THE MEAN VALUE OF THE ARTIN L-SERIES AND ITS DERIVATIVE OF A CUBIC FIELD

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1. Introduction. Let K be a non-abelian cubic field of discriminant D, and $\zeta_K(s)$ its Dedekind zeta-function. Set $\psi(s)=\zeta_K(s)/\zeta(s)$. Then it is known that $\psi(s)$ is the Artin L-series associated with the field K. It is also known that $\psi(s)$ is an entire function of order 1.

If K is not a totally real field then $\psi(s)$ satisfies the functional equation

$$\psi(1-s)=\frac{2}{\sqrt{D}}\left(\frac{\sqrt{D}}{2\pi}\right)^{2s}\sin\pi s\,\Gamma^2(s)\psi(s).$$

If K is a totally real field then $\psi(s)$ satisfies the functional equation

$$\psi(1-s)=\frac{4}{\sqrt{D}}\left(\frac{\sqrt{D}}{2\pi}\right)^{2s}\cos^2\tfrac{1}{2}\pi s\,\Gamma^2(s)\psi(s).$$

Barrucand, in [**1**], has given asymptotic formulae for certain coefficient sums of $\psi(s)$. Here, using these results, and the methods of [**2**], [**5**] we prove the following:

THEOREM 1.

$$\int_0^\infty|\psi(\tfrac{1}{2}+it)|^2e^{-\delta t}\,dt=\frac{2A}{\delta}\log\frac{1}{\delta}+O\left(\frac{1}{\delta}\right)$$

for sufficiently small $\delta>0$.

$\left(\text{The positive constant } A=\dfrac{6\mathrm{L}d_3(1)\psi(1)D(2)E(2)}{\pi^2D(1)E(1)} \text{ is defined in [1, p. 962-A].}\right)$

COROLLARY 1.

$$\int_0^T|\psi(\tfrac{1}{2}+it)|^2\,dt\sim 2AT\log T.$$

THEOREM 2.

$$\int_0^\infty|\psi'(\tfrac{1}{2}+it)|^2e^{-\delta t}\,dt=\frac{8A}{3\delta}\log^3\frac{1}{\delta}+O\left(\frac{1}{\delta}\log^2\frac{1}{\delta}\right)$$

for sufficiently small $\delta>0$.

COROLLARY 2.

$$\int_0^T|\psi'(\tfrac{1}{2}+it)|^2\,dt\sim\tfrac{8}{3}AT\log^3T.$$

2. Lemmas.

LEMMA 1 (Van der Corput [**4**, p. 61]). *Let $F(x)$ and $G(x)$ be real functions, $G(x)/F'(x)$ monotonic and $F'(x)/G(x) \geq m > 0$, or $F'(x)/G(x) \leq -m < 0$, throughout the interval (a, b). Then*

$$\left| \int_a^b G(x) e^{iF(x)}\, dx \right| \leq \frac{4}{m}.$$

LEMMA 2 (Euler Summation [**4**, p. 13]). *Let $\phi(x)$ be a real function with a continuous derivative in the interval (a, b). If, for $a \leq x \leq b$, $\phi'(x) \geq 0$ or $\phi'(x) \leq 0$, then*

$$\sum_{a \leq n \leq b} \phi(n) = \int_a^b \phi(x)\, dx + O(|\phi(a)| + |\phi(b)|).$$

The proof of the following lemmas follows easily from [**1**] and [**2**, p. 124], and will be omitted.

LEMMA 3. *Let $\psi(s) = \sum_{n=1}^{\infty} a(n) n^{-s}$ $(\sigma > 1)$. Then*

$$\sum_{n \leq x} \frac{a^2(n)}{n} = A \log x + O(1)$$

and

$$\sum_{n \leq x} \frac{a^2(n) \log n}{n} = \tfrac{1}{2} A \log^2 x + O(\log x).$$

(The constant A has been defined previously.)

LEMMA 4. *Let $\psi'(s) = \sum_{n=1}^{\infty} b(n) n^{-s}$ $(\sigma > 1)$. (Thus $b(n) = -a(n) \log n$.) Then, for sufficiently small $\beta > 0$, we have*

$$\sum_{n=1}^{\infty} \frac{a^2(n)}{n} e^{-n\beta} = A \log \frac{1}{\beta} + O(1),$$

$$\sum_{n=1}^{\infty} \frac{b^2(n)}{n} e^{-n\beta} = \tfrac{1}{3} A \log^3 \frac{1}{\beta} + O\left(\log^2 \frac{1}{\beta}\right)$$

and

$$\sum_{n=1}^{\infty} \frac{a^2(n) \log n}{n} e^{-n\beta} = O\left(\log^2 \frac{1}{\beta}\right).$$

LEMMA 5.

$$\sum_{n=1}^{\infty} \frac{\log^2 n}{n} e^{-n\beta} = \tfrac{1}{3} \log^3 \frac{1}{\beta} + O\left(\log^2 \frac{1}{\beta}\right)$$

and

$$\sum_{n=1}^{\infty} \frac{\log n}{n} e^{-n\beta} = O\left(\log^2 \frac{1}{\beta}\right).$$

3. Proof of Theorem 1. Throughout the rest of the paper we assume K is not totally real. The results and methods for K totally real are exactly the same as for K not totally real.

Now we have, for say $\sigma \geq 0$, and some constant $C > 0$,

$$\psi(s) = O(|t|^C).$$

This follows easily from the functional equation and an application of the Phragmén-Lindelöf Theorem.

Now we consider the integral

$$\frac{1}{2\pi i} \int_{2-i\infty}^{2+i\infty} \Gamma(s)\psi(s) z^{-s}\, ds = \sum_{n=1}^{\infty} \frac{a(n)}{2\pi i} \int_{2-i\infty}^{2+i\infty} \Gamma(s)(nz)^{-s}\, ds = \sum_{n=1}^{\infty} a(n) e^{-nz} \quad (\text{Re } z > 0).$$

Moving the line of integration to $\sigma = \alpha$ $(0 < \alpha < 1)$ we get

$$\frac{1}{2\pi i} \int_{\alpha-i\infty}^{\alpha+i\infty} \Gamma(s)\psi(s) z^{-s}\, ds = \sum_{n=1}^{\infty} a(n) e^{-nz} = \phi_0(z),$$

say. Hence, as in [**4**, p. 137], we have

$$\int_0^{\infty} |\psi(\tfrac{1}{2} + it)|^2 e^{-2\delta t}\, dt = \int_0^{\infty} |\phi_0(ixe^{-i\delta})|^2\, dx + O(1)$$

for sufficiently small $\delta > 0$.

Now we remark that

$$\phi_0\left(\frac{1}{ixe^{-i\delta}}\right) = \frac{2\pi}{\sqrt{D}}\, ixe^{-i\delta} \phi_0\left(\frac{4\pi^2}{D}\, ixe^{-i\delta}\right).$$

This transformation formula may be proven as in [**4**, p. 142], using the functional equation.

Now, as in [**2**, pp. 125–126], we have

$$\begin{aligned}
\int_0^{2\pi/\sqrt{D}} |\phi_0(ixe^{-i\delta})|^2\, dx &= \int_{\sqrt{D}/(2\pi)}^{\infty} \left|\phi_0\left(\frac{i}{x} e^{-i\delta}\right)\right|^2 \frac{dx}{x^2} \\
&= \int_{\sqrt{D}/(2\pi)}^{\infty} \left|\phi_0\left(\frac{1}{ixe^{-i\delta}}\right)\right|^2 \frac{dx}{x^2} \\
&= \frac{4\pi^2}{D} \int_{\sqrt{D}/(2\pi)}^{\infty} \left|\phi_0\left(\frac{4\pi^2}{D}\, ixe^{-i\delta}\right)\right|^2 dx \\
&= \int_{2\pi/\sqrt{D}}^{\infty} |\phi_0(ixe^{-i\delta})|^2\, dx.
\end{aligned}$$

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Now

$$\begin{aligned}
&\int_{2\pi/\sqrt{D}}^{\infty} |\phi_0(ixe^{-i\delta})|^2\,dx \\
&= \int_{2\pi/\sqrt{D}}^{\infty} \sum_{n=1}^{\infty} a(n) \sum_{m=1}^{\infty} a(m) e^{-n(ixe^{-i\delta})} e^{-m(-ixe^{i\delta})}\,dx \\
&= \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \frac{a(n)a(m)}{ine^{-i\delta}-ime^{i\delta}} e^{2\pi(-ine^{-i\delta}+ime^{i\delta})/\sqrt{D}} \\
&= \frac{1}{2\sin\delta}\sum_{n=1}^{\infty}\frac{a^2(n)}{n} e^{-4\pi n\sin\delta/\sqrt{D}} \\
&\quad +2\sum_{m=2}^{\infty} a(m)\sum_{n=1}^{m-1} a(n)\frac{(m+n)\sin\delta\cos[2\pi(m-n)\cos\delta/\sqrt{D}]}{(m+n)^2\sin^2\delta+(m-n)^2\cos^2\delta} e^{-2\pi(m+n)\sin\delta/\sqrt{D}} \\
&\quad -2\sum_{m=2}^{\infty} a(m)\sum_{n=1}^{m-1} a(n)\frac{(m-n)\cos\delta\sin[2\pi(m-n)\cos\delta/\sqrt{D}]}{(m+n)^2\sin^2\delta+(m-n)^2\cos^2\delta} e^{-2\pi(m+n)\sin\delta/\sqrt{D}} \\
&= A_1(\delta)+2A_2(\delta)-2A_3(\delta),
\end{aligned}$$

say. By Lemma 4,

$$A_1(\delta)=\frac{A}{2\delta}\log\left(\frac{1}{\delta}\right)+O\left(\frac{1}{\delta}\right).$$

Also $A_2(\delta)$ may be evaluated, as in [**4**, p. 145], to give

$$A_2(\delta)=O\left(\frac{1}{\delta}\right).$$

The sum $A_3(\delta)$ is slightly more complicated, and may be evaluated as in [**3**, p. 150] to give

$$A_3(\delta)=O\left(\frac{1}{\delta}\right).$$

Collecting these extimates, we obtain

$$\int_{2\pi/\sqrt{D}}^{\infty} |\phi_0(ixe^{-i\delta})|^2\,dx=\frac{A}{2\delta}\log\frac{1}{\delta}+O\left(\frac{1}{\delta}\right),$$

and this gives

$$\int_0^{\infty} |\psi(\tfrac{1}{2}+it)|^2 e^{-\delta t}\,dt=\frac{2A}{\delta}\log\frac{1}{\delta}+O\left(\frac{1}{\delta}\right).$$

Corollary 1 now follows from the theorem of [**4**, p. 136].

4. Proof of Theorem 2. We consider the integral

$$\frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty}\Gamma(s)\psi'(s)z^{-s}\,ds=\sum_{n=1}^{\infty}\frac{b(n)}{2\pi i}\int_{2-i\infty}^{2+i\infty}\Gamma(s)(nz)^{-s}\,ds=\sum_{n=1}^{\infty} b(n)e^{-nz}\quad (\operatorname{Re} z>0).$$

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Again, moving the line of integration to $\sigma=\alpha$ $(0<\alpha<1)$, we get

$$\frac{1}{2\pi i}\int_{\alpha-i\infty}^{\alpha+i\infty}\Gamma(s)\psi'(s)z^{-s}\,ds=\sum_{n=1}^{\infty}b(n)e^{-nz}=\phi_1(z),$$

say.

We remark that

$$\phi_1\left(\frac{1}{ixe^{-i\delta}}\right)=[ixe^{-i\delta}]\left[\frac{4\pi}{\sqrt{D}}\right]\left[\tfrac{1}{2}\phi_1\left(\frac{4\pi^2}{D}\,ixe^{-i\delta}\right)+\log\left(\frac{\sqrt{D}}{2\pi}\right)\phi_0\left(\frac{4\pi^2}{D}\,ixe^{-i\delta}\right)\right.$$
$$\left.-\log x\phi_0\left(\frac{4\pi^2}{D}\,ixe^{-i\delta}\right)+i\delta\phi_0\left(\frac{4\pi^2}{D}\,ixe^{-i\delta}\right)\right]+O(x^{\alpha}).$$

This transformation formula may be proven using the functional equation, as in the first part.

Now, as in the first part,

$$\int_0^{2\pi/\sqrt{D}}|\phi_1(ixe^{-i\delta})|^2\,dx=\int_{\sqrt{D}/(2\pi)}^{\infty}|\phi_1(ixe^{-i\delta})-2\log(2\pi/\sqrt{D})\phi_0(ixe^{-i\delta})$$
$$-2\log x\phi_0(ixe^{-i\delta})+2i\delta\phi_0(ixe^{-i\delta})+O(x^{\alpha-1})|^2\,dx$$
$$=\int_{\sqrt{D}/(2\pi)}^{\infty}|\phi_1(ixe^{-i\delta})|^2\,dx$$
$$-2\int_{\sqrt{D}/(2\pi)}^{\infty}\log x\phi_1(ixe^{-i\delta})\phi_0(-ixe^{i\delta})\,dx$$
$$-2\int_{\sqrt{D}/(2\pi)}^{\infty}\log x\phi_1(-ixe^{i\delta})\phi_0(ixe^{-i\delta})\,dx$$
$$+4\int_{\sqrt{D}/(2\pi)}^{\infty}\log^2 x\,|\phi_0(ixe^{-i\delta})|^2\,dx+O\left(\frac{1}{\delta}\log^2\frac{1}{\delta}\right)$$
$$=\int_{2\pi/\sqrt{D}}^{\infty}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}b(n)b(m)e^{-n(ixe^{-i\delta})}e^{-m(-ixe^{i\delta})}\,dx$$
$$-2\int_{2\pi/\sqrt{D}}^{\infty}\log x\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}b(n)a(m)e^{-n(-ixe^{i\delta})}e^{-m(ixe^{-i\delta})}\,dx$$
$$-2\int_{2\pi/\sqrt{D}}^{\infty}\log x\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}b(n)a(m)e^{-n(ixe^{-i\delta})}e^{-m(-ixe^{i\delta})}\,dx$$
$$+4\int_{2\pi/\sqrt{D}}^{\infty}\log^2 x\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}a(n)a(m)e^{-n(ixe^{-i\delta})}e^{-m(-ixe^{i\delta})}\,dx$$
$$+O\left(\frac{1}{\delta}\log^2\frac{1}{\delta}\right).$$

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Now let us look at the terms with $n=m$ in the above sum. They equal

$$\sum_{n=1}^{\infty} a^2(n) \int_{2\pi/\sqrt{D}}^{\infty} \log^2(nx^2) e^{-2nx\sin\delta}\, dx.$$

This last sum equals, upon an integration by parts,

$$\frac{1}{2\sin\delta} \sum_{n=1}^{\infty} \frac{\log^2(4\pi^2 n/D)a^2(n)}{n} e^{-4\pi n \sin\delta/\sqrt{D}} + \sum_{n=1}^{\infty} \frac{a^2(n)}{2n\sin\delta} \int_{2\pi/\sqrt{D}}^{\infty} \frac{4\log(nx^2)}{x} e^{-2nx\sin\delta}\, dx$$
$$= B_1(\delta) + B_2(\delta), \text{ say.}$$

By Lemma 4,

$$B_1(\delta) = \frac{A}{6\delta} \log^3 \frac{1}{\delta} + O\left(\log^2 \frac{1}{\delta}\right).$$

Substituting $3nx\sqrt{D}/(2\pi)$ for x, we find

$$\begin{aligned} B_2(\delta) &= \sum_{n=1}^{\infty} \frac{4a^2(n)}{n\sin\delta} \int_{3n}^{\infty} \frac{\log x}{x} e^{-4\pi x \sin\delta/(3\sqrt{D})}\, dx \\ &\quad - \sum_{n=1}^{\infty} \frac{2a^2(n)}{n\sin\delta} \int_{3n}^{\infty} \frac{\log n}{x} e^{-4\pi x \sin\delta/(3\sqrt{D})}\, dx \\ &\quad - \sum_{n=1}^{\infty} \frac{2a^2(n)}{n\sin\delta} \int_{3n}^{\infty} \frac{\log(9D/(4\pi^2))}{x} e^{-4\pi x \sin\delta/(3\sqrt{D})}\, dx \\ &= B_{21}(\delta) - B_{22}(\delta) - B_{23}(\delta), \end{aligned}$$

say.

Now by Lemma 2, we find

$$B_{21}(\delta) = \sum_{n=1}^{\infty} \frac{4a^2(n)}{n\sin\delta} \sum_{m=3n}^{\infty} \frac{\log m}{m} e^{-4\pi m \sin\delta/(3\sqrt{D})} + O\left(\sum_{n=1}^{\infty} \frac{a^2(n)\log n}{n^2 \sin\delta} e^{-4\pi n \sin\delta/\sqrt{D}}\right).$$

By interchanging the order of summation, we obtain

$$B_{21}(\delta) = \frac{4}{\sin\delta} \sum_{m=3}^{\infty} \frac{\log m}{m} e^{-4\pi m \sin\delta/(3\sqrt{D})} \sum_{n=1}^{[m/3]} \frac{a^2(n)}{n} + O\left(\sum_{n=1}^{\infty} \frac{a^2(n)\log n}{n^2\sin\delta} e^{-4\pi n \sin\delta/\sqrt{D}}\right).$$

By Lemmas 3, 4, and 5,

$$B_{21}(\delta) = \frac{4A}{3\delta} \log^3 \frac{1}{\delta} + O\left(\frac{1}{\delta} \log^2 \frac{1}{\delta}\right).$$

Similarly, by Lemmas 3, 4, and 5,

$$B_{22}(\delta) = \frac{A}{3\delta} \log^3 \frac{1}{\delta} + O\left(\frac{1}{\delta} \log^2 \frac{1}{\delta}\right),$$

$$B_{23}(\delta) = O\left(\frac{1}{\delta} \log^2 \frac{1}{\delta}\right).$$

Thus

$$B_1(\delta)+B_2(\delta)=\frac{7A}{6\delta}\log^3\frac{1}{\delta}+O\left(\frac{1}{\delta}\log^2\frac{1}{\delta}\right).$$

We now consider the terms with $n\neq m$, which are

$$\int_{2\pi/\sqrt{D}}^{\infty}\sum_{\substack{n=1\\ n\neq m}}^{\infty}\sum_{m=1}^{\infty}b(n)b(m)e^{-n(ixe^{-i\delta})}e^{-m(-ixe^{i\delta})}\,dx$$

$$-2\int_{2\pi/\sqrt{D}}^{\infty}\log x\sum_{\substack{n=1\\ n\neq m}}^{\infty}\sum_{m=1}^{\infty}b(n)a(m)e^{-n(-ixe^{i\delta})}e^{-m(ixe^{-i\delta})}\,dx$$

$$-2\int_{2\pi/\sqrt{D}}^{\infty}\log x\sum_{\substack{n=1\\ n\neq m}}^{\infty}\sum_{m=1}^{\infty}b(n)a(m)e^{-n(ixe^{-i\delta})}e^{-m(-ixe^{i\delta})}\,dx$$

$$+4\int_{2\pi/\sqrt{D}}^{\infty}\log^2 x\sum_{\substack{n=1\\ n\neq m}}^{\infty}\sum_{m=1}^{\infty}a(n)a(m)e^{-n(ixe^{-i\delta})}e^{-m(-ixe^{i\delta})}\,dx$$

$$=C_1(\delta)-2C_2(\delta)-2C_3(\delta)+4C_4(\delta),$$

say.

Let us first consider $C_4(\delta)$.

$$C_4(\delta)=\sum_{\substack{n=1\\ n\neq m}}^{\infty}\sum_{m=1}^{\infty}a(n)a(m)\int_{2\pi/\sqrt{D}}^{\infty}\log^2 xe^{-x[(m+n)\sin\delta+i(n-m)\cos\delta]}\,dx.$$

Upon an integration by parts,

$$C_4(\delta)=\sum_{\substack{n=1\\ n\neq m}}^{\infty}\sum_{m=1}^{\infty}\frac{a(n)a(m)}{(m+n)\sin\delta+i(n-m)\cos\delta}\log^2\left(\frac{2\pi}{\sqrt{D}}\right)e^{-2\pi[(m+n)\sin\delta+i(n-m)\cos\delta]/\sqrt{D}}$$

$$+2\sum_{\substack{n=1\\ n\neq m}}^{\infty}\sum_{m=1}^{\infty}\frac{a(n)a(m)}{(m+n)\sin\delta+i(n-m)\cos\delta}\int_{2\pi/\sqrt{D}}^{\infty}\frac{\log x}{x}e^{-x(m+n)\sin\delta}e^{i(-x)(n-m)\cos\delta}\,dx$$

$$=C_{41}(\delta)+2C_{42}(\delta),$$

say.

We consider first, $C_{42}(\delta)$.

$$C_{42}(\delta)=\sum_{\substack{n=1\\ n\neq m}}^{\infty}\sum_{m=1}^{\infty}\frac{a(n)a(m)(m+n)\sin\delta}{(m+n)^2\sin^2\delta+(n-m)^2\cos^2\delta}\int_{2\pi/\sqrt{D}}^{\infty}\frac{\log x}{x}e^{-x(m+n)\sin\delta}e^{i(-x)(n-m)\cos\delta}\,dx$$

$$-\sum_{\substack{n=1\\ n\neq m}}^{\infty}\sum_{m=1}^{\infty}\frac{ia(n)a(m)(n-m)\cos\delta}{(m+n)^2\sin^2\delta+(n-m)^2\cos^2\delta}\int_{2\pi/\sqrt{D}}^{\infty}\frac{\log x}{x}e^{-x(m+n)\sin\delta}e^{i(-x)(n-m)\cos\delta}\,dx$$

$$=C_{421}(\delta)-C_{422}(\delta),$$

say.

$$C_{421}(\delta) = \sum_{m=2}^{\infty}\sum_{n=1}^{m-1} \frac{a(n)a(m)(m+n)\sin\delta}{(m+n)^2\sin^2\delta+(n-m)^2\cos^2\delta}\int_{2\pi/\sqrt{D}}^{\infty}\frac{\log x}{x}e^{-x(m+n)\sin\delta}e^{i(-x)(n-m)\cos\delta}\,dx$$
$$+\sum_{n=2}^{\infty}\sum_{m=1}^{n-1} \frac{a(n)a(m)(m+n)\sin\delta}{(m+n)^2\sin^2\delta+(n-m)^2\cos^2\delta}\int_{2\pi/\sqrt{D}}^{\infty}\frac{\log x}{x}e^{-x(n+m)\sin\delta}e^{i(-x)(n-m)\cos\delta}\,dx$$
$$= C_{4211}(\delta)+C_{4212}(\delta),$$

say.

$$C_{4211}(\delta) = O\left(\sum_{m=2}^{\infty}\sum_{n=1}^{m-1}\frac{|a(n)|\,|a(m)|\,2m\sin\delta}{(n-m)^2\cos^2\delta}\left|\int_{2\pi/\sqrt{D}}^{\infty}\frac{\log x}{x}e^{-x(m+n)\sin\delta}e^{i(-x)(n-m)\cos\delta}\,dx\right|\right)$$

and upon substituting $ex\sqrt{D}/(2\pi)$ for x, we obtain

$$C_{4211}(\delta) = O\left(\sum_{m=2}^{\infty}\sum_{n=1}^{m-1}\frac{|a(n)|\,|a(m)|\,2m\sin\delta}{(n-m)^2\cos^2\delta}\left|\int_{e}^{\infty}\frac{\log x}{x}e^{-2\pi x(m+n)\sin\delta/(e\sqrt{D})}e^{i(-2\pi x)(n-m)\cos\delta(e\sqrt{D})}\,dx\right|\right)$$
$$+O\left(\sum_{m=2}^{\infty}\sum_{n=1}^{m-1}\frac{|a(n)|\,|a(m)|\,2m\sin\delta}{(n-m)^2\cos^2\delta}\left|\int_{e}^{\infty}\frac{1}{x}e^{-2\pi x(m+n)\sin\delta/(e\sqrt{D})}e^{i(-2\pi x)(n-m)\cos\delta/(e\sqrt{D})}\,dx\right|\right)$$
$$= O\left(\sum_{m=2}^{\infty}\sum_{n=1}^{m-1}\frac{|a(n)|\,|a(m)|\,2m\sin\delta}{(n-m)^2\cos^2\delta}\,\frac{e^{-2\pi(m+n)\sin\delta/\sqrt{D}}}{(m-n)\cos\delta}\right),$$

by Lemma 1. This sum may be evaluated as in [**4**, p. 145] to give

$$C_{4211}(\delta) = O\left(\frac{1}{\delta}\right).$$

Similarly $C_{4212}(\delta) = O\left(\frac{1}{\delta}\right)$, and so $C_{421}(\delta) = O\left(\frac{1}{\delta}\right)$.

By the same procedure as above, we find $C_{422}(\delta) = O\left(\frac{1}{\delta}\right)$, and so $C_{42}(\delta) = O\left(\frac{1}{\delta}\right)$. We now consider $C_{41}(\delta)$.

$$C_{41}(\delta) = \sum_{n=1}^{\infty}\sum_{\substack{m=1\\ n\neq m}}^{\infty}\frac{a(n)a(m)(m+n)\sin\delta\cos[2\pi(n-m)\cos\delta/\sqrt{D}]}{(m+n)^2\sin^2\delta+(m-n)^2\cos^2\delta}e^{-2\pi(m+n)\sin\delta/\sqrt{D}}$$
$$+\sum_{n=1}^{\infty}\sum_{\substack{m=1\\ n\neq m}}^{\infty}\frac{a(n)a(m)(m-n)\cos\delta\sin[2\pi(n-m)\cos\delta/\sqrt{D}]}{(m+n)^2\sin^2\delta+(m-n)^2\cos^2\delta}e^{-2\pi(m+n)\sin\delta/\sqrt{D}}$$
$$= C_{411}(\delta)+C_{412}(\delta),$$

say.

$C_{411}(\delta)$ may be evaluated as in [**4**, p. 145], to give

$$C_{411}(\delta) = O\left(\frac{1}{\delta}\right).$$

The second sum, again, is slightly more complicated and may be evaluated as in [3, p. 150] to give

$$C_{412}(\delta)=O\left(\frac{1}{\delta}\right).$$

Thus $C_{41}(\delta)=O\left(\frac{1}{\delta}\right)$, and so $C_4(\delta)=O\left(\frac{1}{\delta}\right)$.

Proceeding as above, we find similarly

$$C_1(\delta)=O\left(\frac{1}{\delta}\log^2\frac{1}{\delta}\right),$$

$$C_2(\delta)=O\left(\frac{1}{\delta}\log\frac{1}{\delta}\right),$$

$$C_3(\delta)=O\left(\frac{1}{\delta}\log\frac{1}{\delta}\right).$$

Collecting all the estimates, we obtain

$$\int_0^{2\pi/\sqrt{D}}|\phi_1(ixe^{-i\delta})|^2\,dx=\frac{7A}{6\delta}\log^3\frac{1}{\delta}+O\left(\frac{1}{\delta}\log^2\frac{1}{\delta}\right).$$

Now $\int_{2\pi/\sqrt{D}}^{\infty}|\phi_1(ixe^{-i\delta})|^2\,dx$ may be evaluated as before to give

$$\int_{2\pi/\sqrt{D}}^{\infty}|\phi_1(ixe^{-i\delta})|^2\,dx=\frac{1}{2\sin\delta}\sum_{n=1}^{\infty}\frac{b^2(n)}{n}e^{-4\pi n\sin\delta/\sqrt{D}}+O\left(\frac{1}{\delta}\log^2\frac{1}{\delta}\right)$$

$$=\frac{A}{6\delta}\log^3\frac{1}{\delta}+O\left(\frac{1}{\delta}\log^2\frac{1}{\delta}\right).$$

Thus

$$\int_0^{\infty}|\phi_1(ixe^{-i\delta})|^2\,dx=\frac{4A}{3\delta}\log^3\frac{1}{\delta}+O\left(\frac{1}{\delta}\log^2\frac{1}{\delta}\right),$$

and this gives

$$\int_0^{\infty}|\psi'(\tfrac{1}{2}+it)|^2\,e^{-\delta t}\,dt=\frac{8A}{3\delta}\log^3\frac{1}{\delta}+O\left(\frac{1}{\delta}\log^2\frac{1}{\delta}\right).$$

Corollary 2 now follows as before.

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