(t) - \bar{x}^0| = e^{-t}|x_0 - \bar{x}^0| \leq e^{-t} \leq e^{-T} \leq d$. Therefore, by the choice of d, for all $t \geq T$ we have $m(y(t), \rho) > 0$, or $W(y(t), \rho) - W(\phi(\rho), \rho) > -c(\rho)/2$. Hence,

$$\begin{aligned}
\mathcal{W}(x(\cdot), \rho) - \mathcal{W}(y(\cdot), \rho) &= \rho \int_0^T e^{-\rho t}(W(x(t), \rho) - W(y(t), \rho))\mathrm{d}t \\
&\quad + \rho \int_T^\infty e^{-\rho t}(W(x(t), \rho) - W(y(t), \rho))\mathrm{d}t \\
&< \rho \int_0^T e^{-\rho t} 2M \mathrm{d}t + \rho \int_T^\infty e^{-\rho t}(-c(\rho)/2)\mathrm{d}t \\
&= (1 - e^{-\rho T})2M - e^{-\rho T}c(\rho)/2 \\
&\leq (1 - e^{-\rho_1 T})2M - e^{-\rho_1 T} \min_{\rho' \in [0, \rho_0]} c(\rho')/2 < 0,
\end{aligned}$$

which contradicts the optimality of $x(\cdot)$. □

We are now ready to prove our main theorem.

Proof of Theorem A.3. Assume A1′, A2, and A3, and let J and $\phi(\cdot)$ be as in Lemma A.6. By Lemma A.7, we can take an $\bar{\varepsilon} > 0$ such that for any $\rho \in J$, if $|x - \phi(\rho)| < \bar{\varepsilon}$, then $W(x, \rho) > W(x^c, \rho)$ for all $x^c \in \mathcal{C}(\rho) \setminus \{\phi(\rho)\}$. Then for any $\rho \in J$, $\rho > 0$, we have $\lim_{t \to \infty} x(t) = \phi(\rho)$ for any equilibrium path $x(\cdot)$ with $x(0) \in B_{\bar{\varepsilon}}(\phi(\rho))$ by Lemma A.5, which means that $\phi(\rho)$ is absorbing.

Let $\bar{\rho} = \bar{\rho}(\bar{\varepsilon}) > 0$, $\bar{\rho}(\bar{\varepsilon}) \in J$, be as in Lemma A.9. Fix any $\rho \in (0, \bar{\rho}]$ and any $x_0 \in \Delta$. Let $x(\cdot)$ be a solution to the optimization problem (A1), which is an equilibrium path from x_0 by Lemma A.8. By Lemma A.9, there exists $t \geq 0$ such that $|x(t) - \phi(\rho)| < \bar{\varepsilon}$, and therefore, by Lemma A.7, $W(x(t), \rho) > W(x^c, \rho)$ for all $x^c \in \mathcal{C}(\rho) \setminus \{\phi(\rho)\}$. It follows from Lemma A.5 that we have $\lim_{t' \to \infty} x(t') = \phi(\rho)$. Hence, $\phi(\rho)$ is globally accessible. □

A.1.4 CASE OF STRICTLY CONCAVE POTENTIAL

If the potential function is strictly concave, all the equilibrium paths converge to the unique potential maximizer.

Proposition A.10. *Fix any $\rho > 0$, and suppose that $Q(\cdot, \rho)$ has a potential function $W(\cdot, \rho)$ strictly concave on Δ. Then from any state in Δ, any equilibrium path converges to the unique maximizer of $W(\cdot, \rho)$ on Δ.*

Proof. Let $\bar{x}^\rho \in \Delta$ be the unique maximizer of $W(\cdot, \rho)$ on Δ. By the strict concavity of $W(\cdot, \rho)$, (i) $\bar{x}^\rho$ is a unique KKT point for the problem max $W(x, \rho)$ subject to $x \in \Delta$ and thus is a unique equilibrium state of $W(\cdot, \rho)$, and (ii) for each face of Δ, $W(\cdot, \rho)$ has at most one critical point, so that all the critical points are isolated.

Let $x(\cdot)$ be any equilibrium path. By Lemma A.4(2), $\omega(x(\cdot)) \subset \mathcal{C}(\rho)$, but $\omega(x(\cdot))$ is connected and hence is a singleton by (ii) above, which implies that $x(\cdot)$ converges to some critical state. Since the limit must be an equilibrium state by Proposition A.2(2), it follows that $x(\cdot)$ converges to $\bar{x}^\rho$ by (i). □

A.2 RELEVANCE OF CAPITAL ACCUMULATION

In this section, to demonstrate the relevance of capital accumulation in fostering agglomeration in stable equilibrium states, we compare the analysis in the main text with the hypothetical case where saving is prohibited. For the budget constraint, instead of (5) we impose

$$\phi_i(x_i(t))c_i(\tau, t) = r_i(t)k_i(\tau, t), \tag{A2}$$

where $k_i(\tau, t) = k_0$ for all $\tau \geq t$. The equilibrium rental rate is determined in the same manner:

$$r_i(t) = Z_i(x(t)) + \mu. \tag{A3}$$

The expected lifetime payoff is thus given by

$$\begin{aligned}\check{V}_i(x(\,\cdot\,), \tau) &= (1+\rho)\int_\tau^\infty e^{-(1+\rho)(t-\tau)} \ln \frac{(Z_i(x(t)) + \mu)k_0}{\phi_i(x_i(t))} \mathrm{d}t \\ &= (1+\rho)\int_\tau^\infty e^{-(1+\rho)(t-\tau)} \left[\ln\left(Z_i(x(t)) + \mu\right) - \ln \phi_i(x_i(t))\right] \mathrm{d}t + \ln k_0,\end{aligned}$$

and the stationary payoff by

$$\check{Q}_i(x) = \ln\left(Z_i(x) + \mu\right) - \ln \phi_i(x_i),$$

which is independent of the discount rate ρ here.

As in the main analysis, we restrict our attention to the case where $\check{Q} = (\check{Q}_i)_{i\in S}$ admits a potential function. The necessary and sufficient condition for the existence of a potential function is

$$\frac{z_{ij} - z_{ik}}{Z_i(x) + \mu} + \frac{z_{jk} - z_{ji}}{Z_j(x) + \mu} + \frac{z_{ki} - z_{kj}}{Z_k(x) + \mu} = 0$$

for all $i, j, k \in S$ and $x \in \Delta$ in view of the triangular integrability condition (see Section A.1.1). To simplify the analysis, we assume that the following sufficient condition to hold:

$$z_{ij} = z_i \text{ for all } j \neq i, \tag{A4}$$

where we assume that intraregional spillovers are stronger than interregional spillovers, that is,

$$z_{ii} \geq z_i \tag{A5}$$

for all $i \in S$. We further focus on the case where the returns to capital are sufficiently large relative to time discounting so that

$$z_i + \mu \geq 1 + \rho \tag{A6}$$

for all $i \in S$, which, by (A3), implies that the equilibrium rental rate $r_i(t)$ is always no smaller than the effective discount rate $1 + \rho$. Under these assumptions, the stationary payoff function is written as

$$\check{Q}_i(x) = \ln\left((z_{ii} - z_i)x_i + z_i + \mu\right) - \ln \phi_i(x_i),$$

and the potential function is given by

$$\check{W}(x) = \sum_{i\in S} \int_0^{x_i} \left[\ln\left((z_{ii} - z_i)y + z_i + \mu\right) - \ln \phi_i(y)\right] \mathrm{d}y.$$

On the other hand, under the assumption (A4), the potential function (28) in the main analysis becomes, modulo a constant,

$$W(x) = \sum_{i\in S} \left[\frac{1}{2(1+\rho)} \left((z_{ii} - z_i)x_i^2 + 2z_i x_i\right) - \int_0^{x_i} \ln \phi_i(y)\mathrm{d}y\right].$$

Now we want to demonstrate that agglomeration is "more likely" with capital accumulation than without. Recall from Section 3.4 that strong agglomeration forces lead to the convexity of the potential. We thus compare the convexity of the potential functions W and $\check{W}$ with and without capital accumulation and show that the former is more likely to be convex than the latter.

Consider the function $W(x) - \check{W}(x)$ (as a function defined on a neighborhood of Δ). Its Hessian is a diagonal matrix with diagonal elements

$$\frac{z_{ii} - z_i}{1 + \rho} - \frac{z_{ii} - z_i}{(z_{ii} - z_i)x_i + z_i + \mu} = \frac{(z_{ii} - z_i)[(z_{ii} - z_i)x_i + z_i + \mu - (1+\rho)]}{(1+\rho)[(z_{ii} - z_i)x_i + z_i + \mu]} \geq 0$$

for any $x \in \Delta$, where the inequality follows from (A5) and (A6). This implies that the function $W(x) - \check{W}(x)$ is convex on Δ. Therefore, if $\check{W}(x)$ is convex, then so is $W(x) = \check{W}(x) + (W(x) - \check{W}(x))$.

Proposition A.11. *Assume (A4)–(A6). Then $W(x)$ is convex on Δ whenever $\check{W}(x)$ is convex on Δ.*

Hence, in view of our stability theorem, as long as the discount rate is sufficiently small and capital returns are not too small, full agglomeration is more likely to be the stable equilibrium state with capital accumulation than without capital accumulation.

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