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KILLING FRAMES AND S-CURVATURE OF HOMOGENEOUS FINSLER SPACES*

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Abstract. In this paper, we first deduce a formula of S-curvature of homogeneous Finsler spaces in terms of Killing vector fields. Then we prove that a homogeneous Finsler space has isotropic S-curvature if and only if it has vanishing S-curvature. In the special case that the homogeneous Finsler space is a Randers space, we give an explicit formula which coincides with the previous formula obtained by the second author using other methods.

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1. Introduction. The notion of S-curvature was introduced by Z. Shen in [**8**] in his study of volume comparison in Finsler geometry. S-curvature is an important quantity in Finsler geometry in that it has some mysterious interrelations with other quantities such as flag curvature, Ricci scalar, etc. Shen showed that the Bishop–Gromov volume comparison theorem holds for Finsler spaces with vanishing S-curvature. Therefore, it is also significant to characterize Finsler spaces with vanishing S-curvature.

The goal of this paper is to give an explicit formula of S-curvature of a homogeneous Finsler space in terms of Killing vector fields. Let (M, F) be a connected Finsler space. Then the group of isometries of (M, F), denoted by $I(M, F)$, is a Lie transformation group on M with respect to the compact-open topology (see [**6**]). A vector field X on a Finsler space (M, F) is called a Killing vector field, if the local one-parameters groups of transformations generated by X consists of local isometries of (M, F). A Killing vector field X of (M, F) can be equivalently described as follows. Any vector field X on M can naturally define a vector field $\tilde{X}$ on TM. The vector field X generates a flow of diffeomorphisms ρ_t on M, with the corresponding flow of diffeomorphisms $\tilde{\rho}_t$ on TM. Then the value of $\tilde{X}$ at $(x, y) \in TM$ is just $\frac{d}{dt}[\tilde{\rho}_t(x, y)]|_{t=0}$. Obviously, X is a Killing vector field for F if and only if $\tilde{X}(F) = 0$.

The space (M, F) is called homogeneous if the action of $I(M, F)$ on M is transitive. In this case, M can be written as a coset space $I(M, F)/I(M, F)_x$, where $I(M, F)_x$ is

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the isotropic subgroup of $I(M, F)$ at a point of $x \in M$. Since M is connected, the unit connected component of $I(M, F)$, denoted by G, is also transitive on M. Let H be the isotropic of G at the point x. Then we have $M = G/H$. Moreover, the Finsler metric F can be viewed as an G-invariant Finsler metric on G/H.

Since (M, F) is homogeneous, given any tangent vector $v \in T_y(M)$, $y \in M$, there exists a Killing vector field X such that $X|_y = v$. Therefore, if we can get a formula for $S(X)$, where X is an arbitrary Killing vector field, then the S-curvature of (M, F) is completely determined. The main result of this paper is a formula of the S-curvature as described above. As an application, we show that a homogeneous Finsler space has isotropic S-curvature if and only if it has vanishing S-curvature. This generalizes the similar results on homogeneous Randers spaces and homogeneous (α, β)-spaces in [**5**] and [**7**].

In Section 3, we present some preliminaries on Finsler spaces and S-curvature. Section 4 is devoted to deducing the formula of S-curvature. In Section 5, we apply our formula to homogeneous Randers spaces, and show that the formula coincides with the previous one obtained in [**5**] in this special case.

2. Preliminaries. In this section, we recall some known results on Finsler spaces, for details we refer the readers to [**1**, **3**, **4**] and [**9**].

A Finsler metric on a manifold M is a function $F : TM\backslash\{0\} \to \mathbb{R}^+$ satisfying the following properties:

(1) F is smooth on $TM\backslash\{0\}$.
(2) F is positively homogeneous of degree 1, namely, $F(\lambda y) = \lambda F(y)$, for any $\lambda > 0$ and $y \in TM\backslash\{0\}$.
(3) For any standard local coordinate system $(TU, (x, , y)$ of TM, where $x = (x^i)$ in an small open neighbourhood $U \subset M$, and $y = y^i \partial_{x^i} \in TM_x$, the fundamental tensor $g_{ij}(y) = \frac{1}{2}[F^2]_{y^i y^j}$ is positive definite whenever $y \neq 0$.

Riemannian metrics are a special class of Finsler metrics widely studied by mathematicians. Their fundamental tensors only depends on x, which are regarded as the metrics themselves.

Randers metrics are the most well-known non-Riemannian Finsler metric. They are defined as $F = \alpha + \beta$, where α is a Riemannian metric and β is a 1-form, whose α-norm $||\beta(x)||_\alpha$ is less than 1 everywhere. Randers metrics are generalized to (α, β)-metrics of the form $F = \alpha\phi(\beta/\alpha)$.

3. Killing frames and the geodesic spray. A Killing frame for a Finsler manifold (M, F) is a set of local vector fields X_i, $i = 1, \ldots, n = \dim M$, defined on an open subset U around a given point, such that

(1) the values $X_i(x)$, $\forall i$, give bases for each tangent space $T_x(M)$, $x \in U$, and
(2) in U, each X_i satisfies $\tilde{X}_i(F) = 0$, in other words, the X_i's are local Killing vector fields in U.

Though Killing frames are rare in the general study of Finsler geometry, they can be easily found for a homogeneous Finsler space at any given point. Let the homogeneous Finsler space (M, F) be presented as $M = G/H$, where H is the isotropy subgroup for the given x. The tangent space TM_x can be identified as the quotient $\mathfrak{m} = \mathfrak{g}/\mathfrak{h}$, where $\mathfrak{g}$ and $\mathfrak{h}$ are the Lie algebras of G and H, respectively. Take any basis $\{v_1, \ldots, v_n\}$ of $\mathfrak{m}$, with the pre-images $\{\hat{v}_1, \ldots, \hat{v}_n\}$ in $\mathfrak{g}$. Then the Killing vector fields $\{X_1, \ldots, X_n\}$ on M corresponding to $\hat{v}_i$s defines a Killing frame around x. The choice of $\hat{v}_i$s or X_is identifies the quotient space $\mathfrak{m}$ with a subspace of $\mathfrak{g}$, and then we can write the decomposition of linear space

$$\mathfrak{g} = \mathfrak{h} + \mathfrak{m}. \tag{3.1}$$

For the Killing frame $\{X_1, \ldots, X_n\}$ around $x \in M$, a set of y-coordinates $y = (y^i)$ can be defined by $y = y^i X_i$. Accordingly, we have the fundamental tensor $g_{ij} = \frac{1}{2}[F^2]_{y^i y^j}$, and the inverse matrix of (g_{ij}) is denoted as (g^{ij}). When both the Killing frame and the local coordinates $\{x = (\bar{x}^{\bar{i}}), y = \bar{y}^{\bar{j}}\partial_{\bar{x}^{\bar{j}}}\}$ are used, the terms and indices for the local coordinates are marked with bars, and the indices with bars are moved up and down by the fundamental tensors $\bar{g}^{\bar{i}\bar{j}}$ or $\bar{g}_{\bar{i}\bar{j}}$ for the local coordinates. Let $f_{\bar{i}}^{i}$ and $f_i^{\bar{i}}$, $\forall i$ and $\bar{i}$, be the transition functions such that around x,

$$\partial_{\bar{x}^{\bar{i}}} = f_{\bar{i}}^{i} X_i \text{ and } X_i = f_i^{\bar{i}} \partial_{\bar{x}^{\bar{i}}}. \tag{3.2}$$

We summarize some easy and useful identities which show how the transition functions exchange the indices with and without bars:

$$\bar{y}^{\bar{i}} = f_i^{\bar{i}} y^i \quad\text{and}\quad y^i = f_{\bar{i}}^{i} \bar{y}^{\bar{i}} \tag{3.3}$$

$$\partial_{\bar{y}^{\bar{i}}} = f_{\bar{i}}^{i} \partial_{y^i} \quad\text{and}\quad \partial_{y^i} = f_{\bar{i}}^{i} \partial_{\bar{y}^{\bar{i}}}, \tag{3.4}$$

$$\bar{g}_{\bar{i}\bar{j}} = f_{\bar{i}}^{i} g_{ij} f_{\bar{j}}^{j} \quad\text{and}\quad g_{ij} = f_i^{\bar{i}} \bar{g}_{\bar{i}\bar{j}} f_j^{\bar{j}}, \tag{3.5}$$

$$\bar{g}^{\bar{i}\bar{j}} = f_i^{\bar{i}} g^{ij} f_j^{\bar{j}} \quad\text{and}\quad g^{ij} = f_{\bar{i}}^{i} \bar{g}^{\bar{i}\bar{j}} f_{\bar{j}}^{j}. \tag{3.6}$$

To apply Killing frames to the study of Finsler geometry, we start with the geodesic spray.

THEOREM 3.1. *Let $\{X_1, \ldots, X_n\}$ be a Killing frame around $x \in M$ for the Finsler metric F. Then for $y = \tilde{y}^i X_i(x) \in TM_x$, the geodesic spray $G(x, y)$ can be presented as*

$$G(x, y) = y^i \tilde{X}_i + \frac{1}{2} g^{il} c_{lj}^{k} [F^2]_{y^k} y^j \partial_{y^i}, \tag{3.7}$$

where c_{lj}^{k} are defined by $[X_l, X_j](x) = c_{lj}^{k} X_k(x)$.

If we use the local coordinates $\{x = (\bar{x}^i)$ and $y = \bar{y}^{\bar{i}}\partial_{\bar{x}^{\bar{i}}}\}$, then a direct calculation shows that c_{ij}^{k}s can be presented as

$$c_{ij}^{k} = [(f_i^{\bar{i}} \partial_{\bar{x}^{\bar{i}}} f_j^{\bar{j}} - f^{\bar{i}} \partial_{\bar{x}^{\bar{i}}} f_i^{\bar{j}}) f_{\bar{j}}^{k}](x). \tag{3.8}$$

Now consider the case that $M = G/H$ is a homogeneous Finsler space, where H is the isotropy group of $x \in M$. Let the Killing vector fields X_i's be defined by $\hat{v}_i \in \mathfrak{g}$, $\forall i$. Then the tangent space TM_x can be identified with the n-dimensional subspace $\mathfrak{m}$ spanned by the values of all the $\hat{v}_i$'s at x. With respect to the decomposition $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$, there is a projection map $\mathrm{pr} : \mathfrak{g} \to \mathfrak{m}$. Note that for the bracket operation $[\cdot, \cdot]$ on $\mathfrak{g}$, we have $[\cdot, \cdot]_{\mathfrak{m}} = \mathrm{pr}[\cdot, \cdot]$. Then c_{ij}^{k}s can be determined by

$$[\hat{v}_i, \hat{v}_j]_{\mathfrak{m}} = -c_{ij}^{k} \hat{v}_k. \tag{3.9}$$

The proof of Theorem 3.1 needs local coordinates $\{x = (\bar{x}^i)$ and $y = \bar{y}^{\bar{i}}\partial_{\bar{x}^{\bar{i}}}\}$ around x. We first need to see how to present each $\tilde{X}_i$ with the local coordinates.

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LEMMA 3.2. *For any vector field* $X = f^{\bar{i}}\partial_{\bar{x}^{\bar{i}}}$ *around* x,

$$\tilde{X}(x, y) = f^{\bar{i}}\partial_{\bar{x}^{\bar{i}}} + \bar{y}^{\bar{i}}\partial_{\bar{x}^{\bar{i}}} f^{\bar{j}}\partial_{\bar{y}^{\bar{j}}}, \tag{3.10}$$

for any $y = \bar{y}^{\bar{i}}\partial_{\bar{x}^{\bar{i}}} \in TM_x$.

Proof. Let ρ_t and $\tilde{\rho}_t$ be the flows of diffeomorphisms X generates on M and TM, respectively. For each i, the flow curve $\tilde{\rho}_t(\partial_{\bar{x}^{\bar{i}}}|_x)$ can be presented as

$$(\rho_t(x), \partial_{\bar{x}^{\bar{i}}} + t\partial_{\bar{x}^{\bar{i}}} f_{\bar{j}}\partial_{\bar{x}^{\bar{j}}} + o(t)), \tag{3.11}$$

so the flow curve $\tilde{\rho}_t(x, y)$ for $y = \bar{y}^{\bar{i}} \in TM_x$ has the local coordinates

$$(\rho_t(x), [\bar{y}^{\bar{j}} + t\bar{y}^{\bar{i}}\partial_{\bar{x}^{\bar{i}}} f^{\bar{j}} + o(t)]\partial_{\bar{x}^{\bar{j}}}). \tag{3.12}$$

Differentiating with respect to t and considering the values at $t = 0$, we get (3.10). □

Now, we use the above lemma to recalculate the terms of the geodesic spray

$$G = \bar{y}^{\bar{i}}\partial_{\bar{x}^{\bar{i}}} - \frac{1}{2}\bar{g}^{\bar{i}\bar{l}}([F^2]_{\bar{x}^{\bar{j}}\bar{y}^{\bar{l}}}\bar{y}^{\bar{j}} - [F^2]_{\bar{x}^{\bar{l}}})\partial_{\bar{y}^{\bar{i}}}. \tag{3.13}$$

By (3.10) and the property that $\tilde{X}_i(F) = 0$, $\forall i$, we have the following equations which hold on a neighbourhood around x:

$$\begin{aligned}
\bar{y}^{\bar{i}}\partial_{\bar{x}^{\bar{i}}} &= y^i f_i^{\bar{i}}\partial_{\bar{x}^{\bar{i}}} \\
&= y^i(\tilde{X}_i - \bar{y}^{\bar{i}}\partial_{\bar{x}^{\bar{i}}} f_i^{\bar{j}}\partial_{\bar{y}^{\bar{j}}}) \\
&= y^i\tilde{X}_i - f_k^{\bar{i}}\partial_{\bar{x}^{\bar{i}}} f_i^{\bar{j}} f_{\bar{j}}^{l} y^i y^k \partial_{y^l} \\
&= y^i\tilde{X}_i - f_k^{\bar{i}}\partial_{\bar{x}^{\bar{i}}} f_j^{\bar{j}} f_{\bar{j}}^{i} y^j y^k \partial_{y^i},
\end{aligned} \tag{3.14}$$

$$\begin{aligned}
\bar{g}^{\bar{i}\bar{l}}[F^2]_{\bar{x}^{\bar{l}}}\partial_{\bar{y}^{\bar{i}}} &= g^{il} f_l^{\bar{l}}[F^2]_{\bar{x}^{\bar{l}}}\partial_{y^i} \\
&= -g^{il}(y^{\bar{i}}\partial_{\bar{x}^{\bar{i}}} f_l^{\bar{j}}[F^2]_{\bar{y}^{\bar{j}}})\partial_{y^i} \\
&= -g^{il} f_j^{\bar{i}}\partial_{\bar{x}^{\bar{i}}} f_l^{\bar{j}} f_{\bar{j}}^{k}[F^2]_{y^k} y^j \partial_{y^i},
\end{aligned} \tag{3.15}$$

and

$$\begin{aligned}
\bar{g}^{\bar{i}\bar{l}}[F^2]_{\bar{x}^{\bar{j}}\bar{y}^{\bar{l}}}\bar{y}^{\bar{j}}\partial_{\bar{y}^{\bar{i}}} &= g^{il}[f_j^{\bar{j}}[F^2]_{\bar{x}^{\bar{j}}}]_{y^l} y^j \partial_{y^i} \\
&= -g^{il}(\bar{y}^{\bar{i}}\partial_{\bar{x}^{\bar{i}}} f_j^{\bar{j}}[F^2]_{\bar{y}^{\bar{j}}})_{y^l} y^j \partial_{y^i} \\
&= -g^{il} f_l^{\bar{i}}\partial_{\bar{x}^{\bar{i}}} f_j^{\bar{j}}[F^2]_{\bar{y}^{\bar{j}}} y^j \partial_{y^i} - g^{il} f_k^{\bar{i}}\partial_{\bar{x}^{\bar{i}}} f_j^{\bar{j}} f_{\bar{j}}^{h}[F^2]_{y^h y^l} y^j y^k \partial_{y^i} \\
&= -g^{il} f_l^{\bar{i}}\partial_{\bar{x}^{\bar{i}}} f_j^{\bar{j}} f_{\bar{j}}^{k}[F^2]_{y^k} y^j \partial_{y^i} - 2 f_k^{\bar{i}}\partial_{\bar{x}^{\bar{i}}} f_j^{\bar{j}} f_{\bar{j}}^{i} y^j y^k \partial_{y^i}.
\end{aligned} \tag{3.16}$$

By (3.14)–(3.16) and (3.8), we get

$$\begin{aligned}
G(x, y) &= y^i\tilde{X}_i + \frac{1}{2}g^{il}(f_l^{\bar{i}}\partial_{\bar{x}}^{\bar{i}} - f_j^{\bar{i}}\partial_{\bar{x}^{\bar{i}}} f_l^{\bar{j}}) f_{\bar{j}}^{k}[F^2]_{y^k} y^j \partial_{y^i} \\
&= y^i\tilde{X}_i + \frac{1}{2}g^{il} c_{lj}^{k}[F^2]_{y^k} y^j \partial_{y^i}.
\end{aligned} \tag{3.17}$$

This completes the proof of Theorem 3.1.

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4. The formula of S-curvature for a homogeneous Finsler space. The formula (3.7) of the geodesic spray can be immediately applied to get a formula of S-curvature. Suppose in a local coordinate system, $x = (\bar{x}^{\bar{i}})$ and $y = \bar{y}^{\bar{i}}\partial_{\bar{x}^{\bar{i}}} \in TM_x$, with $y \neq 0$. Then the distortion function is defined by

$$\tau(x, y) = \ln\sqrt{\frac{\det(\bar{g}_{\bar{p}\bar{q}})}{\sigma(x)}}, \tag{4.18}$$

where $\sigma(x)$ is defined by

$$\sigma(x) = \frac{\mathrm{Vol}(B^n)}{\mathrm{Vol}\{(y^i) \in \mathbb{R}^n | F_x(y^i b_i) < 1\}},$$

where Vol means the volume of a subset in the standard Euclidean space $\mathbb{R}^n$ and B^n is the open ball of radius 1. The function $\sigma(x)$ can be used to define the Busemann–Hausdorff volume $\sigma(x)d\bar{x}^1 \cdots \bar{x}^n$. The S-curvature of the nonzero tangent vector (x, y), denoted as $S(x, y)$, is defined to be the derivative of τ in the direction of the geodesics of $G(x, y)$, with initial vector y.

Notice that the distortion function $\tau(x, y)$ is only determined by the metric F, not relevant to the choice of local coordinates or frames. If there is Killing frame $\{X_1, \ldots, X_n\}$ around x, then it is not hard to see that $\tilde{X}_i\tau = 0$, $\forall i$. Thus, only the derivative of τ in the direction $\frac{1}{2}g^{il}c_{lj}^k[F^2]_{y^k}y^j\partial_{y^i}$ remains to appear in the S-curvature formula. Notice also that in the expression of τ, $\sigma(x)$ is a function of x only. This observation leads to the following formula for the S-curvature.

THEOREM 4.1. *Let $\{X_1, \ldots, X_n\}$ be a Killing frame around x, then for any $y \neq 0$ in TM_x, the S-curvature at (x, y) can be presented with the notations for the Killing frame as*

$$S(x, y) = \frac{1}{2}g^{il}c_{lj}^k[F^2]_{y^k}y^j I_i, \tag{4.19}$$

where $I_i = \ln\sqrt{\det(g_{pq})}$ are the coefficients of the mean Cartan torsion with respect to the basis the Killing frame induced in TM_x.

Now assume $M = G/H$ is homogeneous, with H being the isotropy group at x. In Section 3, we have seen the existence of Killing frames around x. Each Killing frame $\{X_1, \ldots, X_n\}$ determines a decomposition $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$, where X_i is determined by $\hat{v}_i$ in $\mathfrak{m}$. Let the operation $[\cdot, \cdot]_{\mathfrak{m}}$ be defined as before. The gradient field of $\ln\sqrt{\det(g_{pq})}$ with respect to the fundamental tensor on $TM_x\backslash 0$ is the $\mathfrak{m}$-valued function

$$g^{il}I_i\hat{v}_l = g^{il}[\ln\sqrt{\det(g_{pq})}]_{y^i}\hat{v}_l. \tag{4.20}$$

We will denote it as $\nabla^{g_{ij}}\ln\sqrt{\det(g_{pq})}(y)$ for $y \in \mathfrak{m}$. Let $\langle\cdot, \cdot\rangle_y$ be the inner product defined by the fundamental tensor g_{ij} at y. Then by (3.9), we can rewrite (4.19) as

$$\begin{aligned} S(x, y) &= g^{il}c_{lj}^k g_{kh}y^h y^j I_i \\ &= \langle [y, \nabla^{g_{ij}}\ln\sqrt{\det(g_{pq})}(y)]_{\mathfrak{m}}, y\rangle_y, \end{aligned} \tag{4.21}$$

which gives a more beautiful formula for the S-curvature of a homogenous Finsler space. The formula (4.21) is relevant to the choice of $\mathfrak{m}$ rather than the specified basis of $\mathfrak{m}$ to generate the Killing frame. To summarize, we have proved the following

THEOREM 4.2. *Let M be a homogeneous Finsler space G/H, where H is the isotropy subgroup of $x \in M$. Fix any complement $\mathfrak{m}$ of $\mathfrak{h}$ in $\mathfrak{g}$, with the corresponding $[\cdot,\cdot]_\mathfrak{m}$. Then for any nonzero $y \in \mathfrak{m} = TM_x$, we have*

$$S(x, y) = \langle [y, \nabla^{g_{ij}} \ln \sqrt{\det(g_{pq})}(y)]_\mathfrak{m}, y\rangle_y. \tag{4.22}$$

A Finsler metric F is said to have isotropic S-curvature when the S-curvature has the form $S = (n+1)c(x)F$ for some function $c(x)$ on M. An immediate application of Theorem 4.2 is the following corollary.

COROLLARY 4.3. *A homogeneous Finsler space is of isotropic S-curvature if and only if it has vanishing S-curvature.*

Proof. We need to only consider the S-curvature at a fixed point x. The function $\ln\sqrt{\det(g_{pq})}$ is homogeneous of degree 0, so it must reach its maximum or minimum at some nonzero y, where the gradient field vanishes. Then by (4.21), $S(x, y) = 0$. If the S-curvature is isotropic, i.e., if $S = (n+1)c(x)F$, then $c(x)$ must be 0. This proves the "only if" part of the corollary. The "if" part is obvious. □

As another corollary, we obtain an important property of homogeneous Einstein–Randers spaces. Recall that an Einstein–Randers space must have constant S-curvature (see [**2**]). Therefore we have

COROLLARY 4.4. *A Homogeneous Einstein–Randers space must have vanishing S-curvature.*

5. Homogeneous Randers spaces. In this last section, we will apply the S-curvature formula (4.21) to calculate the S-curvatures of homogeneous Randers spaces. It turns out that the resulting formula coincides with the one given in [**5**]. Although the situation will be a little more complicated, the same technique can be transported to the (α, β)-metrics, and we can get the same formula as in [**7**]. Though the calculation here seems longer, it is still acceptable. More importantly, the calculation does not use the complicated S-curvature formula for Randers metric or (α, β)-metrics at all. Hopefully, with the help of (4.21), we can obtain the explicit formula of S-curvature for many more generalized classes of homogeneous Finsler metrics.

Let $F = \alpha + \beta$ be a homogeneous Randers metric on $M = G/H$, where H is the isotropy group of $x \in TM$. Let $\mathfrak{m}$ be a complement of $\mathfrak{h}$ in $\mathfrak{g}$. Identify $\mathfrak{m}$ with TM_x as above. Then α is determined by an linear metric on $\mathfrak{m}$ and β is determined by a vector in $\mathfrak{m}^*$, which is invariant under the adjoint action of H. We still denote them as α and β. Let $\langle , \rangle$ be the inner product induced by α gives on $\mathfrak{m}$ and suppose β is defined by $\beta(\cdot) = \langle \cdot, u\rangle$, where $u \in \mathfrak{m}$ is a fixed vector of H (see [**5**]).

For each $v \in TM_x = \mathfrak{m}$ with $\alpha(v) = 1$, we choose an orthonormal basis $\{\hat{v}_1, \ldots, \hat{v}_n\}$ of $\mathfrak{m}$ with respect to α, such that $u = b\hat{v}_1$ and $v = a\hat{v}_1 + a'\hat{v}_2$ with $a^2 + a'^2 = 1$. Then α on $\mathfrak{m}$ is simplified as $\alpha = \sqrt{{y^1}^2 + \cdots + {y^n}^2}$ and $\beta = by^1$. All the notations are defined for this Killing frame around x defined by the $\hat{v}_i$s.

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The fundamental tensor at v is given by

$$\begin{aligned} g_{11} &= 1 + b^2 + 3ba - ba^3, \\ g_{12} &= ba'^3, \\ g_{22} &= 1 + ba^3, \\ g_{ii} &= 1 + ba, \end{aligned}$$

and all other g_{ij} vanishes at v. Similarly, the tensor g^{ij} is given by

$$\begin{aligned} g^{11} &= (1 + ba)^{-3}(1 + ba^3), \\ g^{12} &= -(1 + ba)^{-3} ba'^3, \\ g^{22} &= (1 + ba)^{-3}(1 + b^2 + 3ba - ba^3), \\ g^{ii} &= (1 + ba)^{-1}, \end{aligned}$$

and all other g^{ij} vanishes at v. The coefficients of the Cartan torsion are then given by

$$\begin{aligned} I_1 &= \frac{n+1}{2(1+ba)} ba'^2, \\ I_2 &= -\frac{n+1}{2(1+ba)} baa', \end{aligned}$$

and all other I_is vanish as v. Therefore at the vector v, we have

$$\begin{aligned} \nabla^{g_{ij}} \ln \sqrt{\det(g_{pq})} &= \frac{(n+1)ba'^2}{2(1+ba)^3} \hat{v}_1 - \frac{(n+1)ba'(a+b)}{2(1+ba)^3} \hat{v}_2 \\ &= \frac{n+1}{2(1+ba)^2} u - \frac{(n+1)b(a+b)}{2(1+ba)^3} v, \end{aligned}$$

and

$$\langle [v, \nabla^{g_{ij}} \ln \sqrt{\det(g_{pq})(v)}]_{\mathfrak{m}}, v \rangle_v = \frac{n+1}{2(1+ba)^2} \langle [v, u]_{\mathfrak{m}}, v \rangle_v. \tag{5.23}$$

The calculation of the fundamental tensor at v indicates that

$$\langle [v, u]_{\mathfrak{m}}, v \rangle_v = acg_{11} + (ad + a'c) g_{12} + a'dg_{22},$$

where $[v, u]_{\mathfrak{m}} = c\hat{v}_1 + d\hat{v}_2$, i.e.,

$$\langle [y, u]_{\mathfrak{m}}, u \rangle = bc, \tag{5.24}$$

$$\langle [y, u]_{\mathfrak{m}}, v \rangle = ac + a'd. \tag{5.25}$$

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Thus, we have $a'bd = b\langle[v, u]_{\mathfrak{m}}, v\rangle - a\langle[v, u]_{\mathfrak{m}}, u\rangle$, and

$$\begin{aligned}
&\langle[v, \nabla^{g_{ij}} \ln\sqrt{\det(g_{pq})(v)}]_{\mathfrak{m}}, v\rangle_v, \\
&= \frac{n+1}{2(1+ba)^2}[ac(1+b^2+3ba-ba^3)+(ad+a'c)ba'^3+a'd(1+ba^3)] \\
&= \frac{n+1}{2(1+ba)^2}[(ac+a'd)+(ab+3a^2-a^4-a^2a'^2+a'^4)bc+(aa'^2+a^3)abd] \\
&= \frac{n+1}{2(1+ba)}(\langle[v, u]_{\mathfrak{m}}, u\rangle + \langle[v, u]_{\mathfrak{m}}, v\rangle). \qquad (5.26)
\end{aligned}$$

Notice that $F(v) = 1 + ba$ and $\alpha(v) = 1$. Therefore for general v, the homogeneity of the S-curvature indicates that the formula should be adjusted to

$$S(x, v) = \frac{n+1}{2F(v)}(\alpha(v)\langle[v, u]_{\mathfrak{m}}, v\rangle + \langle[v, u]_{\mathfrak{m}}, v\rangle). \qquad (5.27)$$

This formula coincides with the one obtained in [**5**].

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