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ON THE EXCEPTIONAL SET OF TRANSCENDENTAL ENTIRE FUNCTIONS IN SEVERAL VARIABLES

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Abstract

We prove that any subset of $\overline{\mathbb{Q}}^m$ (closed under complex conjugation and which contains the origin) is the exceptional set of uncountably many transcendental entire functions over $\mathbb{C}^m$ with rational coefficients. This result solves a several variables version of a question posed by Mahler for transcendental entire functions [*Lectures on Transcendental Numbers*, Lecture Notes in Mathematics, 546 (Springer-Verlag, Berlin, 1976)].

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1. Introduction

An analytic function f over a domain $\Omega \subseteq \mathbb{C}$ is said to be an *algebraic function* over $\mathbb{C}(z)$ if there exists a nonzero polynomial $P \in \mathbb{C}[X, Y]$ for which $P(z, f(z)) = 0$, for all $z \in \Omega$. A function which is not algebraic is called a *transcendental function*.

The study of the arithmetic behaviour of transcendental functions started in 1886 with a letter of Weierstrass to Strauss, proving the existence of such functions taking $\mathbb{Q}$ into itself. Weierstrass also conjectured the existence of a transcendental entire function f for which $f(\overline{\mathbb{Q}}) \subseteq \overline{\mathbb{Q}}$ (as usual, $\overline{\mathbb{Q}}$ denotes the field of all algebraic numbers). Motivated by results of this kind, he defined the *exceptional set* of an analytic function $f : \Omega \to \mathbb{C}$ as

$$S_f = \{\alpha \in \overline{\mathbb{Q}} \cap \Omega : f(\alpha) \in \overline{\mathbb{Q}}\}.$$

Thus, Weierstrass' conjecture can be rephrased as: *does there exist a transcendental entire function f such that $S_f = \overline{\mathbb{Q}}$?* This conjecture was settled in 1895 by Stäckel [4],

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who proved, in particular, that for any $\Sigma \subseteq \overline{\mathbb{Q}}$, there exists a transcendental entire function f for which $\Sigma \subseteq S_f$.

In his classical book [1], Mahler introduced the problem of studying S_f for various classes of functions. After discussing a number of examples, Mahler posed several problems about the admissible exceptional sets for analytic functions, one of which is as follows. Here $B(0,\rho)$ denotes the closed ball with centre 0 and radius ρ in $\mathbb{C}$.

PROBLEM 1.1. Let $\rho \in (0,\infty]$ be a real number. Does there exist for any choice of $S \subseteq \overline{\mathbb{Q}} \cap B(0,\rho)$ (closed under complex conjugation and such that $0 \in S$) a transcendental analytic function $f \in \mathbb{Q}[[z]]$ with radius of convergence ρ for which $S_f = S$?

In 2016, Marques and Ramirez [3] proved that the answer to this question is 'yes' provided that $\rho = \infty$ (that is, for entire functions). Indeed, they proved the following more general result about the arithmetic behaviour of certain entire functions.

LEMMA 1.2 [3, Theorem 1.3]. *Let A be a countable set and let $\mathbb{K}$ be a dense subset of $\mathbb{C}$. For each $\alpha \in A$, fix a dense subset $E_\alpha \subseteq \mathbb{C}$. Then there exist uncountably many transcendental entire functions $f \in \mathbb{K}[[z]]$ such that $f(\alpha) \in E_\alpha$ for all $\alpha \in A$.*

This result was improved by Marques and Moreira in [2] giving an affirmative answer to Mahler's Problem 1.1 for any $\rho \in (0,\infty]$.

In this paper, we consider Mahler's Problem 1.1 in the context of transcendental entire functions of several variables. Although the previous definitions extend to the context of several variables in a very natural way, we shall include them here for the sake of completeness.

An analytic function f over a domain $\Omega \subseteq \mathbb{C}^m$ (we also say that f is *entire* if $\Omega = \mathbb{C}^m$) is said to be *algebraic* over $\mathbb{C}(z_1,\ldots,z_m)$ if it is a solution of a polynomial functional equation

$$P(z_1,\ldots,z_m, f(z_1,\ldots,z_m)) = 0 \quad \text{for all } (z_1,\ldots,z_m) \in \Omega,$$

for some nonzero polynomial $P \in \mathbb{C}[z_1,\ldots,z_m,z_{m+1}]$. A function which is not algebraic is called a transcendental function. (We remark that an entire function in several variables is algebraic if and only if it is a polynomial function just as in the case of one variable.) Let $\mathbb{K}$ be a subset of $\mathbb{C}$ and let f be an analytic function on the polydisc $\Delta(0,\rho) := B(0,\rho_1) \times \cdots \times B(0,\rho_m) \subseteq \mathbb{C}^m$ for some $\rho = (\rho_1,\ldots,\rho_m) \in (0,\infty]^m$. We say that $f \in \mathbb{K}[[z_1,\ldots,z_m]]$ if

$$f(z_1,\ldots,z_m) = \sum_{(k_1,\ldots,k_m)\in\mathbb{Z}_{\geq 0}^m} c_{k_1,\ldots,k_m} z_1^{k_1}\cdots z_m^{k_m},$$

with $c_{k_1,\ldots,k_m} \in \mathbb{K}$ for all $(k_1,\ldots,k_m) \in \mathbb{Z}_{\geq 0}^m$ and for all $(z_1,\ldots,z_m) \in \Delta(0,\rho)$.

The exceptional set S_f of an analytic function $f : \Omega \subseteq \mathbb{C}^m \to \mathbb{C}$ is defined as

$$S_f := \{(\alpha_1,\ldots,\alpha_m) \in \Omega \cap \overline{\mathbb{Q}}^m : f(\alpha_1,\ldots,\alpha_m) \in \overline{\mathbb{Q}}\}.$$

For example, let $f : \mathbb{C}^2 \to \mathbb{C}$ and $g : \mathbb{C}^2 \to \mathbb{C}$ be the transcendental entire functions given by

$$f(w,z) = e^{w+z} \quad \text{and} \quad g(w,z) = e^{wz}.$$

By the Hermite–Lindemann theorem,

$$S_f = \{(\alpha, -\alpha) : \alpha \in \overline{\mathbb{Q}}\} \quad \text{and} \quad S_g = (\overline{\mathbb{Q}} \times \{0\}) \cup (\{0\} \times \overline{\mathbb{Q}}).$$

In general, if $P_1(X,Y), \ldots, P_n(X,Y) \in \overline{\mathbb{Q}}[X,Y]$, then the function

$$f(w,z) = \exp\Big(\prod_{k=1}^{n} P_k(w,z)\Big)$$

has the exceptional set given by

$$S_f = \bigcup_{k=1}^{n} \{(\alpha,\beta) \in \overline{\mathbb{Q}}^2 : P_k(\alpha,\beta) = 0\}.$$

We refer the reader to [1, 5] (and references therein) for more about this subject.

In the main result of this paper, we shall prove that every subset S of $\overline{\mathbb{Q}}^m$ (under some mild conditions) is the exceptional set of uncountably many transcendental entire functions of several variables with rational coefficients.

THEOREM 1.3. *Let m be a positive integer. Then, every subset S of $\overline{\mathbb{Q}}^m$, closed under complex conjugation and such that $(0, \ldots, 0) \in S$, is the exceptional set of uncountably many transcendental entire functions $f \in \mathbb{Q}[[z_1, \ldots, z_m]]$.*

To prove this theorem, we shall provide a more general result about the arithmetic behaviour of a transcendental entire function of several variables.

THEOREM 1.4. *Let X be a countable subset of $\mathbb{C}^m$ and let $\mathbb{K}$ be a dense subset of $\mathbb{C}$. For each $u \in X$, fix a dense subset $E_u \subseteq \mathbb{C}$ and suppose that if $(0, \ldots, 0) \in X$, then $E_{(0,\ldots,0)} \cap \mathbb{K} \neq \emptyset$. Then there exist uncountably many transcendental entire functions $f \in \mathbb{K}[[z_1, \ldots, z_m]]$ such that $f(u) \in E_u$ for all $u \in X$.*

Theorem 1.4 is a several variables extension of the one-variable result due to Marques and Ramirez [3, Theorem 1.3].

2. Proofs

2.1. Proof that Theorem 1.4 implies Theorem 1.3. In the statement of Theorem 1.4, choose $X = \overline{\mathbb{Q}}^m$ and $\mathbb{K} = \mathbb{Q}^* + i\mathbb{Q}$. Write $S = \{u_1, u_2, \ldots\}$ and $\overline{\mathbb{Q}}^m / S = \{v_1, v_2, \ldots\}$ (one of them may be finite) and define

$$E_u := \begin{cases} \overline{\mathbb{Q}} & \text{if } u \in S, \\ \mathbb{K} \cdot \pi^n & \text{if } u = v_n. \end{cases}$$

By Theorem 1.4, there exist uncountably many transcendental entire functions

$$f(z_1,\dots,z_m)=\sum_{k_1\geq 0,\dots,k_m\geq 0} c_{k_1,\dots,k_m} z_1^{k_1}\cdots z_m^{k_m}$$

in $\mathbb{K}[[z_1,\dots,z_m]]$ such that $f(u)\in E_u$ for all $u\in\overline{\mathbb{Q}}^m$. Define $\psi(z_1,\dots,z_m)$ as

$$\psi(z_1,\dots,z_m):=\frac{f(z_1,\dots,z_m)+\overline{f(\overline{z_1},\dots,\overline{z_m})}}{2}.$$

By the properties of the conjugation of power series,

$$\psi(z_1,\dots,z_m)=\sum_{(k_1,\dots,k_m)\in\mathbb{Z}_{\geq 0}^m}\mathrm{Re}(c_{k_1,\dots,k_m})z_1^{k_1}\cdots z_m^{k_m}$$

is a transcendental entire function in $\mathbb{Q}[[z_1,\dots,z_m]]$ since $\mathrm{Re}(c_{k_1,\dots,k_m})$ is rational and nonzero for all $(k_1,\dots,k_m)\in\mathbb{Z}_{\geq 0}^m$ by construction. (Here, as usual, $\mathrm{Re}(z)$ denotes the real part of the complex number z.)

Therefore, it suffices to prove that $S_\psi=S$. In fact, since S is closed under complex conjugation, if $u\in S$, then $\overline{u}\in S$ and thus $f(u)$ and $\overline{f(\overline{u})}$ are algebraic numbers and so is $\psi(u)$. (Observe also that $f(0,\dots,0)=c_{0,\dots,0}\in\overline{\mathbb{Q}}$.) In the case in which $u=v_n$, for some n, we can distinguish two cases. When $v_n\in\mathbb{R}^m$, then $\psi(u)=\mathrm{Re}(f(v_n))$ is transcendental, since $f(v_n)\in\mathbb{K}\cdot\pi^n$. For $v_n\notin\mathbb{R}^m$, we have $\overline{v_n}=v_l$ for some $l\neq n$. Thus, there exist nonzero algebraic numbers γ_1,γ_2 such that

$$\psi(v_n)=\frac{\gamma_1\pi^n+\gamma_2\pi^l}{2},$$

which is transcendental, since $\overline{\mathbb{Q}}$ is algebraically closed and π is transcendental. In conclusion, $\psi\in\mathbb{Q}[[z_1,\dots,z_m]]$ is a transcendental entire function whose exceptional set is S.

2.2. Proof of Theorem 1.4. Let us proceed by induction on m. The case $m=1$ is covered by Lemma 1.2. Suppose that the theorem holds for all positive integers $k\in[1,m-1]$. That is, if $\mathbb{K}$ is a dense subset of $\mathbb{C}$, X is a countable subset of $\mathbb{C}^k$ and E_u is a dense subset in $\mathbb{C}$ for each $u\in X$, then there exist uncountably many transcendental entire functions $f\in\mathbb{K}[[z_1,\dots,z_k]]$ such that $f(u)\in E_u$ for all $u\in X$, for any integer $k\in[1,m-1]$.

Now, let X be a countable subset of $\mathbb{C}^m$ and E_u a fixed dense subset of $\mathbb{C}$ for all $u\in X$. Without loss of generality, we can assume that $(0,\dots,0)\in X$. In this case, by hypothesis, $\mathbb{K}\cap E_{(0,\dots,0)}\neq\emptyset$. To apply the induction hypothesis, we consider the partition of X given by

$$X=\bigcup_{S\in\mathcal{P}_m}X_S,$$

where $\mathcal{P}_m$ denotes the powerset of $[1,m]=\{1,\dots,m\}$ and X_S denotes the set of all $z=(z_1,\dots,z_m)$ in $X\subseteq\mathbb{C}^m$ such that $z_i\neq 0$ if and only if $i\in S$. In particular, $X_\emptyset=\{(0,\dots,0)\}$ and $X_{[1,m]}=X\cap(\mathbb{C}\setminus\{0\})^m$.

Given $S = \{i_1, \dots, i_k\}$ in $\mathcal{Q}_m = \mathcal{P}_m \setminus \{\emptyset, [1, m]\}$ and $z = (z_1, \dots, z_m)$ in $\mathbb{C}^m$, we denote by z_S the element $(z_{i_1}, \dots, z_{i_k}) \in \mathbb{C}^k$. To simplify the exposition, we will assume that $i_1 < \cdots < i_k$ for all $S \in \mathcal{Q}_m$. Our goal is to show that there exist uncountably many ways to construct a transcendental entire function $f \in \mathbb{K}[[z_1, \dots, z_m]]$ given by

$$f(z_1, \dots, z_m) = a_0 + \bigg(\sum_{S \in \mathcal{Q}_m} \bigg(\prod_{i \in S} z_i \bigg) f_S(z_S) \bigg) + f^*(z_1, \dots, z_m),$$

where $a_0 \in E_{(0,\dots,0)} \cap \mathbb{K}$ and, for each $S = \{i_1, \dots, i_k\} \in \mathcal{Q}_m$, the function $f_S : \mathbb{C}^k \to \mathbb{C}$ is a transcendental entire function such that

$$f_S(u_S) \in \frac{1}{\alpha_{i_1} \cdots \alpha_{i_k}} \cdot (E_u - \Theta_{S,u})$$

for all $u = (\alpha_1, \dots, \alpha_m) \in X_S$ with

$$\Theta_{S,u} = a_0 + \sum_{T \in \mathcal{Q}_m, T \neq S} \bigg(\prod_{i \in T} \alpha_i \bigg) f_T(u_T) \in \mathbb{C}.$$

By the induction hypothesis, f_S exists for all $S \in \mathcal{Q}_m$ (noting that if E_u is a dense subset of $\mathbb{C}$, then $(\alpha_{i_1} \cdots \alpha_{i_k})^{-1} \cdot (E_u - \Theta_{S,u})$ is also a dense set). Moreover, we want the function $f^*(z_1, \dots, z_m) \in \mathbb{K}[[z_1, \dots, z_m]]$ to satisfy the condition

$$f^*(u) \in \bigg(E_u - a_0 - \sum_{S \in \mathcal{Q}_m} \bigg(\prod_{i \in S} \alpha_i \bigg) f_S(u_S) \bigg) \tag{2.1}$$

for all $u = (\alpha_1, \dots, \alpha_m) \in X_{[1,m]}$, and $f^*(z_1, \dots, z_m) = 0$ whenever $z_i = 0$ for some i with $1 \le i \le m$. Under these conditions, it is easy to see that if $S \in \mathcal{Q}_m$ and $u \in X_S$, then $f^*(u) = 0$ and $f(u) \in E_u$.

To construct the function $f^* : \mathbb{C}^m \to \mathbb{C}$, let us consider an enumeration $\{u_1, u_2, \dots\}$ of $X_{[1,m]}$, where we write $u_j = (\alpha_1^{(j)}, \dots, \alpha_m^{(j)})$. We construct a function $f^* \in \mathbb{K}[[z_1, \dots, z_m]]$ given by

$$f^*(z_1, \dots, z_m) = \sum_{n=m}^{\infty} P_n(z_1, \dots, z_m) = \sum_{i_1 \ge 1, \dots, i_m \ge 1} c_{i_1,\dots,i_m} z_1^{i_1} \cdots z_m^{i_m},$$

where P_n is a homogeneous polynomial of degree n and the coefficients $c_{i_1,\dots,i_m} \in \mathbb{K}$ will be chosen so that f^* will satisfy the desired conditions.

The first condition is

$$|c_{i_1,\dots,i_m}| < s_{i_1+\cdots+i_m} := \frac{1}{\binom{i_1+\cdots+i_m-1}{m-1}(i_1 + \cdots + i_m)!},$$

where $c_{i_1,\dots,i_n} \neq 0$ for infinitely many m-tuples of integers $i_1 \ge 1, \dots, i_m \ge 1$. These conditions will be used to guarantee that f^* is an entire function. Let $L(P)$ denote the length of the polynomial $P(z_1, \dots, z_m) \in \mathbb{C}[z_1, \dots, z_m]$ given by the sum of the absolute values of its coefficients. Since

$$|P_n(z_1, \dots, z_m)| \le L(P_n) \max\{1, |z_1|, \dots, |z_m|\}^n,$$

for all $n \geq m$ and $(z_1, \ldots, z_m)$ belonging to the open ball $B(0, R)$,

$$|P_n(z_1, \ldots, z_m)| < \frac{\binom{n-1}{m-1}}{\binom{n-1}{m-1} n!} \max\{1, R\}^n = \frac{\max\{1, R\}^n}{n!},$$

since $P_n(z_1, \ldots, z_m)$ has at most $\binom{n-1}{m-1}$ monomials of degree n. Hence, the series $\sum_{n \geq m} P_n(z_1, \ldots, z_m)$ converges uniformly in any of these balls. Thus, f^* is a transcendental entire function such that $f^*(0, z_2, \ldots, z_m) = f^*(z_1, 0, z_3, \ldots, z_m) = f^*(z_1, z_2, \ldots, 0) = 0$.

To obtain the coefficients $c_{i_1,\ldots,i_m} \in \mathbb{K}$ such that f^* satisfies the condition (2.1), we consider a hyperplane $\pi(n, j)$ for positive integers n and j with $1 \leq j \leq n$, given by

$$\pi(n, j) : \mu_{n,1}^{(j)} z_1 + \cdots + \mu_{n,m}^{(j)} z_m - \lambda_n^{(j)} = 0,$$

and such that if u_j, u_{n+1} and the origin are noncollinear, then $\pi(n, j)$ is a hyperplane containing u_j and parallel to the line passing through the origin and the point u_{n+1}, and, if u_j, u_{n+1} and the origin are collinear, then $\pi(n, j)$ is a hyperplane containing u_j and perpendicular to the line passing through the origin and the point u_{n+1}. Note that in both cases, $\lambda_n^{(j)} \neq 0$ and u_{n+1} does not belong to any hyperplane $\pi(n, j)$ with $1 \leq j \leq n$.

Now, we define the polynomials $A_0(z_1, \ldots, z_m) := z_1 \cdots z_m$ and

$$A_n(z_1, \ldots, z_m) := \prod_{j=1}^{n} (\mu_{n,1}^{(j)} z_1 + \cdots + \mu_{n,m}^{(j)} z_m - \lambda_n^{(j)})$$

for all $n \geq 1$. By the definition of $\pi(n, j)$, we have $A_n(u_j) = 0$ for $1 \leq j \leq n$. Since u_{n+1} and the origin do not belong to $\pi(n, j)$, we also have $A_n(0, \ldots, 0) \neq 0$ and $A_n(u_{n+1}) \neq 0$ for all $n \geq 1$. Thus, we can define the function

$$f_{1,0}^*(z_1, \ldots, z_m) := \delta_{1,0} A_0(z_1, \ldots, z_m) = \delta_{1,0} z_1 \cdots z_m$$

such that $\Theta_1 + f_{1,0}^*(u_1) \in E_{u_1}$ and $0 < |\delta_{1,0}| < s_m/m$, where

$$\Theta_j := a_0 + \sum_{S \in Q_m} \Big(\prod_{i \in S} \alpha_i^{(j)} \Big) f_S(u_{j,S}),$$

and $u_{j,S} = (\alpha_{i_1}^{(j)}, \ldots, \alpha_{i_k}^{(j)})$ for $S = \{i_1, \ldots, i_k\}$, for all integers $j \geq 1$.

Since $\mathbb{K}$ is a dense subset of $\mathbb{C}$, we can choose $\delta_{1,1}$ such that the coefficient $c_{1,1,\ldots,1}$ of $z_1 \cdots z_m$ in the function

$$f_{1,1}^*(z_1, \ldots, z_m) := f_{1,0}^*(z_1, \ldots, z_m) + \delta_{1,1} z_1 \cdots z_m A_1^{(1)}(z_1, \ldots, z_m)$$

belongs to $\mathbb{K}$ with $|c_{1,1,\ldots,1}| < s_m$. Therefore, we take

$$f_1^*(z_1, \ldots, z_m) := f_{1,1}^*(z_1, \ldots, z_m),$$

where $P_1(z_1, \ldots, z_m) = c_{1,1,\ldots,1} z_1 \cdots z_m$.

Recursively, we can construct a function $f^*_{n,0}(z_1, \ldots, z_m)$ given by

$$f^*_{n,0}(z_1, \ldots, z_m) := f^*_{n-1}(z_1, \ldots, z_m) + \delta_{n,0} z_1^n z_2 \cdots z_m A_{n-1}(z_1, \ldots, z_m)$$

where we take $\delta_{n,0} \neq 0$ in the ball $B(0, s_{n+m-1}/(n+m-1))$ such that

$$\Theta_n + f^*_{n,0}(u_n) \in E_{u_n}.$$

This is possible since E_{u_n} is a dense subset of $\mathbb{C}$ and all coordinates of u_n are nonzero.

Since $\mathbb{K}$ is a dense subset of $\mathbb{C}$, if we consider the ordering of the monomials of degree $n+m-1$ given by the lexicographical order of the exponents, then we can choose $\delta_{n,l}$ such that the coefficient $c_{j_1,\ldots,j_m}$ of the lth monomial $z_1^{j_1} \cdots z_m^{j_m}$ in

$$f^*_{n,l}(z_1, \ldots, z_m) := f^*_{n,l-1}(z_1, \ldots, z_m) + \delta_{n,l} z_1^{j_1} \cdots z_m^{j_m} A_n(z_1, \ldots, z_m)$$

belongs to $\mathbb{K}$ with $|c_{j_1,\ldots,j_m}| < s_{n+m-1}$. Thus, we define

$$f^*_n(z_1, \ldots, z_m) := f^*_{n,L}(z_1, \ldots, z_m),$$

where $L = \binom{n+m-2}{m-1}$ is the number of distinct monomials of degree $n+m-1$. Then $f^*_n(z_1, \ldots, z_m)$ is a polynomial function such that $c_{j_1,\ldots,j_m} \in \mathbb{K}$ for every m-tuple $(j_1, \ldots, j_m)$ such that $j_1 + \cdots + j_m \leq n+m-1$.

Finally, this construction implies that the functions f^*_n converge to a transcendental entire function $f^* \in \mathbb{K}[[z_1, \ldots, z_m]]$ as $n \to \infty$ such that

$$f^*(u_j) = f^*_n(u_j) = f^*_j(u_j)$$

for all $n \geq j \geq 1$. Let $f : \mathbb{C}^m \to \mathbb{C}$ be the entire function given by

$$f(z_1, \ldots, z_m) = a_0 + \bigg(\sum_{S \in Q_m} \bigg(\prod_{i \in S} z_i\bigg) f_S(z_S)\bigg) + f^*(z_1, \ldots, z_m).$$

Then $f(u) \in E_u$ for all $u \in X \subset \mathbb{C}^m$. Since f is an entire function that is not a polynomial, it follows that f is transcendental. Note that there are uncountably many ways to choose the constants $\delta_{n,j}$. This completes the proof.

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