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KEISLER'S THEOREM AND CARDINAL INVARIANTS

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Abstract. We consider several variants of Keisler's isomorphism theorem. We separate these variants by showing implications between them and cardinal invariants hypotheses. We characterize saturation hypotheses that are stronger than Keisler's theorem with respect to models of size $\aleph_1$ and $\aleph_0$ by CH and $\mathrm{cov}(\mathsf{meager}) = \mathfrak{c} \wedge 2^{<\mathfrak{c}} = \mathfrak{c}$ respectively. We prove that Keisler's theorem for models of size $\aleph_1$ and $\aleph_0$ implies $\mathfrak{b} = \aleph_1$ and $\mathrm{cov}(\mathsf{null}) \leq \mathfrak{d}$ respectively. As a consequence, Keisler's theorem for models of size $\aleph_0$ fails in the random model. We also show that for Keisler's theorem for models of size $\aleph_1$ to hold it is not necessary that $\mathrm{cov}(\mathsf{meager})$ equals $\mathfrak{c}$.

§1. Introduction. The method of ultrapowers is one of the most important ways to construct models. Ultrapowers are models obtained by properly equating the elements of product sets of the models using ultrafilters. We consider the problem of when there exists an ultrafilter $\mathcal{U}$ on ω such that for two models $\mathcal{A}, \mathcal{B}$ in a countable language $\mathcal{L}$, the respective ultrapowers $\mathcal{A}^\omega/\mathcal{U}, \mathcal{B}^\omega/\mathcal{U}$ are isomorphic. Since ultrapowers are elementary extensions of original models, if $\mathcal{A}^\omega/\mathcal{U}$ and $\mathcal{B}^\omega/\mathcal{U}$ are isomorphic, then $\mathcal{A}$ and $\mathcal{B}$ must be elementarily equivalent. Keisler showed, under CH, conversely if $\mathcal{A}$ and $\mathcal{B}$ are elementarily equivalent and have size $\leq \mathfrak{c}$, then for every ultrafilter $\mathcal{U}$ over ω, $\mathcal{A}^\omega/\mathcal{U}$ and $\mathcal{B}^\omega/\mathcal{U}$ are isomorphic. The purpose of this paper is to give necessary conditions and sufficient conditions for when Keisler's theorem holds in a model where CH does not hold, and to separate the variants of Keisler's theorem using those conditions.

CONVENTION. *All ultrafilters considered in this paper are nonprincipal.*

DEFINITION 1.1. Let κ be a cardinal.

(1) We say $\mathrm{KT}(\kappa)$ holds if for every countable language $\mathcal{L}$ and $\mathcal{L}$-structures $\mathcal{A}, \mathcal{B}$ of size $\leq \kappa$ which are elementarily equivalent, there exists an ultrafilter $\mathcal{U}$ over ω such that $\mathcal{A}^\omega/\mathcal{U} \simeq \mathcal{B}^\omega/\mathcal{U}$.
(2) We say $\mathrm{SAT}(\kappa)$ holds if there exists an ultrafilter $\mathcal{U}$ over ω such that for every countable language $\mathcal{L}$ and every sequence of $\mathcal{L}$-structures $(\mathcal{A}_i)_{i\in\omega}$ with each $\mathcal{A}_i$ of size $\leq \kappa$, $\prod_{i\in\omega} \mathcal{A}_i/\mathcal{U}$ is saturated.

$\mathrm{SAT}(\kappa)$ implies $\mathrm{KT}(\kappa)$ for every $\kappa \leq \mathfrak{c}$ by the fact that two saturated structures which are elementarily equivalent and have the same size are isomorphic. Golshani and Shelah [6] proved $\neg\,\mathrm{KT}(\aleph_2)$ and later we will prove $\neg\mathrm{SAT}(\aleph_2)$ in Theorem 2.2. So this implication $\mathrm{SAT}(\kappa) \Rightarrow \mathrm{KT}(\kappa)$ holds formally for every κ.

Keisler [7] proved $\mathrm{CH} \Rightarrow \mathrm{SAT}(\mathfrak{c})$. The result $\neg\,\mathrm{KT}(\aleph_2)$ of Golshani and Shelah implies $\mathrm{KT}(\mathfrak{c}) \Rightarrow \mathrm{CH}$. So CH, $\mathrm{SAT}(\mathfrak{c})$, and $\mathrm{KT}(\mathfrak{c})$ are equivalent. Golshani and

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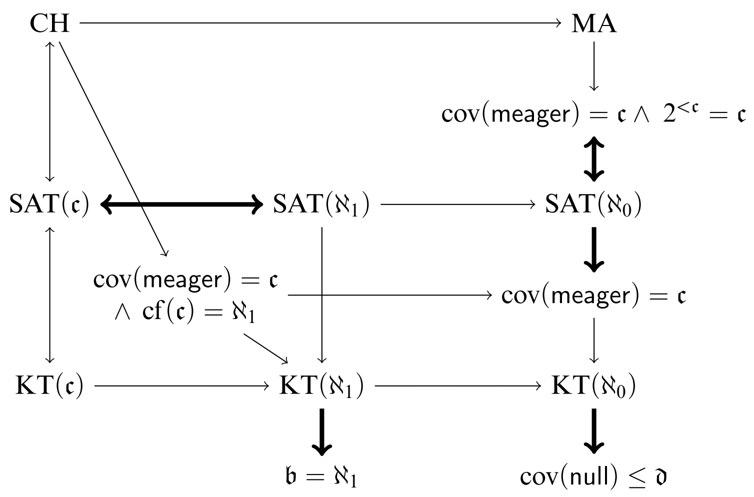

FIGURE 1. Implications; thick arrows indicate our results.

Shelah also proved that $\mathrm{cov}(\mathrm{meager}) = \mathfrak{c} \wedge \mathrm{cf}(\mathfrak{c}) = \aleph_1$ implies $\mathrm{KT}(\aleph_1)$. Another classical result is the theorem by Ellentuck–Rucker [5] which shows that $\mathrm{MA}(\sigma$-centered) implies $\mathrm{SAT}(\aleph_0)$. Moreover, Shelah [9] showed $\mathrm{Con}(\neg\,\mathrm{KT}(\aleph_0))$ by showing that $\mathfrak{d} < \mathfrak{v}^{\forall}$ implies $\neg\,\mathrm{KT}(\aleph_0)$ and that the former is consistent.

In this paper, we prove the implications indicated by thick lines in Figure 1.

In the rest of this section, we recall basic notions related to cardinal invariants.

DEFINITION 1.2. (1) If X, Y are sets and R is a subset of $X \times Y$, we call a triple (X, Y, R) a *relational system*.
(2) For a relational system $\mathcal{A} = (X, Y, R)$, define $\mathcal{A}^{\perp} = (Y, X, \hat{R})$, where $\hat{R} = \{(y, x) \in Y \times X : \neg(x\ R\ y)\}$.
(3) For a relational system $\mathcal{A} = (X, Y, R)$, define $\|\mathcal{A}\| = \min\{|B| : B \subseteq Y \wedge (\forall x \in X)(\exists y \in B)(x\ R\ y)\}$.
(4) For relational systems $\mathcal{A} = (X, Y, R), \mathcal{B} = (X', Y', S)$, we call a pair (φ, ψ) a Galois–Tukey morphism from $\mathcal{A}$ to $\mathcal{B}$ if $\varphi\colon X \to X'$, $\psi\colon Y' \to Y$ and $(\forall x \in X)(\forall y \in Y')(\varphi(x)\ S\ y \implies x\ R\ \psi(y))$ hold.

FACT 1.3 [4, Theorem 4.9]. *If there is a Galois–Tukey morphism (φ, ψ) from $\mathcal{A}$ to $\mathcal{B}$, then $\|\mathcal{A}\| \leq \|\mathcal{B}\|$ and $\|\mathcal{B}^{\perp}\| \leq \|\mathcal{A}^{\perp}\|$ hold.*

DEFINITION 1.4. (1) For $f, g \in \omega^{\omega}$, define $f <^* g$ iff $(\forall^{\infty} n)(f(n) < g(n))$.
(2) For $c \in (\omega + 1)^{\omega}$ and $h \in \omega^{\omega}$, define $\prod c = \prod_{n\in\omega} c(n)$ and $S(c, h) = \prod_{n\in\omega} [c(n)]^{\leq h(n)}$.
(3) For $x \in \prod c$ and $\varphi \in S(c, h)$, define $x \in^* \varphi$ iff $(\forall^{\infty} n)(x(n) \in \varphi(n))$ and define $x \in^{\infty} \varphi$ iff $(\exists^{\infty} n)(x(n) \in \varphi(n))$.

DEFINITION 1.5. (1) Define $\mathbf{D} = (\omega^{\omega}, \omega^{\omega}, <^*)$, $\mathfrak{d} = \|\mathbf{D}\|$ and $\mathfrak{b} = \|\mathbf{D}^{\perp}\|$.
(2) For a poset $(P, <)$, define a relational system $\mathbf{Cof}(P, <)$ by $\mathbf{Cof}(P, <) = (P, P, <)$. Then we have $\mathrm{cf}(P, <) = \|\mathbf{Cof}(P, <)\|$.

(3) For $c \in (\omega+1)^\omega, h \in \omega^\omega$, define $\mathbf{Lc}(c,h) = (\prod c, S(c,h), \in^*)$, $\mathfrak{c}^\forall_{c,h} = \|\mathbf{Lc}(c,h)\|$ and $\mathfrak{v}^\forall_{c,h} = \|\mathbf{Lc}(c,h)^\perp\|$.
(4) Define $\mathbf{wLc}(c,h) = (\prod c, S(c,h), \in^\infty)$, $\mathfrak{c}^\exists_{c,h} = \|\mathbf{wLc}(c,h)\|$, and $\mathfrak{v}^\exists_{c,h} = \|\mathbf{wLc}(c,h)^\perp\|$.
(5) For an ideal I on X, define $\mathbf{Cov}(I) = (X, I, \in)$, $\mathrm{cov}(I) = \|\mathbf{Cov}(I)\|$, and $\mathrm{non}(I) = \|\mathbf{Cov}(I)^\perp\|$.

By the definition of the norm $\|\cdot\|$, the next lemma is obvious.

LEMMA 1.6. (1) $\mathfrak{c}^\forall_{c,h} = \min\{|S| : S \subseteq S(c,h), (\forall x \in \prod c)(\exists \varphi \in S)(\forall^\infty n)(x(n) \in \varphi(n))\}$.
(2) $\mathfrak{c}^\exists_{c,h} = \min\{|S| : S \subseteq S(c,h), (\forall x \in \prod c)(\exists \varphi \in S)(\exists^\infty n)(x(n) \in \varphi(n))\}$.
(3) $\mathfrak{v}^\forall_{c,h} = \min\{|X| : X \subseteq \prod c, (\forall \varphi \in S(c,h))(\exists x \in X)(\exists^\infty n)(x(n) \notin \varphi(n))\}$.
(4) $\mathfrak{v}^\exists_{c,h} = \min\{|X| : X \subseteq \prod c, (\forall \varphi \in S(c,h))(\exists x \in X)(\forall^\infty n)(x(n) \notin \varphi(n))\}$.

DEFINITION 1.7. (1) Define $\mathfrak{v}^\forall = \min\{\mathfrak{v}^\forall_{c,h} : c, h \in \omega^\omega, \lim_{n\to\infty} h(n) = \infty\}$.
(2) Define $\mathfrak{c}^\exists = \min\{\mathfrak{c}^\exists_{c,h} : c, h \in \omega^\omega, \sum_{n\in\omega} h(n)/c(n) < \infty\}$.

FACT 1.8 [3, Lemma 3.5 and Theorem 3.12]. *Let $\langle \mathcal{A}_i : i \in \omega\rangle$ be a sequence of structures in a language $\mathcal{L}$ such that each $\mathcal{A}_i$ has size $\leq \mathfrak{c}$. Let $\mathcal{U}$ be an ultrafilter over ω. Then the ultraproduct $\prod_{i\in\omega} \mathcal{A}_i/\mathcal{U}$ has size either finite or $\mathfrak{c}$.*

§2. SAT($\aleph_1$) and KT($\aleph_1$).

In this section, we prove that SAT($\aleph_1$) is equivalent to CH and that KT($\aleph_1$) implies $\mathfrak{b} = \aleph_1$.

THEOREM 2.1. SAT($\aleph_1$) *implies CH.*

PROOF. Assume SAT($\aleph_1$) and $\neg$CH. Take an ultrafilter $\mathcal{U}$ over ω that witnesses SAT($\aleph_1$). Let $\mathcal{A}_* = (\omega_1, <)^\omega/\mathcal{U}$. For $\alpha < \omega_1$, put $\alpha_* = [\langle \alpha, \alpha, \alpha, \dots\rangle]$. Define a set p of formulas with a free variable x by

$$p = \{\ulcorner \alpha_* < x \urcorner : \alpha < \omega_1\}.$$

This p is finitely satisfiable and the number of parameters occurring in p is $\aleph_1 < \mathfrak{c} = |\mathcal{A}_*|$ by $\neg$CH. Thus, by SAT($\aleph_1$), we can take $f : \omega \to \omega_1$ such that $[f]$ realizes p. Put $\beta = \sup_{n\in\omega} f(n)$. Now we have $\{n \in \omega : \beta < f(n)\} \in \mathcal{U}$ and this contradicts the definition of β. ⊣

THEOREM 2.2. $\neg$SAT($\aleph_2$) *holds.*

PROOF. Take an ultrafilter $\mathcal{U}$ over ω that witnesses SAT($\aleph_2$). Let $\mathcal{A}_* = (\omega_2, <)^\omega/\mathcal{U}$. For $\alpha < \omega_1$, put $\alpha_* = [\langle \alpha, \alpha, \alpha, \dots\rangle]$. Define a set p of formulas with a free variable x by

$$p = \{\ulcorner \alpha_* < x < (\omega_1)_* \urcorner : \alpha < \omega_1\}.$$

The remaining argument is the same as Theorem 2.1. ⊣

DEFINITION 2.3. Let $\mathfrak{mcf} = \min\{\mathrm{cf}(\omega^\omega/\mathcal{U}) : \mathcal{U}$ an ultrafilter over $\omega\}$.

The order of $\omega^\omega/\mathcal{U}$ is the almost domination order modulo $\mathcal{U}$ and $\mathrm{cf}(\omega^\omega/\mathcal{U})$ is the dominating number of this relation. So it is clear that $\mathfrak{b} \leq \mathfrak{mcf} \leq \mathfrak{d}$.

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LEMMA 2.4 [6, Claim 2.2]. *Let $\mathcal{A}$ be a structure in a language $\mathcal{L} = \{<\}$. Suppose that $a \in \mathcal{A}$ has cofinality ω_1. Let $\mathcal{U}$ be an ultrafilter over ω. Then $a_* = [\langle a, a, a, \dots \rangle]$ has cofinality ω_1 in $\mathcal{A}^\omega/\mathcal{U}$.*

PROOF. Take an increasing cofinal sequence $\langle x_\alpha : \alpha < \omega_1 \rangle$ of points in $\mathcal{A}$ below a. Then $\langle x_\alpha^* : \alpha < \omega_1 \rangle$ is an increasing cofinal sequence in $\mathcal{A}_*$, where $x_\alpha^* = [\langle x_\alpha, x_\alpha, x_\alpha, \dots \rangle]$ for each $\alpha < \omega_1$. This can be shown by regularity of ω_1. ⊣

LEMMA 2.5 [6, Claim 2.4]. *Let $\mathcal{U}$ be an ultrafilter over ω and $\mathcal{B}_* = (\mathbb{Q}, <)^\omega/\mathcal{U}$. Then for every $a, b \in \mathcal{B}_*$, there is an automorphism on $\mathcal{B}_*$ that sends a to b.*

PROOF. Consider the map $F : \mathbb{Q}^3 \to \mathbb{Q}$ defined by $F(x, y, z) = x - y + z$. Then we have

$$(\forall y, z \in \mathbb{Q})(\text{the map } x \mapsto F(x, y, z) \text{ is an automorphism on } (\mathbb{Q}, <) \text{ that sends } y \text{ to } z).$$

This statement can be written by a first-order formula in the language $\mathcal{L}' = \{<, F\}$. Thus the same statement is true in $(\mathbb{Q}, <, F)^\omega/\mathcal{U}$. The map $F_* : \mathcal{B}_*^3 \to \mathcal{B}_*$ induced by F satisfies that

$$(\forall y, z \in \mathcal{B}_*)$$
$$(\text{the map } x \mapsto F(x, y, z) \text{ is an automorphism on } (\mathcal{B}_*, <) \text{ that sends } y \text{ to } z).$$ ⊣

THEOREM 2.6. $\mathrm{KT}(\aleph_1)$ *implies* $\mathfrak{mcf} = \aleph_1$.

PROOF. This proof is based on [6, Theorem 2.1]. Assume that $\mathfrak{mcf} \geq \aleph_2$. We shall show $\neg\,\mathrm{KT}(\aleph_1)$.

Let $\mathcal{L} = \{<\}$, $\mathcal{A} = (\mathbb{Q}, <)$ and $\mathcal{B} = (\mathbb{Q} + ((\omega_1 + 1) \times \mathbb{Q}_{\geq 0}), <_\mathcal{B})$. Here $<_\mathcal{B}$ is defined by a lexicographical order and a disjoint union order. $\mathcal{A}$ and $\mathcal{B}$ are dense linear ordered sets, so by completeness of DLO, we have $\mathcal{A} \equiv \mathcal{B}$. Take an ultrafilter $\mathcal{U}$ over ω. Put $\mathcal{A}_* = \mathcal{A}^\omega/\mathcal{U}, \mathcal{B}_* = \mathcal{B}^\omega/\mathcal{U}$.

There is a point a in $\mathcal{B}$ such that $\mathrm{cf}(\mathcal{B}_a) = \aleph_1$, where $\mathcal{B}_a = \{x \in \mathcal{B} : x < a\}$. Then $a_* \in \mathcal{B}_*$ has cofinality $\aleph_1$ by Lemma 2.4. Here $a_* = [\langle a, a, a, \dots \rangle]$. On the other hand, we shall show every point in $\mathcal{A}_*$ has cofinality $\geq \mathfrak{mcf}$. If we do this, since we assumed $\mathfrak{mcf} \geq \aleph_2$, we will have $\mathcal{A}_* \not\cong \mathcal{B}_*$.

By Lemma 2.5, it suffices to consider the point $0_* = [\langle 0, 0, 0, \dots \rangle]$. Since $\mathbb{Q}$ is symmetrical, we consider $\mathrm{cf}((\mathbb{Q}_{>0})^\omega/\mathcal{U}, >_\mathcal{U})$.

Now we construct a Galois–Tukey morphism (φ, ψ): $\mathbf{Cof}(\omega^\omega/\mathcal{U}) \to \mathbf{Cof}((\mathbb{Q}_{>0})^\omega/\mathcal{U}, >_\mathcal{U})$ by

$$\varphi : \omega^\omega/\mathcal{U} \to (\mathbb{Q}_{>0})^\omega/\mathcal{U}; [f] \mapsto [\langle 1/(f(n) + 1) : n \in \omega \rangle],$$
$$\psi : (\mathbb{Q}_{>0})^\omega/\mathcal{U} \to \omega^\omega/\mathcal{U}; [g] \mapsto [\langle \lfloor 1/g(n) - 1 \rfloor : n \in \omega \rangle].$$

So we have $\mathrm{cf}((\mathbb{Q}_{>0})^\omega/\mathcal{U}, >_\mathcal{U}) \geq \mathrm{cf}(\omega^\omega/\mathcal{U}, <_\mathcal{U})$.

Thus we have $\mathrm{cf}((\mathbb{Q}_{>0})^\omega/\mathcal{U}, >_\mathcal{U}) \geq \mathfrak{mcf}$. We are done. ⊣

COROLLARY 2.7. $\mathrm{KT}(\aleph_1)$ *implies* $\mathfrak{b} = \aleph_1$.

PROOF. This follows from Theorem 2.6 and the fact that $\mathfrak{b} \leq \mathfrak{mcf}$. ⊣

§3. $\mathrm{SAT}(\aleph_0)$ and $\mathrm{KT}(\aleph_0)$.

In this section, we first briefly mention consistency of $\mathrm{KT}(\aleph_0) + \neg\,\mathrm{KT}(\aleph_1)$. And we prove that $\mathrm{SAT}(\aleph_0)$ is equivalent to $\mathrm{cov}(\mathsf{meager}) = \mathfrak{c} \wedge 2^{<\mathfrak{c}} = \mathfrak{c}$.

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Fact 3.1 [4, Theorem 7.13]. *The statement* $\mathrm{cov}(\mathsf{meager}) = \mathfrak{c}$ *is equivalent to* MA(*countable*), *that is for every countable poset* $\mathbb{P}$ *and a family of dense sets* $\mathcal{D}$ *with* $|\mathcal{D}| < \mathfrak{c}$ *there is a filter* G *of* $\mathbb{P}$ *that intersects all* $D \in \mathcal{D}$.

Theorem 3.2. $\mathrm{cov}(\mathsf{meager}) = \mathfrak{c}$ *implies* $\mathrm{KT}(\aleph_0)$.

Proof. [6, Theorem 3.3] shows that $\mathrm{cov}(\mathsf{meager}) = \mathfrak{c} \wedge \mathrm{cf}(\mathfrak{c}) > \aleph_1$ implies $\mathrm{KT}(\aleph_1)$ and the exact same proof works for $\mathrm{KT}(\aleph_0)$ without the assumption $\mathrm{cf}(\mathfrak{c}) > \aleph_1$.

Here we sketch the proof.

Let $\mathcal{L}$ be a countable language and $\mathcal{A}^0$ and $\mathcal{A}^1$ are countable $\mathcal{L}$-structures which are elementarily equivalent.

Enumerate $(\mathcal{A}^i)^\omega$ for $i = 0, 1$ as

$$(\mathcal{A}^i)^\omega = \{f^i_\alpha : \alpha < \mathfrak{c}\}.$$

By a back-and-forth method, we construct a sequence of triples $\langle (\mathcal{U}_\alpha, g^0_\alpha, g^1_\alpha) : \alpha < \mathfrak{c}\rangle$ satisfying:

(1) $g^0_\alpha \in \mathcal{A}^0$.
(2) $g^1_\alpha \in \mathcal{A}^1$.
(3) $\mathcal{U}_\alpha$ is a filter over ω generated by $\aleph_0 + |\alpha|$ sets.
(4) $(\mathcal{U}_\alpha : \alpha < \mathfrak{c})$ is an increasing continuous sequence.
(5) If $\varphi(x_0, \dots, x_{n-1}$ is an $\mathcal{L}$-formula and $\beta_0, \dots, \beta_n \leq \alpha$, then the set

$$\{k \in \omega : \mathcal{M}^0 \models \varphi(g^0_{\beta_0}(k), \dots, g^0_{\beta_{n-1}}(k)) \iff \mathcal{M}^1 \models \varphi(g^1_{\beta_0}(k), \dots, g^1_{\beta_{n-1}}(k))\}$$

belongs to $\mathcal{U}_{\alpha+1}$.

In the construction, when α is even, we put $g^0_\alpha = f^0_\gamma$ where γ is the least ordinal $f^0_\gamma \notin \{g^0_\beta : \beta < \alpha\}$. And $\mathbb{P}$ is the poset of finite partial functions from ω to $\mathcal{A}^1$. Take a generating set $\mathcal{F}$ of $\mathcal{U}_\alpha$ of size $\aleph_0 + |\alpha|$. Then by using MA(countable), take a $\mathbb{P}$-generic filter G with respect to the following family of dense sets of $\mathbb{P}$:

$$D_n = \{p \in \mathbb{P} : n \in \mathrm{dom}\, p\} \text{ (for } n \in \omega)$$

and

$$\begin{aligned} E_{X, \langle \varphi_\iota : \iota \in I\rangle, \langle \gamma^\iota_1, \dots, \gamma^\iota_{n_\iota} : \iota \in I\rangle} = \{ & p \in \mathbb{P} : (\exists k \in \mathrm{dom}(p) \cap X)(\forall \iota \in I) \\ & (M^0 \models \varphi_\iota(g^0_{\gamma^\iota_1}(k), \dots g^0_{\gamma^\iota_{n_\iota}}(k), g^0_\alpha(k)) \Leftrightarrow \\ & M^1 \models \varphi_\iota(g^1_{\gamma^\iota_1}(k), \dots g^1_{\gamma^\iota_{n_\iota}}(k), p(k))\}), \end{aligned}$$

where $X \in \mathcal{F}$, $\langle \varphi_\iota : \iota \in I\rangle$ is a finite sequence of $\mathcal{L}$-formulas and $\gamma^\iota_1, \dots, \gamma^\iota_{n_\iota}$ for $\iota \in I$ are ordinals less than α. Then putting $g^1_\alpha = \bigcup G$ satisfies the induction hypothesis.

Then the appropriate construction guarantees that $\mathcal{U} = \bigcup_{\alpha < \mathfrak{c}} \mathcal{U}_\alpha$ is an ultrafilter and that the function

$$\langle ([g^0_\alpha]_\mathcal{U}, [g^1_\alpha]_\mathcal{U}) : \alpha < \mathfrak{c}\rangle$$

is isomorphic from $(M^0)^\omega/\mathcal{U}$ to $(M^1)^\omega/\mathcal{U}$. ⊣

Corollary 3.3. *Assume* Con(ZFC). *Then* $\mathrm{Con}(\mathrm{ZFC} + \mathrm{KT}(\aleph_0) + \neg\,\mathrm{KT}(\aleph_1))$.

Proof. $\mathrm{MA} + \neg\mathrm{CH}$ implies $\mathrm{KT}(\aleph_0) \wedge \neg\,\mathrm{KT}(\aleph_1)$ by Theorems 2.6 and 3.2. ⊣

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Fact 3.4 [2, Lemma 2.4.2]. $\mathrm{cov}(\mathsf{meager}) = \mathfrak{d}^{\exists}_{\langle \omega : n \in \omega \rangle, \mathrm{id}}$. *In other words,* $\mathrm{cov}(\mathsf{meager}) \geq \kappa$ *holds iff* $(\forall X \subseteq \omega^\omega$ *of size* $< \kappa)(\exists S \in \prod_{i \in \omega} [\omega]^{\leq i})(\forall x \in X)(\exists^\infty n)(x(n) \in S(n))$ *holds.*

Theorem 3.5. $\mathrm{SAT}(\aleph_0)$ *implies* $\mathrm{cov}(\mathsf{meager}) = \mathfrak{c}$.

Proof. Take an ultrafilter $\mathcal{U}$ that witnesses $\mathrm{SAT}(\aleph_0)$. Fix $X \subseteq \omega^\omega$ of size $< \mathfrak{c}$. Define a language $\mathcal{L}$ by $\mathcal{L} = \{\subseteq\}$ and for each $i \in \omega$, define an $\mathcal{L}$-structure $\mathcal{A}_i$ by $\mathcal{A}_i = ([\omega]^{\leq i}, \subseteq)$. For each $x \in \omega^\omega$, let $S_x = \langle \{x(i)\} : i \in \omega \rangle$. In the ultraproduct $\mathcal{A}_* = \prod_{i \in \omega} \mathcal{A}_i / \mathcal{U}$, define a set p of formulas of one free variable S by

$$p = \{\ulcorner [S_x] \subseteq S \urcorner : x \in X\}.$$

This p is finitely satisfiable. In order to check this, let $x_0, \dots, x_n$ be finitely many members of X. Define S by $S(m) = \{x_0(m), \dots, x_n(m)\}$ for $m \geq n$. We don't need to care about $S(m)$ for $m < n$. Then this S satisfies $[S_{x_i}] \subseteq [S]$ for all $i \leq n$. Moreover, the number of parameters of p is $< \mathfrak{c}$.

So by $\mathrm{SAT}(\aleph_0)$, we can take $[S] \in \mathcal{A}_*$ that realizes p. Then S fulfills $(\forall x \in X)(\{n \in \omega : x(n) \in S(n)\} \in \mathcal{U})$. Thus $(\forall x \in X)(\exists^\infty n)(x(n) \in S(n))$. ⊣

Theorem 3.6. $\mathrm{SAT}(\aleph_0)$ *implies* $2^{<\mathfrak{c}} = \mathfrak{c}$.

Proof. Take an ultrafilter $\mathcal{U}$ over ω that witnesses $\mathrm{SAT}(\aleph_0)$. Fix $\kappa < \mathfrak{c}$.

Put $\mathcal{L} = \{\subseteq\}$ and define an $\mathcal{L}$-structure $\mathcal{A}$ by $\mathcal{A} = ([\omega]^{<\omega}, \subseteq)$. Put $\mathcal{A}^* = \mathcal{A}^\omega / \mathcal{U}$.

Define a map $\imath \colon \omega^\omega / \mathcal{U} \to \mathcal{A}^*$ by $\imath([x]) = [\langle \{x(n)\} : n \in \omega \rangle]$. By Fact 1.8, we have $|\omega^\omega / \mathcal{U}| = \mathfrak{c}$. Take a subset F of $\omega^\omega / \mathcal{U}$ of size κ.

For each $X \subseteq F$, let p_X be a set of formulas with a free variable z defined by

$$p_X = \{\ulcorner \imath(y) \subseteq z \urcorner : y \in X\} \cup \{\ulcorner \imath(y) \not\subseteq z \urcorner : y \in F \setminus X\}.$$

Each p_X is finitely satisfiable. In order to check this, take $[x_0], \dots, [x_n] \in X$ and $[y_0], \dots, [y_m] \in F \setminus X$. Put $z(i) = \{x_0(i), \dots, x_n(i)\}$. Then $\imath([x_0]), \dots, \imath([x_n]) \subseteq_{\mathcal{U}} [z]$. In order to prove $\imath([y_j]) \not\subseteq_{\mathcal{U}} [z]$ for each $j \leq m$, suppose that $\{i \in \omega : y_j(i) \in z(i)\} \in \mathcal{U}$. Then for each $i \in \omega$, there is a $k_i \leq n$ such that $\{i \in \omega : y_j(i) = x_{k_i}(i)\} \in \mathcal{U}$. Then there is a $k \leq n$ such that $\{i \in \omega : y_j(i) = x_k(i)\} \in \mathcal{U}$. This implies $[y_j] = [x_k]$, which is a contradiction.

By $\mathrm{SAT}(\aleph_0)$, for each $X \subseteq F$, take $[z_X] \in \mathcal{A}^*$ that realizes p_X. For $X, Y \subseteq F$ with $X \neq Y$, we have $[z_X] \neq [z_Y]$. So $2^\kappa = |\{[z_X] : X \subseteq F\}| \leq |\mathcal{A}^*| = \mathfrak{c}$. Therefore we have proved $2^{<\mathfrak{c}} = \mathfrak{c}$. ⊣

Theorem 3.7. $\mathrm{cov}(\mathsf{meager}) = \mathfrak{c} \wedge 2^{<\mathfrak{c}} = \mathfrak{c}$ *implies* $\mathrm{SAT}(\aleph_0)$.

Proof. This proof is based on [5, Theorem 1].

Let $\langle b_\alpha : \alpha < \mathfrak{c} \rangle$ be an enumeration of ω^ω. Let $\langle (\mathcal{L}_\xi, \mathcal{B}_\xi, \Delta_\xi) : \xi < \mathfrak{c} \rangle$ be an enumeration of triples $(\mathcal{L}, \mathcal{B}, \Delta)$ such that $\mathcal{L}$ is a countable language, $\mathcal{B} = \langle \mathcal{A}_i : i \in \omega \rangle$ is a sequence of $\mathcal{L}$-structures with universe ω, and Δ is a subset of $\mathrm{Fml}(\mathcal{L}^+)$ with $|\Delta| < \mathfrak{c}$. Here $\mathcal{L}^+ = \mathcal{L} \cup \{c_\alpha : \alpha < \mathfrak{c}\}$ where the c_α's are new constant symbols and $\mathrm{Fml}(\mathcal{L}^+)$ is the set of all $\mathcal{L}^+$ formulas with one free variable. Here we used the assumption $2^{<\mathfrak{c}} = \mathfrak{c}$. And ensure each $(\mathcal{L}, \mathcal{B}, \Delta)$ occurs cofinally in this sequence.

For $\mathcal{B}_\xi = \langle \mathcal{A}^\xi_i : i \in \omega \rangle$, put $\mathcal{B}_\xi(i) = (\mathcal{A}^\xi_i, b_0(i), b_1(i), \dots)$, which is an $\mathcal{L}^+$-structure.

Let $\langle X_\xi : \xi < \mathfrak{c} \rangle$ be an enumeration of $\mathcal{P}(\omega)$.

We construct a sequence $\langle F_\xi : \xi < \mathfrak{c} \rangle$ of filters inductively so that the following properties hold:

(1) F_0 is the filter consisting of all cofinite subsets of ω.
(2) $F_\xi \subseteq F_{\xi+1}$ and $F_\xi = \bigcup_{\alpha<\xi} F_\alpha$ for ξ limit.
(3) $X_\xi \in F_{\xi+1}$ or $\omega \setminus X_\xi \in F_{\xi+1}$.
(4) F_ξ is generated by $< \mathfrak{c}$ members.
(5) If

$$\text{for all } \Gamma \subseteq \Delta_\xi \text{ finite, } \{i \in \omega : \Gamma \text{ is satisfiable in } \mathcal{B}_\xi(i)\} \in F_\xi, \qquad (*)$$

then there is an $f \in \omega^\omega$ such that for all $\varphi \in \Delta_\xi$, $\{i \in \omega : f(i) \text{ satisfies } \varphi \text{ in } \mathcal{B}_\xi(i)\} \in F_{\xi+1}$.

Suppose we have constructed F_ξ. We construct $F_{\xi+1}$. Let F'_ξ be a generating subset of F_ξ with $\left|F'_\xi\right| < \mathfrak{c}$. If $(*)$ is false, let $F_{\xi+1}$ be the filter generated by $F'_\xi \cup \{X_\xi\}$ or $F'_\xi \cup \{\omega \setminus X_\xi\}$. Suppose $(*)$.

Put $\mathbb{P} = \mathrm{Fn}(\omega, \omega) = \{p : p \text{ is a finite partial function from } \omega \text{ to } \omega\}$. For $n \in \omega$, put

$$D_n = \{p \in \mathbb{P} : n \in \operatorname{dom} p\}.$$

For $A \in F'_\xi$ and $\varphi_1, \dots, \varphi_n \in \Delta_\xi$, put

$$E_{A,\varphi_1,\dots,\varphi_n} = \{p \in \mathbb{P} : (\exists i \in \operatorname{dom} p \cap A)(p(i) \text{ satisfies } \varphi_1, \dots, \varphi_n \text{ in } \mathcal{B}_\xi(i))\}.$$

Each D_n is clearly dense. In order to show that each $E_{A,\varphi_1,\dots,\varphi_n}$ is dense, take $p \in \mathbb{P}$. By (*) and the property $A \in F_\xi$, we can take $i \in A \setminus \operatorname{dom} p$ and $k \in \omega$ such that k satisfies $\varphi_1, \dots, \varphi_n$ in $\mathcal{B}_\xi(i)$. Put $q = p \cup \{(i, k)\}$. This is an extension of p in $E_{A,\varphi_1,\dots,\varphi_n}$.

By using MA(countable), take a generic filter $G \subseteq \mathbb{P}$ with respect to above dense sets. Put $f = \bigcup G$. Then $F''_\xi := F'_\xi \cup \{Y_\varphi : \varphi \in \Delta_\xi\}$ satisfies finite intersection property, where $Y_\varphi = \{i \in \omega : f(i) \text{ satisfies } \varphi \text{ in } \mathcal{B}_\xi(i)\}$. In order to check this, let $A \in F'_\xi$ and $\varphi_1, \dots, \varphi_n \in \Delta_\xi$. Then by genericity, we can take $p \in G \cap E_{A,\varphi_1,\dots,\varphi_n}$. So we can take $i \in \operatorname{dom} p \cap A$ such that $p(i)$ satisfies $\varphi_1, \dots, \varphi_n$ in $\mathcal{B}_\xi(i)$. Then we have $i \in A \cap Y_{\varphi_1} \cap \dots \cap Y_{\varphi_n}$.

Let $F_{\xi+1}$ be the filter generated by $F''_\xi \cup \{X_\xi\}$ or $F''_\xi \cup \{\omega \setminus X_\xi\}$.

We have constructed $\langle F_\xi : \xi < \mathfrak{c} \rangle$. In order to check that the resulting ultrafilter $F = \bigcup_{\xi<\mathfrak{c}} F_\xi$ witnesses $\mathrm{SAT}(\aleph_0)$, let $\mathcal{L}$ and $\mathcal{B} = \langle \mathcal{A}_i : i \in \omega \rangle$ satisfy the assumption of the theorem. Let Δ be a subset of $\mathrm{Fml}(\mathcal{L}^+)$ with $|\Delta| < \mathfrak{c}$. Assume that for all $\Gamma \subseteq \Delta$ finite, $X_\Gamma := \{i \in \omega : \Gamma \text{ is satisfiable in } \mathcal{B}_\xi(i)\} \in F$. By the regularity of $\mathfrak{c}$, we have $\alpha < \mathfrak{c}$ such that for all $\Gamma \subseteq \Delta$ finite, $X_\Gamma \in F_\alpha$. Let $\xi \geq \alpha$ be satisfying $(\mathcal{L}_\xi, \mathcal{B}_\xi, \Delta_\xi) = (\mathcal{L}, \mathcal{B}, \Delta)$. Then by (5), there is an $f \in \omega$ such that for all $\varphi \in \Delta$, $\{i \in \omega : f(i) \text{ satisfies } \varphi \text{ in } \mathcal{B}(i)\} \in F$. Thus $\prod_{i\in\omega} \mathcal{A}_i / F$ is saturated. ⊣

§4. $\mathrm{KT}(\aleph_0)$ **implies** $\mathfrak{c}^\exists \leq \mathfrak{d}$**.** In this section, we will show the following theorem. This proof is based on [9, Theorem 1.1] and [1, Theorem 3.7].

THEOREM 4.1. $\mathrm{KT}(\aleph_0)$ *implies* $\mathfrak{c}^\exists \leq \mathfrak{d}$.

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DEFINITION 4.2. Define a language $\mathcal{L}$ by $\mathcal{L} = \{E, U, V\}$, where E is a binary predicate and U, V are unary predicates. We say an $\mathcal{L}$-structure $M = (|M|, E^M, U^M, V^M)$ is a *bipartite directed graph* if the following conditions hold:

(1) $U^M \cup V^M = |M|$.
(2) $U^M \cap V^M = \varnothing$.
(3) $(\forall x, y \in |M|)(x\ E^M\ y \to (x \in U^M \text{ and } y \in V^M))$.

DEFINITION 4.3. For $n, k \in \omega$ with $k \leq n$, define a bipartite directed graph $\Delta_{n,k}$ as follows:

(1) $U^{\Delta_{n,k}} = \{1, 2, 3, \dots, n\}$.
(2) $V^{\Delta_{n,k}} = [\{1, 2, 3, \dots, n\}]^{\leq k} \setminus \{\varnothing\}$.
(3) For $u \in U^{\Delta_{n,k}}, v \in V^{\Delta_{n,k}}$, $u\ E^{\Delta_{n,k}}\ v$ iff $u \in v$.

DEFINITION 4.4. For $n \in \omega$, let $G_n = \Delta_{n^3,n}$. Let Γ be the disjoint union of $(G_n : n \geq 2)$.

We define a natural order $\lhd$ on Γ by $x \lhd y$ if $m < n$ for $x \in G_m, y \in G_n$. Then Γ is a bipartite directed graph with an order $\lhd$. Put $\mathcal{L}' = \mathcal{L} \cup \{\lhd\}$. From now on, we consider $\mathcal{L}'$-structures which are elementarily equivalent to Γ.

DEFINITION 4.5. Let Γ_{NS} be a countable non-standard elementary extension of Γ.

When we say connected components, we mean the connected components when we ignore the orientation of the edges.

LEMMA 4.6. *Let M be an $\mathcal{L}'$-structure that is elementarily equivalent to* Γ. *Then the connected components of M are precisely the maximal antichains of M with respect to* $\lhd$.

PROOF. Suppose that $A \subseteq M$ is connected but not an antichain. Then we can find elements $a_0, \dots, a_n \in M$ such that

$$M \models (a_0 E a_1 \vee a_1 E a_0) \wedge \cdots \wedge (a_{n-1} E a_n \vee a_n E a_{n-1}) \wedge \\ (a_0 \text{ and } a_n \text{ are comparable with respect to } \lhd).$$

By elementarity, we have $n + 1$ many elements in Γ that satisfies the same formula. This is a contradiction. So every connected subset in M is an antichain.

Note that any two connected vertexes in Γ have a path of length at most 4. Thus we have

$$\Gamma \models (\forall a, b)((a \text{ and } b \text{ are incomparable with respect to } \lhd) \\ \to (\text{there is a path between } a \text{ and } b \text{ with length at most } 4)).$$

By elementarity, the same formula holds in M. So every antichain in M is connected.

Therefore the connected components of M are precisely the maximal antichains of M with respect to $\lhd$. ⊣

Therefore, $\lhd$ induces an order on the connected components of M and it is denoted also by $\lhd$.

LEMMA 4.7. *Every infinite connected component C of Γ_{NS} satisfies the following*:

$$(\forall F \subseteq C \cap U \text{ finite})(\exists v \in C \cap V)(v \text{ has an edge to each point in } F).$$

PROOF. Let $F = \{u_1, \dots, u_n\}$. Observe that

$$\Gamma \models (\forall x_1) \dots (\forall x_n)[x_1, \dots, x_n \text{ are points in } U \text{ and belong to the same connected component and the index of this connected component is } \geq n \rightarrow (\exists y)[y \text{ belongs to this component}, y \in V \text{ and } x_1, \dots, x_n \ E\ y]].$$

By elementarity, Γ_{NS} satisfies the same formula. ⊣

LEMMA 4.8. *Let $\langle \Delta_n : n \in \omega \rangle$ be a sequence of bipartite directed graphs with $|U^{\Delta_n}| = |V^{\Delta_n}| = \aleph_0$. Suppose that for each $n \in \omega$,*

$$(\forall F \subseteq U^{\Delta_n} \text{ finite})(\exists v \in V^{\Delta_n})(v \text{ has an edge to each point in } F).$$

Then for every ultraproduct $R := \prod_{n\in\omega} \Delta_n/\mathcal{V}$, we have

$$(\exists \langle v_i : i < \mathfrak{d} \rangle \text{ with each } v_i \in V^R)(\forall u \in U^R)(\exists i < \mathfrak{d})(u\ E^R\ v_i).$$

PROOF. We may assume that each $U^{\Delta_n} = \omega$. Let $\{f_i : i < \mathfrak{d}\}$ be a cofinal subset of $(\omega^\omega, <^*)$. For each $n, m \in \omega$, take $v_{n,m} \in V^{\Delta_n}$ that is connected with first m points in U^{Δ_n}. For $i < \mathfrak{d}$, put

$$v_i = [\langle v_{n,f_i(n)} : n \in \omega \rangle].$$

Let $[u] \in U^R$. Consider u as an element of ω^ω. Take f_i that dominates u. Then we have

$$\{n \in \omega : u(n)\ E^{\Delta_n}\ v_{n,f_i(n)}\} \in \mathcal{V}.$$

Therefore $[u]\ E^R\ v_i$. ⊣

LEMMA 4.9. *Let $\mathcal{V}$ be an ultrafilter over ω and put $Q = (\Gamma_{\mathrm{NS}})^\omega/\mathcal{V}$. Then there exist cofinally many connected components C with respect to $\lhd$ such that*

$$(\exists \langle v_i : i < \mathfrak{d} \rangle \text{ with each } v_i \in C \cap V^Q)(\forall u \in C \cap U^Q)(\exists i < \mathfrak{d})(u\ E^Q\ v_i).$$

PROOF. Fix a connected component C_0 of Q and $[x_0] \in C_0$. Then for each $n \in \omega$, there is an infinite component C_n above $x_0(n)$. Now

$$C = \{[x] \in Q : x \in (\Gamma_{\mathrm{NS}})^\omega \text{ and } (\forall n \in \omega)(x(n) \in C_n)\}$$

is a connected component of Q above C_0. Since C can be viewed as $C = \prod_{n\in\omega} C_n/\mathcal{V}$, the conclusion of the lemma holds for C by Lemmas 4.7 and 4.8. ⊣

LEMMA 4.10. *Let $\kappa < \mathfrak{c}^\exists$ and $\mathcal{U}$ be an ultrafilter over ω and put $P = \Gamma^\omega/\mathcal{U}$. Then for every C in a final segment of connected components of P, we have*

$$(\forall \langle v_i : i < \kappa \rangle \text{ with each } v_i \in C \cap V^P)(\exists u \in C \cap U^P)(\forall i < \kappa)(u \not\mathrel{E}^P v_i).$$

PROOF. Let $f : \omega \to \Gamma$ satisfy $f(n) \in G_n$ for all n. Let C_0 be the connected component that $[f]$ belongs to. Take a connected component C such that $C_0 \lhd C$ and an element $[g] \in C$. Take a function $h : \omega \to \omega$ such that $\{n \in \omega : g(n) \in G_{h(n)}\} \in \mathcal{U}$. Then $A := \{n \in \omega : h(n) \geq n\} \in \mathcal{U}$. Put $h'(n) = \max\{h(n), n\}$.

Take $\langle [v_i] : i < \kappa \rangle$ with each $[v_i] \in C \cap V^P$. Then we have

$$B_i := \{n \in \omega : v_i(n) \in G_{h(n)} \cap V^\Gamma\} \in \mathcal{U}.$$

Take v_i' such that $v_i'(n) = v_i(n)$ for $n \in A \cap B_i$ and $v_i'(n) \in [h'(n)^3]^{\leq h'(n)}$ for $n \in \omega$. The assumption $\kappa < \mathfrak{c}^\exists$ and the calculation

$$\sum_{n \geq 1} \frac{h'(n)}{h'(n)^3} = \sum_{n \geq 1} \frac{1}{h'(n)^2} \leq \sum_{n \geq 1} \frac{1}{n^2} < \infty$$

give an $x \in \prod h'$ such that for all $i < \kappa$, $(\forall^\infty n)(x(n) \notin v_i'(n))$. For each $i < \kappa$, take n_i such that $(\forall n \geq n_i)(x(n) \notin v_i'(n))$.

Take a point $[u] \in C \cap U^P$ such that $u(n) = x(n)$ for all $n \in A$. Then for all $i < \kappa$ we have

$$\{n \in \omega : u(n) \not\!E^\Gamma\ v_i(n)\} \supseteq A \cap B_i \cap [n_i, \omega) \in \mathcal{U}.$$

Therefore $[u] \not\!E^P\ [v_i]$ for all $i < \kappa$. ⊣

Assume that $\mathfrak{d} < \mathfrak{c}^\exists$. Then by Lemmas 4.10 and 4.9, for any two ultrafilters $\mathcal{U}, \mathcal{V}$ over ω, we have $\Gamma^\omega/\mathcal{U} \not\simeq (\Gamma_{\mathrm{NS}})^\omega/\mathcal{V}$. So $\neg\,\mathrm{KT}(\aleph_0)$ holds. We have proved Theorem 4.1.

Fact 4.11 [8, Lemma 2.3]. $\mathrm{cov}(\mathsf{null}) \leq \mathfrak{c}^\exists$.

Corollary 4.12. *In the random model,* $\neg\,\mathrm{KT}(\aleph_0)$ *holds.*

Proof. This corollary holds since $\aleph_1 = \mathfrak{d} < \mathrm{cov}(\mathsf{null}) = \mathfrak{c}$ in the random model. ⊣

Remark 4.13. $\mathfrak{v}^\forall \leq \mathfrak{c}^\exists$ follows from [8, Lemma 2.6]. So the implication $\mathrm{KT}(\aleph_0) \implies \mathfrak{d} \geq \mathfrak{c}^\exists$ strengthens the implication $\mathrm{KT}(\aleph_0) \implies \mathfrak{d} \geq \mathfrak{v}^\forall$.

Remark 4.14. In [9], Shelah constructed a creature forcing that forces the following statements:

(1) There are a finite language $\mathcal{L}$ and countable $\mathcal{L}$-structures $\mathcal{A}, \mathcal{B}$ with $\mathcal{A} \equiv \mathcal{B}$ such that for all ultrafilters $\mathcal{U}, \mathcal{V}$ over ω, we have $\mathcal{A}^\omega/\mathcal{U} \not\simeq \mathcal{B}^\omega/\mathcal{V}$.

(2) There is an ultrafilter $\mathcal{U}$ over ω such that for every countable language $\mathcal{L}$ and any sequence $\langle(\mathcal{A}_n, \mathcal{B}_n) : n \in \omega\rangle$ of pairs of finite $\mathcal{L}$-structures, if $\prod_{n\in\omega} \mathcal{A}_n/\mathcal{U} \equiv \prod_{n\in\omega} \mathcal{B}_n/\mathcal{U}$, then these ultraproducts are isomorphic.

Shelah himself pointed out in [9, Remark 2.2] item 2 holds in the random model. On the other hand, we have proved item 1 also holds in the random model. Therefore both of above two statements hold in the random model.

§5. $\mathrm{KT}(\aleph_1)$ in forcing extensions.

A theorem by Golshani and Shelah [6] states that $\mathrm{cov}(\mathsf{meager}) = \mathfrak{c} \wedge \mathrm{cf}(\mathfrak{c}) = \aleph_1$ implies $\mathrm{KT}(\aleph_1)$. In [6], it was also proved that $\mathrm{cf}(\mathfrak{c}) = \aleph_1$ is not necessary for $\mathrm{KT}(\aleph_1)$. In this section, we prove that $\mathrm{cov}(\mathsf{meager}) = \mathfrak{c}$ is also not necessary for $\mathrm{KT}(\aleph_1)$.

Theorem 5.1. *Let* $\lambda > \aleph_1$ *be a regular cardinal with* $\lambda^{<\lambda} = \lambda$. *Let* $\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha : \alpha < \omega_1\rangle$ *be a finite support forcing iteration. Suppose that for all* $\alpha < \omega_1$, $\Vdash_\alpha$ "$\dot{\mathbb{Q}}_\alpha$ *is ccc and* $|\dot{\mathbb{Q}}_\alpha| \leq \lambda$". *And suppose that for all even* $\alpha < \omega_1$, $\Vdash_\alpha \dot{\mathbb{Q}}_\alpha = \mathbb{C}_\lambda$. *Here* $\mathbb{C}_\lambda$ *denotes the Cohen forcing adjoining* λ *many Cohen reals. Then,* $\Vdash_{\omega_1} \mathrm{KT}(\aleph_1)$.

Proof. This proof is based on [6, Theorem 3.3].

Let G be a $(V, \mathbb{P}_{\omega_1})$-generic filter.

Let $\mathcal{L}$ be a countable language and $M^0 \equiv M^1$ be two $\mathcal{L}$-structures of size $\leq \aleph_1$ in $V[G]$. Take sequences $\langle M_i^l : i < \omega_1 \rangle$ for $l = 0, 1$ that are increasing and continuous such that each M_i^l is countable elementary substructure of M^l and $M^l = \bigcup_{i<\omega_1} M_i^l$. We can take an increasing sequence $\langle \alpha_i : i < \omega_1 \rangle$ of even ordinals such that $M_i^l \in V[G_{\alpha_i+1}]$ for every $l < 2$ and $i < \omega_1$.

For $i < \omega_1$ and $\beta < \lambda$, let c_β^i be the β-th Cohen real added by $\dot{\mathbb{Q}}_{\alpha_i}$.

Take an enumeration $\langle X_\gamma : \gamma < \lambda \cdot \omega_1 \rangle$ of $\mathcal{P}(\omega)$ such that $\langle X_\gamma : \gamma < \lambda \cdot (i+1) \rangle \in V[G_{\alpha_i+1}]$ for every $i < \omega_1$. We can take such a sequence. The reason for this is that we can take $\langle \dot{X}_\gamma : \lambda \cdot i \leq \gamma < \lambda \cdot (i+1) \rangle$ as an enumeration of $\mathbb{P}_{\alpha_i+1}$ nice names for subsets of ω and put $X_\gamma = (\dot{X}_\gamma)^G$.

For each $l < 2$, take an enumeration $\langle f_\gamma^l : \gamma < \lambda \cdot \omega_1 \rangle$ of $(M^l)^\omega$ such that $f_{\lambda \cdot i + \beta}^l \in (M_i^l)^\omega$ for every $i < \omega_1$ and $\beta < \lambda$ and $\langle f_\gamma^l : \gamma < \lambda \cdot (i+1) \rangle \in V[G_{\alpha_i+1}]$.

For $\lambda' < \lambda$, let $G_{\alpha_i,\lambda'}$ denote $G \cap (\mathbb{P}_{\alpha_i} * \mathbb{C}_{\lambda'})$.

Now we construct a sequence of quadruples $\langle (\mathcal{U}_\gamma, g_\gamma^0, g_\gamma^1, \lambda_\gamma) : \gamma < \lambda \cdot \omega_1 \rangle$ by induction so that the following properties hold.

(1) Each $\mathcal{U}_\gamma$ is a filter over ω.
(2) For every $l < 2$, $i < \omega_1$, $\beta < \lambda$, and $\gamma = \lambda \cdot i + \beta$, $g_\gamma^l \in (M_i^l)^\omega \cap V[G_{\alpha_i,\lambda_\gamma}]$.
(3) For every $l < 2$ and $i < \omega_1$, $\langle g_\gamma^l : \gamma < \lambda \cdot (i+1) \rangle \in V[G_{\alpha_i+1}]$.
(4) Each λ_γ is an ordinal below λ. For $\lambda \cdot i \leq \gamma \leq \gamma' < \lambda \cdot (i+1)$, we have $\lambda_\gamma \leq \lambda_{\gamma'}$.
(5) For $i < \omega_1$ and $l < 2$, $\{g_\gamma^l : \gamma < \lambda \cdot i\} = \{f_\gamma^l : \gamma < \lambda \cdot i\}$.
(6) If $\lambda \cdot i \leq \gamma < \lambda \cdot (i+1)$, then $\mathcal{U}_\gamma \in V[G_{\alpha_i}, \lambda_\gamma]$.
(7) If $\gamma < \delta < \lambda \cdot \omega_1$, then $\mathcal{U}_\gamma \subseteq \mathcal{U}_\delta$.
(8) If $\gamma < \lambda \cdot \omega_1$ is a limit ordinal, then $\mathcal{U}_\gamma = \bigcup_{\delta<\gamma} \mathcal{U}_\delta$.
(9) $X_\gamma \in \mathcal{U}_{\gamma+1}$ or $\omega \setminus X_\gamma \in \mathcal{U}_{\gamma+1}$.
(10) If $\varphi(x_1, \dots, x_n)$ is an $\mathcal{L}$-formula, $\gamma = \lambda \cdot i + \beta$, and $\gamma_1, \dots, \gamma_n \leq \gamma$, then $Y_{\varphi,\gamma_1,\dots,\gamma_n}$ defined below belongs to $\mathcal{U}_{\gamma+1}$:

$$Y_{\varphi,\gamma_1,\dots,\gamma_n} = \{k \in \omega : M_i^0 \models \varphi(g_{\gamma_1}^0(k), \dots, g_{\gamma_n}^0(k)) \\ \Leftrightarrow M_i^1 \models \varphi(g_{\gamma_1}^1(k), \dots, g_{\gamma_n}^1(k))\}.$$

(*Construction*) First we let U_0 be the set of cofinite subsets of ω.

Suppose that $\langle \mathcal{U}_\delta : \delta \leq \gamma \rangle$ and $\langle g_\delta^0, g_\delta^1, \lambda_\delta : \delta < \gamma \rangle$ are defined. Now we will define $g_\gamma^0, g_\gamma^1, \lambda_\gamma$ and $\mathcal{U}_{\gamma+1}$. Take i and β such that $\gamma = \lambda \cdot i + \beta$.

Suppose that γ is even.

Let $g_\gamma^0 = f_{\varepsilon_\gamma}^0$, where ε_γ is the minimum ordinal such that $f_{\varepsilon_\gamma}^0$ does not belong to $\{g_\delta^0 : \delta < \gamma\}$.

Take $\lambda' < \lambda$ such that $M_i^0, M_i^1, \langle g_\delta^0 : \delta \leq \gamma \rangle, \langle g_\delta^1 : \delta < \gamma \rangle \in V[G_{\alpha_i,\lambda'}]$. Put $\lambda_\gamma = \lambda' + 1$. Take a bijection $\pi_i^1 : \omega \to M_i^1$ in $V[G_{\alpha_i,\lambda'}]$. Define g_γ^1 by $g_\gamma^1 = \pi_i^1 \circ c_{\lambda'}^i$.

Put $\mathcal{Y} = \{ Y_{\varphi,\gamma_1,\dots,\gamma_n} : \varphi(x_1, \dots, x_n)$ is an $\mathcal{L}$-formula and $\gamma_1, \dots, \gamma_n \leq \gamma\}$. Now we show $\mathcal{U}_\gamma \cup \mathcal{Y}$ has the finite intersection property. In order to show it, let $X \in \mathcal{U}_\gamma$, $\langle \varphi_\iota : \iota \in I \rangle$ is a finite sequence of $\mathcal{L}$-formulas and $\gamma_1^\iota, \dots, \gamma_{n_\iota}^\iota$ for $\iota \in I$ are ordinals that are less than γ. It suffices to show that the set $D \in V[G_{\alpha_i,\lambda'}]$ defined below is a dense subset of $\mathbb{C}$:

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$$D = \{p \in \mathbb{C} : (\exists k \in \mathrm{dom}(p) \cap X)(\forall \imath \in I)$$
$$M_i^0 \models \varphi_\imath(g^0_{\gamma_1^\imath}(k), \dots g^0_{\gamma^\imath_{n_\imath}}(k), g^0_\gamma(k)) \Leftrightarrow M_i^1 \models \varphi_\imath(g^1_{\gamma_1^\imath}(k), \dots g^1_{\gamma^\imath_{n_\imath}}(k), \pi^1_i(p(k)))\}.$$

We now prove this. Let $p \in \mathbb{C}$.

For each $k \in \omega$ and $\imath \in I$, put

$$v(k,\imath) = \begin{cases} 1, & \text{if } M_i^0 \models \varphi_\imath(g^0_{\gamma_1^\imath}(k), \dots, g^0_{\gamma^\imath_{n_\imath}}(k), g^0_\gamma(k)), \\ 0, & \text{otherwise.} \end{cases}$$

And for each $k \in \omega$ put

$$v(k) = \langle v(k,\imath) : \imath \in I \rangle.$$

Then by finiteness of ${}^I 2$, for some $v_0 \in {}^I 2$, we have $\omega \setminus v^{-1}(v_0) \notin \mathcal{U}_\gamma$.

For each $\imath \in I$, put

$$\varphi_\imath^+(x_1^\imath, \dots, x^\imath_{n_\imath}, y) \equiv \begin{cases} \varphi_\imath(x_1^\imath, \dots, x^\imath_{n_\imath}, y), & \text{if } v_0(\imath) = 1, \\ \neg\varphi_\imath(x_1^\imath, \dots, x^\imath_{n_\imath}, y), & \text{otherwise.} \end{cases}$$

Put

$$\psi \equiv \exists y \bigwedge_{\imath \in I} \varphi_\imath^+(x_1^\imath, \dots, x^\imath_{n_\imath}, y).$$

Then by the induction hypothesis (10), $Y_{\psi, \langle \gamma_1^\imath, \dots \gamma^\imath_{n_\imath} : \imath \in I\rangle} \in \mathcal{U}_\gamma$. So take $k \in X \cap v^{-1}(v_0) \cap Y_{\psi, \langle \gamma_1^\imath, \dots \gamma^\imath_{n_\imath} : \imath \in I\rangle} \setminus \mathrm{dom}(p)$.

Since $M_i^0 \models \psi(\langle g^0_{\gamma_1^\imath}(k), \dots g^0_{\gamma^\imath_{n_\imath}}(k) : \imath \in I\rangle)$, we have $M_i^1 \models \psi(\langle g^1_{\gamma_1^\imath}(k), \dots g^1_{\gamma^\imath_{n_\imath}}(k) : \imath \in I\rangle)$.

By the definition of ψ, we can take $y \in M_i^1$ such that $M_i^1 \models \varphi_\imath^+(g^1_{\gamma_1^\imath}(k), \dots, g^1_{\gamma^\imath_{n_\imath}}(k), y)$ for every $\imath \in I$. We now put $q = p \cup \{(k, (\pi_i^1)^{-1}(y))\} \in \mathbb{C}$. This witnesses denseness of D.

Now we define $\mathcal{U}_{\gamma+1}$ as the filter generated by $\mathcal{U}_\gamma \cup \mathcal{Y} \cup \{X_\gamma\}$ or the filter generated by $\mathcal{U}_\gamma \cup \mathcal{Y} \cup \{\omega \setminus X_\gamma\}$.

When γ is odd, do the same construction above except for swapping 0 and 1. Since the above construction below $\lambda \cdot (i+1)$ can be performed in $V[G_{\alpha_i+1}]$, (3) in the induction hypothesis holds. (*End of Construction.*)

Now we put $\mathcal{U} = \bigcup_{\gamma < \lambda\cdot\omega_1} \mathcal{U}_\gamma$, which is an ultrafilter over ω. Then the function

$$\langle([g^0_\gamma]_{\mathcal{U}}, [g^1_\gamma]_{\mathcal{U}}) : \gamma < \lambda \cdot \omega_1\rangle$$

witnesses $(M^0)^\omega/\mathcal{U} \simeq (M^1)^\omega/\mathcal{U}$. ⊣

Corollary 5.2. $\mathrm{Con}(\mathrm{ZFC}) \to \mathrm{Con}(\mathrm{ZFC} + \mathrm{cof}(\mathsf{null}) = \aleph_1 < \mathfrak{c} + \mathrm{KT}(\aleph_1))$.

Proof. Let $\mathbb{A}$ denote the amoeba forcing. Let $\lambda > \aleph_1$ be a regular cardinal with $\lambda^{<\lambda} = \lambda$. Let $\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha : \alpha < \omega_1\rangle$ be a finite support forcing iteration such that for all even $\alpha < \omega_1$ we have $\Vdash_\alpha \dot{\mathbb{Q}}_\alpha = \mathbb{C}_\lambda$ and for all odd $\alpha < \omega_1$ we have $\Vdash_\alpha \dot{\mathbb{Q}}_\alpha = \mathbb{A}$.

Then $\mathbb{P}_{\omega_1} \Vdash \mathrm{KT}(\aleph_1)$ by Theorem 5.1.

Moreover, we have $\mathrm{cof}(\mathsf{null}) = \aleph_1$ since the amoeba forcing $\mathbb{A}$ adds a null set containing all null sets coded in the ground model (see [2, p. 106]). ⊣

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§6. Open questions. The following three questions remain.

QUESTION 6.1. (1) *Does* $\mathrm{KT}(\aleph_1)$ *imply a stronger hypothesis than* $\mathfrak{mcf} = \aleph_1$? *In particular does* $\mathrm{KT}(\aleph_1)$ *imply* $\mathrm{non}(\mathsf{meager}) = \aleph_1$?

(2) *Does* $\mathrm{KT}(\aleph_0)$ *imply a stronger hypothesis than* $\mathfrak{c}^\exists \leq \mathfrak{d}$? *In particular does* $\mathrm{KT}(\aleph_0)$ *imply* $\mathrm{non}(\mathsf{meager}) \leq \mathrm{cov}(\mathsf{meager})$?

(3) *In the Sacks model, does* $\mathrm{KT}(\aleph_0)$ *hold*? (*If in this model* $\neg\,\mathrm{KT}(\aleph_0)$ *holds, we can separate* $\mathrm{KT}(\aleph_0)$ *and* $\mathfrak{c}^\exists \leq \mathfrak{d}$.)

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