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A Necessary Condition for Multipliers of Weak Type $(1, 1)$

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Abstract. Simple necessary conditions for weak type $(1, 1)$ of invariant operators on $L(\mathbb{R}^d)$ and their applications to rational Fourier multiplier are given.

In this note we give necessary conditions for a multiplier operator acting on $L^1(\mathbb{R}^d)$ to be of weak type $(1, 1)$. These conditions can be applied to rational multipliers, which arise naturally in the theory of spaces of differentiable functions (see [2]). For example, the multiplier $\phi(x, y) = \frac{xy}{1+x^2y^2}$ was considered in [2] and was shown not to be of weak type $(1, 1)$. The proof in [2] is based on the fact that the kernel of the operator corresponding to a multiplier that is defined as a function of the product xy (such as ϕ) is itself a function of the product of variables. The main difficulty in [2] was to find the kernel of the multiplier. Our approach in this paper is simpler and more general. Our proof remains entirely on the multiplier side and does not use the algebraic properties of the multiplier. However it gives no satisfactory information of the asymptotic of the norm of the multiplier transform in L^p as p tends to 1.

Let G be a locally compact abelian group, Γ its dual. For $\phi \in L^\infty(\Gamma)$ denote by T_ϕ the $L^2(G)$ multiplier transform defined by ϕ, *i.e.* $T_\phi f = (\phi \hat{f})^\vee$. We put $N_1^{(w)}(\phi) = \sup_{c>0} c \cdot \left| \{t : |T_\phi f(t)| > c\} \right|$. We say that T_ϕ is of weak type $(1, 1)$ iff $N_1(w)(\phi) < \infty$.

Proposition 1 *Let $\phi \colon \mathbb{R}^d \to \mathbb{C}$ be a bounded continuous function. Assume that there exist $a \in \mathbb{R}^d$, a sequence $(a_j)_{j=1}^\infty \subset \mathbb{R}^d$ and $C > 0$ such that*

$$\lim_{j\to\infty} |\langle a, a_j \rangle| = \infty, \tag{1}$$

$$|\phi(a_j)| > C \quad \text{for } j = 1, 2, \dots, \tag{2}$$

$$\lim_{j\to\infty} \phi(x \pm a_j) = 0 \quad \text{for } x \neq \lambda a. \tag{3}$$

Then T_ϕ is not of weak type $(1, 1)$.

Corollary 1 *Let $f \colon \mathbb{R} \to \mathbb{C}$ be a continuous non-constant function satisfying $\lim_{|t|\to\infty} f(t) = 0$ and let $\phi(x, y) = f(xy)$. Then T_ϕ is not of weak type $(1, 1)$.*

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Proof Put $a = (1, 0)$, $s \neq 0$ such that $f(s) \neq 0$ and $a_j = (j, sj^{-1})$ for $j = 1, 2, \ldots$.

Example 1 The multiplier transform of the function $\phi(x, y) = xy(1 + x^2y^2)^{-1}$ is not of weak type $(1, 1)$.

The proof of Proposition 1 is based on the following lemma.

Lemma 1 *Suppose that there exists a sequence $(p_j)_{j=1}^{\infty} \subset \mathbb{R}_+$ with $\lim_{n\to\infty} p_n = \infty$ and that for every $n = 1, 2, \ldots$ and $\varepsilon > 0$ there exists a sequence $(b_j)_{j=1}^{n} \subset \mathbb{R}^d$ satisfying $|b_{j+1}| > 3|b_j|$ for $j = 1, 2, \ldots, n-1$, with the following property. Put*

$$A_n = \{\epsilon_1 b_1 + \cdots + \epsilon_n b_n : \epsilon_j = 0, 1, -1 \text{ for } j = 1, 2, \ldots, n\} \setminus \{0\}$$

and

$$B_n = \{\pm b_1, \ldots, \pm b_n\}.$$

Then

$$\sum_{x \in B_n} |\phi(x)|^2 > p_n, \tag{4}$$

and

$$|\phi(x)| < \varepsilon \quad \text{for } x \in A_n \setminus B_n. \tag{5}$$

Then T_ϕ is not of weak type $(1, 1)$.

Proof Fix $n \in \mathbb{Z}_+$ and $\varepsilon < 3^{-n}$. Let $(b_k)_{k=1}^{n}$ be a sequence satisfying (4) and (5), and let $p \in \mathbb{Z}_+$ be such that all coordinates of vectors pb_k are integers for $k = 1, 2, \ldots, n$. Define $R_n \colon \mathbb{T}^d \to \mathbb{C}$ by

$$R_n(\xi) = \prod_{j=1}^{n} (1 + \cos\langle pb_n, \xi\rangle).$$

Clearly $\|R_n\|_1 = 1$ and $\|T_{\hat{R}_n} \colon L^1(\mathbb{T}^d) \to L^1(\mathbb{T}^d)\| = 1$. Put $\phi_p(z) = \phi(p^{-1}z)$. Obviously $N_1^{(w)}(\phi_p) = N_1^{(w)}(\phi)$. Let $\lambda = \phi_p|\mathbb{Z}^d$. By the weak type transference theorem (*cf.* [1], [3, Proposition 1]), λ is a weak type $(1, 1)$ multiplier on $L^1(\mathbb{T}^d)$ with norm $N_1^{(w)}(\lambda) \le N_1^{(w)}(\phi)$. Therefore $T_\lambda \circ T_{\hat{R}_n}$ is of weak type $(1, 1)$ and

$$N_1^{(w)}(\lambda \hat{R}_n) \le N_1^{(w)}(\phi). \tag{6}$$

Define now the function $g \colon \mathbb{Z}^d \to \mathbb{C}$ by the formula

$$g(m) = \begin{cases} \phi_p(m)\hat{R}_n(m), & \text{if } m = p \cdot \sum_{j=1}^{n} \varepsilon_j b_j \text{ and } \sum_{j=1}^{n} |\varepsilon_j| \ge 2 \\ 0, & \text{otherwise.} \end{cases}$$

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By (5),

$$\sum_{m\in\mathbb{Z}^d}|g(m)| < \sup_{g(m)\neq 0}|\phi_p(m)|\cdot\sum_{m\in\mathbb{Z}^d}|\hat{R}_n(m)| < \varepsilon\cdot 3^n = 1$$

Hence $\|T_g\colon L^1(\mathbb{T}^d)\to L^1(\mathbb{T}^d)\|\leq 1$. Thus, by (6), the operator $T_\lambda\circ T_{\hat{R}_n} - T_g$ is of weak type $(1,1)$ and

$$N_1^{(w)}(\lambda\hat{R}_n - g)\leq 2(N_1^{(w)}(\phi)+1). \tag{7}$$

But $\rho_n = \lambda\hat{R}_n - g = \phi_p\mathbf{1}_{M_n}$, where $M_n = \{0, pb_1, -pb_1, pb_2, -pb_2, \dots, pb_n, -pb_n\}$. Fix now $0 < q < 1$. It is well-known that every operator of weak type $(1,1)$ is bounded from L^1 to L^q. Therefore there exists $C > 0$ such that for every $f\in L^1(\mathbb{T}^d)$ and $n = 1, 2, \dots,$

$$\|T_{\rho_n} f\|_q \leq C\|f\|_1. \tag{8}$$

Clearly $M_n = M_n^+\cup\{0\}\cup M_n^-$ where M_n^+ and M_n^- are Hadamard sequences such that the ratio between any two of their consecutive elements is greater than 2. Therefore M_n is a $\Lambda(2)$ set, *i.e.*

$$\|f\|_2 < K\|f\|_q \tag{9}$$

for every f with supp $\hat{f}\subset M_n$, and the constant $K > 0$ does not depend on $n = 1, 2, \dots$. Formulas (8) and (9) yield together that $\|T_{\rho_n}\colon L^1(\mathbb{T}^d)\to L^2(\mathbb{T}^d)\|\leq CK$ for $n = 1, 2, \dots$. This leads to a contradiction, because for every finite set $E\subset\mathbb{Z}^d$ there exists a trigonometric polynomial $h\in L^1(\mathbb{T}^d)$ with $\|h\|_1 < 2$ such that $\hat{h}(m) = 1$ for $m\in E$. Taking M_n as E we get by (5),

$$C^2K^2\|h\|_1^2 > \|T_{\rho_n}h\|_2^2 > \sum_{j=1}^n|\phi(b_j)|^2 > p_n\to\infty.$$

Proof of Proposition 1 Assume that (1)–(3) holds. Obviously, since ϕ is continuous, we can assume that all coordinates of all points a_j are rational, and moreover, the pairs of vectors a and a_j are linearly independent for $j = 1, 2, \dots$. Let $a_j = \lambda_j a + d_j$ where $\langle a, d_j\rangle = 0$. Then $d_j\neq 0$ for $j = 1, 2, \dots$ and, by (2) and (3), we get $|d_j|\to 0$ as $j\to\infty$. We define the subsequence $(b_j)_{j=1}^n$ inductively. Let us suppose that we have already chosen $b_1, b_2, \dots, b_{m-1}$ with properties (4), (5) and, additionally

$$a \text{ and } b \text{ are linearly independent}\quad\text{for } b\in A_{m-1}.$$

Then, by (3), $\lim_{j\to\infty}|\phi(b\pm a_j)| = 0$ for every $b\in A_{m-1}$. Since A_{m-1} is finite, we can choose k such that $|\phi(b\pm a_k)| < \varepsilon$ for $b\in A_{m-1}$. Moreover, since $\lim_{j\to\infty}|d_j| = 0$ choosing k big enough we get that for every $b\in A_{m-1}$ the vectors a and $b\pm a_k$ are linearly independent and $|a_k| > 3|b_{m-1}|$. Then we put $b_m = a_k$. ■

Lemma 1 has a wide range of application. We show two other possibilities.

Proposition 2 *Let $\phi \in L^\infty(\mathbb{R}^d)$ be a continuous function on $\mathbb{R}^d \setminus \{0\}$. Suppose that there exist: $a \in \mathbb{R}^d$, a sequence $(a_j)_{j=1}^\infty \subset \mathbb{R}^d$ with $\lim_{j\to\infty} |\langle a, a_j\rangle| = \infty$, and a continuous positive function $\psi\colon \mathbb{R} \to \mathbb{R}$ satisfying $\lim_{|t|\to\infty} \psi(t) = 0$, such that*

$$\lim_{n\to\infty} \langle a, a_n\rangle = \infty, \tag{10}$$

$$|\phi(a_n)| > C > 0 \quad \text{for } n = 1, 2, \dots, \tag{11}$$

$$\lim_{n\to\infty} |\phi(x \pm a_n)| < \psi(\langle a, x\rangle) \quad \text{for every } x \in \mathbb{R}^d. \tag{12}$$

Then T_ϕ is not of weak type $(1, 1)$.

Proof Assume that (10)–(12) hold. Since ϕ is continuous, we can assume that all coordinates of all points a_j are rational. We are going to select the sequence $(b_j)_{j=1}^n$ inductively. Suppose that $b_1, b_2, \dots, b_k$ are already chosen and they satisfy

$$|\phi(b_j)| > C \quad \text{for } j = 1, 2, \dots, k$$

and

$$|\langle a, x\rangle| > r_\varepsilon \quad \text{for } x \in A_k.$$

where r_ε is such a number that $|\psi(t)| < \varepsilon$ for $|t| > r_\varepsilon$. Then we put $b_{k+1} = a_N$ where N is sufficiently big to satisfy:

$$|\langle a, a_N\rangle| > 2 \sup_{x\in A_k} |\langle a, x\rangle| + r_\varepsilon.$$

Then, by (12), $|\phi(x)| < \varepsilon$ for $x \in A_{k+1}$. Clearly for $(b_j)_{j=1}^n$ defined in this way we get $\sum_{x\in B_n} |\phi(x)|^2 \geq Cn$. ■

Example 2 Let $f\colon \mathbb{R} \to \mathbb{R}$ be an odd increasing unbounded function such that $\lim_{x\to\infty} f(x)x^{-1} = 0$. Then T_ϕ is not of weak type $(1, 1)$ for $\phi(x, y) = \frac{1}{1+(y-f(x))^2}$.

Proposition 3 *Let $\mathbb{R}^d = R^p \times \mathbb{R}^q$ and let $\phi \in L^\infty(\mathbb{R}^d)$ be a continuous function on $\mathbb{R}^d \setminus 0$. Suppose that there exists an odd function $\lambda\colon \mathbb{R}^p \to \mathbb{R}^q$ such that*

$$\liminf_{|x|\to\infty} \bigl| \phi\bigl(x, \lambda(x)\bigr) \bigr| > 0, \tag{13}$$

and for every $c_1, c_2, \dots, c_k \in \mathbb{R}$ $(k = 2, 3, \dots)$ with $0 < |c_1| < |c_2| < \cdots < |c_k|$,

$$\lim_{|x|\to\infty} \phi\Bigl(\sum c_j x, \sum \lambda(c_j x)\Bigr) = 0. \tag{14}$$

Then T_ϕ is not of weak type $(1, 1)$.

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Proof Assume that (13) and (14) hold. Let $x_j = 3^j x_0 \in \mathbb{R}^p$ and $a_j = \big(x_j, \lambda(x_j)\big) \in \mathbb{R}^d$ for $j = 1, 2, \dots$. We are going to show now that the sequence defined by $b_k = a_{j+k}$ for $k = 1, 2, \dots, n$ satisfies (4) and (5) provided j is chosen sufficiently big. Indeed, (4) follows directly from (13) and for (5) we have for $x = \sum_{k=m}^{n} \epsilon_k b_k \neq 0$

$$\begin{aligned}\phi\Big(\sum_{k=1}^{n} \epsilon_k b_k\Big) &= \phi\Big(\sum_{k=1}^{n} \epsilon_k a_{k+j}\Big)\\ &= \phi\Big(\sum_{k=1}^{n} \epsilon_k x_{k+j}, \sum_{k=1}^{n} \lambda(\epsilon_k x_{k+j})\Big)\\ &= \phi\Big(\sum_{k=1}^{n} \epsilon_k 3^k x_j, \sum_{k=1}^{n} \lambda(\epsilon_k 3^k x_j)\Big)\end{aligned}$$

By (14) the last expression tends to 0 as $j \to \infty$. Thus we can find an index j such that $\phi(x) < \varepsilon$ for every choice of (ϵ_k). ■

Corollary 3 *Let* $\lambda\colon \mathbb{R} \to \mathbb{R}$ *and* $\mu\colon \mathbb{R} \to \mathbb{R}$ *satisfy: (i)* λ *is an odd and increasing function, (ii) that for every* $c_1, \dots c_k \in \mathbb{R}$ $(k = 2, 3, \dots)$, *with* $0 < |c_1| < |c_2| < \cdots < |c_k|$,

$$\lim_{|x|\to\infty} \frac{\mu(x)}{\lambda(\sum c_j x) - \sum \lambda(c_j x)} = 0.$$

Then for every continuous function $f\colon \mathbb{R} \to \mathbb{R}$ *with* $\lim_{|x|\to\infty} f(x) = 0$ *the multiplier transform of the function* $\phi(x, y) = f(\frac{y-\lambda(x)}{\mu(x)})$ *is not of weak type* $(1, 1)$.

Example 3 The multiplier transform of the function $\phi(x, y) = \frac{(|x|+1)^{1/2}}{(|x|+1)^{1/2}+(y-x^{1/3})^2}$ is not of weak type $(1, 1)$.

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