REGULAR RINGS AND MODULES

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1. Introduction

P. M. Cohn [7] calls a submodule P of the left A-module M *pure* iff $0 \to E \otimes P \to E \otimes M$ is exact for all right modules E. This definition of purity, which Cohn [7] has shown to be equivalent to the usual one when A is a PID (= Principal Ideal Domain), was studied in [9] and [10]. Here we show that the ring A is (von Neumann) regular iff every left (or right) ideal is pure. This leads us to define regular modules as modules all of whose submodules are pure. The ring A is then regular iff all its left (or right) A-modules are regular. A regular socle, analogous to the usual socle is defined. For commutative A, some localization theorems are proved, and used to settle a conjecture of Bass [1] concerning commutative perfect rings.

Most of the results in this paper are contained in the author's doctoral thesis [9], at McGill University. The author would like to thank his research director, Dr. J. Lambek, for his generous encouragement and continued interest.

Throughout this paper A will be an associative ring with 1, but not necessarily commutative. All modules are unitary, and $\otimes$ means $\otimes_A$. We use "*fg*" for finitely generated and '*fp*' for finitely presented.

2. Pure left ideals

THEOREM 1. *Let P be a submodule of the left A-module M, and consider the following conditions*:

(1) *M/P is flat.*

(1)′ *P is pure in M.*

(2) *$KM \cap P = KP$ for all rt. ideals K.*

(2)′ *$KM \cap P = KP$ for all fg rt. ideals K.*

(2)″ *$KM \cap P = KP$ for all principal rt. ideals K.*

(3) *$aM \cap P = aP$ for all a in A.*

Then we always have the following implications:

$$(1) \Rightarrow (1)' \Rightarrow (2) \Leftrightarrow (2)' \Rightarrow (2)'' \Leftrightarrow (3).$$

If M is flat we have $(1) \Leftrightarrow (1)' \Leftrightarrow (2)$.

PROOF. $(1) \Rightarrow (1)'$: is given by Cohn [7]. Since KP is always contained in $KM \cap P$, we need only show the opposite inclusion in each case. We shall also use the fact, due to Cohn [7], that P is pure in M iff

$$\Sigma a_{ij} m_j \in P \Rightarrow \Sigma a_{ij} m_j = \Sigma a_{ij} p_j$$

for some $p_j \in P$, where $a_{ij} \in A$, $m_j \in M$ and $i \in I, j \in J$, two finite index sets, and the summation is over the repeated indice.

$(1)' \Rightarrow (2)$: If $p = \Sigma k_j m_j$ (j in J, a finite set) is a typical element of $KM \cap P$ then $p = \sum k_j p_j \in KP$ since P is pure in M. Therefore $KM \cap P$ is contained in KP.

$(2) \Rightarrow (2)'$: is obvious.

$(2)' \Rightarrow (2)$: If $p = \Sigma k_j m_j$ (j in J, a finite set) is a typical element of $KM \cap P$, let K' be the *fg* rt. ideal generated by the k_j. Then p is contained in $K'M \cap P = K'P$, which is contained in KP. Therefore $KM \cap P$ is contained in KP. The remaining implications are obvious.

If M is flat, the equivalences $(1) \Leftrightarrow (1)' \Leftrightarrow (2)$ are given in ([4], Cor., p. 33).

COROLLARY. (1) *The left ideal P of A is pure in A iff $KP = K \cap P$ for all (finitely generated) right ideals K of A.*

(2) *Every pure left ideal P is idempotent.*

(3) *Let P be a left ideal. If $K \cap P$ is idempotent for all fg right ideals K, then P is pure in A.*

PROOF. (1) A is a flat A-module.

(2) Let $P' = PA \geqq P$. Then $P^2 = P'P = P' \cap P = P$.

(3) $K \cap P = (K \cap P)^2 = (K \cap P)(K \cap P) \leqq KP$.

3. Regular rings

The ring A will be called (von Neumann) *regular* iff $a \in aAa$ for all $a \in A$. Since this concept is left-right symmetric, all results about left ideals or modules will have analogues for right ideals and modules, which will be assumed and used, although they have not been stated explicitly.

THEOREM 2. *The ring A is regular iff every left ideal is pure.*

PROOF. $\Rightarrow$: In (4) it is shown that A is regular iff every A-module is flat. Therefore for any left ideal I of A, we have A/I flat and hence I pure in A by Theorem 1.

$\Leftarrow$: For any a in A the left ideal Aa is pure in A and therefore $aA \cap Aa = aAa$ by Theorem 1. But $a \in aA \cap Aa$.

COROLLARY. (1) *If A is regular, every left ideal is idempotent.*

(2) *The converse holds if A is commutative.*

PROOF. These results follow immediately from the corollary of Theorem 1.

THEOREM 3. *For any ring A, consider the following conditions*:

(1) *A is a regular ring.*

(2) *A/I is a regular ring for every two-sided ideal I of A.*

(3) *A/I is a semi-primitive ring for every two-sided ideal I of A.*

(4) *A/I is a semi-prime ring for every two-sided ideal I of A.*

Then we always have $(1) \Rightarrow (2) \Rightarrow (3) \Rightarrow (4)$. *If A is commutative*, $(4) \Rightarrow (1)$.

PROOF. The implications $(1) \Rightarrow (2) \Rightarrow (3) \Rightarrow (4)$ are well-known and can be found in Bourbaki [3] and Lambek [14]. $(4) \Rightarrow (1)$ (A commutative): Suppose K and J are any two ideals of A and let $I = KJ$. Then $\hat{K}\hat{J} = 0$ in the ring $B = A/I$, where ^ denotes the image in B. Since B is semi-prime, $\hat{K} \cap \hat{J} = 0$ (see Lambek [14], p. 56). Hence $K \cap J$ is contained in KJ. Since we always have the opposite inclusion, A is regular by Theorem 2.

4. Regular modules

A left A-module R will be called (von Neumann) *regular* iff every submodule is pure. This generalizes the idea of regular ring, as the following theorem shows:

THEOREM 4. *For any ring A, the following conditions are equivalent*:

(1) *A is a regular ring.*

(2) *Every left A-module is regular.*

(3) *The left A-module A is a regular module.*

PROOF. $(1) \Rightarrow (2)$: Since A is a regular ring, every *rt.* A-module F is flat. Hence if D is any submodule of the left A-module E, the sequence

$$0 \to D \otimes F \to E \otimes F \to E/D \otimes F \to 0$$

is exact and D is pure in E. Therefore E is regular.

$(2) \Rightarrow (3)$: is obvious.

$(3) \Rightarrow (1)$: If A is a regular left A-module, then all the left ideals of A are pure and A is a regular ring by Theorem 2.

REMARKS. (1) This theorem shows that any theorem about regular modules implies a theorem about modules over regular rings.

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(2) Maddox [15] calls a module E *absolutely pure* iff E is pure in any module containing it. It is easy to see that A is regular iff every left A-module is absolutely pure.

PROPOSITION 1. *R is a regular module iff every fg submodule is pure.*

PROOF. $\Rightarrow$: is clear.

$\Leftarrow$: Every submodule of R is the direct limit of fg submodules of R, i.e. the direct limit of pure submodules, and hence pure, by Corollary 2 of Theorem 1 in [10].

In (10) it was shown that E is pure in F iff maps from fp(= finitely presented) modules M to F/E can be lifted to F.

THEOREM 5. *Suppose we have an exact commutative diagram of left A-modules:*

$$\begin{tikzcd} 0 \arrow[r] & E \arrow[r] \arrow[d] & F \arrow[r] \arrow[d] & G \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & E' \arrow[r] & F' \arrow[r] & G' \arrow[r] & 0 \end{tikzcd}$$

with the map $G \to G'$ an isomorphism. Then E pure in F implies E' pure in F'.

PROOF. Any map from an fp module M to G' can be lifted first to G by means of the isomorphism $G \to G'$, and then to F since E is pure in F. Using the map $F \to F'$ we then have a lifting of the map $M \to G'$ to F', and E' is therefore pure in F'.

COROLLARY 1. *Let P and Q be two submodules of M. Then*

(1) *$(P \cap Q)$ pure in $Q \Rightarrow P$ pure in $(P+Q)$.*

(2) *$(P+Q)$ pure in M and $(P \cap Q)$ pure in $Q \Rightarrow P$ pure in M.*

(3) *$(P+Q)$ pure in M and $(P \cap Q)$ pure in $M \Rightarrow P$ pure in M and Q pure in M.*

(4) *$P \cap Q$ pure in $P+Q \Rightarrow P$ and Q are both pure in $P+Q$.*

PROOF. We have an exact commutative diagram.

$$\begin{tikzcd} 0 \arrow[r] & P \cap Q \arrow[r] \arrow[d] & Q \arrow[r] \arrow[d] & Q/(P \cap Q) \arrow[r] \arrow[d, "c"] & 0 \\ 0 \arrow[r] & P \arrow[r] & P+Q \arrow[r] & (P+Q)/P \arrow[r] & 0 \end{tikzcd}$$

where all homomorphisms arise from the natural injections, and c is an isomorphism.

(1) is a straightforward application of the theorem.

(2) By (1), P is pure in $(P+Q)$. But $(P+Q)$ is pure in M and hence P is pure in M by Proposition 1 in (10).

(3) $(P \cap Q)$ pure in $M \Rightarrow (P \cap Q)$ pure in P and $(P \cap Q)$ pure in Q. Apply (2).

(4) Apply (3) with $M = P+Q$.

COROLLARY 2. *For all i in I, any index set, let N_i be a submodule of a fixed module M and let $N = \Sigma N_i$ (i in I). For each k in I define $\hat{N}_k = \Sigma N_i$ (i in I, $i \neq k$). Then for all k in I, N pure in M and $(N_k \cap \hat{N}_k)$ pure in $\hat{N}_k \Rightarrow N_k$ pure in M.*

PROOF. Apply Corollary 1 with $P = N_k$ and $Q = \hat{N}_k$.

REMARK. Examples are given in (9) to show that the converses are false.

THEOREM 6. *Let $0 \to R \to S \to T \to 0$ be an exact sequence of left A-modules. Then S is a regular module iff both R and T are regular modules, and R is pure in S.*

PROOF. In the course of this proof, we shall refer several times to Proposition 1 of (10). For convenience, this will be denoted (*) for this proof only.

$\Rightarrow$. R is regular since every submodule of R is a submodule of S, hence pure in S, and therefore pure in R by (*). Every submodule of T has the form V/R with $R \leqq V \leqq S$. But V is then pure in S and therefore V/R is pure in $S/R = T$ by (*). Hence T is regular. And R is pure in S, since S is regular.

$\Leftarrow$. Let V be any submodule of S. Then $(V+R)/R$ is pure in $S/R = T$, since T is regular. But R is pure in S. Hence $V+R$ is pure in S by (*). Also $V \cap R$ is pure in R, since R is regular. Hence $V \cap R$ is pure in S, since R is pure in S, again by (*). Therefore, both $V+R$ and $V \cap R$ are pure in S. Hence V is pure in S by Corollary 1 of Theorem 5.

THEOREM 7. *Let $R = \Sigma R_i$ (i in I, any index set) be left A-modules. Then R is a regular module iff R_i is a regular module for all i in I.*

PROOF. $\Rightarrow$. For each i in I, R_i is regular by Theorem 6 since it is a submodule of R.

$\Leftarrow$: Since R is a homomorphic image of $S = \oplus R_i$ (i in I), it suffices to show that S is regular by Theorem 6. We will use Proposition 1: Let P be any *fg* submodule of S. Then P is a submodule of $T = \oplus R_i$ (i in J, J some finite subset of I). Since T is a direct summand of S, it is pure in S. Hence if we show that T is regular, we are finished because then P will be pure in T and hence in S, by the transitivity of purity (see Proposition 1 in [10]). We have reduced the problem to proving the following lemma:

LEMMA. *If $T = \oplus_1^n R_i$, then T is regular if each R_i is regular.*

PROOF. We use induction. For each $k < n$, let $T(k) = \oplus_1^k R_i$. Clearly $T(1) = R_1$ is regular. Assume $T(k)$ is regular; then

$$0 \to R_{k+1} \to T(k+1) \to T(k) \to 0$$

is exact with $T(k)$ and R_{k+1} regular and R_{k+1} pure in $T(k+1)$ (since it is a direct summand). Hence $T(k+1)$ is regular by Theorem 6.

COROLLARY. *For any left A-module R, the following conditions are equivalent*:

(1) *R is a regular module.*

(2) *Ax is a regular module for all x in R and Ax is pure in R.*

(3) *Ax is a regular module for all x in R.*

PROOF. The proof is clear since $R = \sum Ax$ (x in R).

5. Regular projective modules

THEOREM 8. *Suppose* $O \to P \to F \to E \to O$ *is exact with F free. Then the following conditions are equivalent*:

(1) *P is pure in F.*

(2) *E is flat.*

(3) *Given any x in P, there exists a homomorphism* $u: F \to P$ *such that* $u(x) = x$.

(4) *Given any* x_i *in P,* $1 \leqq i \leqq m$, *there exists a homomorphism* $u: F \to P$ *such that* $u(x_i) = x_i$ *for all i.*

PROOF. The equivalence of (1) and (2) has been shown in (10). The equivalence of (2), (3) and (4) has been shown by Chase ([6], Prop.2.2), who attributes the result to Villamayor.

COROLLARY. *Suppose* $O \to P \to Q \to E \to O$ *is exact with Q projective and P pure in Q. Then given* x_i *in P,* $1 \leqq i \leqq m$, *there exists a homomorphism* $u: Q \to P$ *such that* $u(x_i) = x_i$ *for all i.*

PROOF. Since Q is projective, there exists $F = Q \oplus Q'$ with F free. And P pure in Q, Q pure in $F \to P$ pure in F by Proposition 1 of [10]. By the theorem, there exists $w: F \to P$ such that $w(x_i) = x_i$. Let $u = w|Q$. Then $u: Q \to P$ and $u(x_i) = w(x_i) = x_i$ for all i, since x_i is in P.

THEOREM 9. *Suppose* $O \to P \to Q \to F \to O$ *is exact with P fg and Q projective. Then P is pure iff P is a direct summand.*

PROOF. Since any direct summand is pure, it suffices to show the converse. Suppose then that P is pure and let x_i in P, ($1 \leqq i \leqq m$). generate P. Then there exists $u: Q \to P$ such that $u(x_i) = x_i$ for all i. If $j: P \to Q$ is the natural injection, then we have $uj = l_p$, whence the sequence splits.

COROLLARY 1. *If Q is a regular projective then every fg submodule is a direct summand.*

PROOF. Every (fg) submodule is pure.

COROLLARY 2. (Osofsky [17]). *A is regular iff every fg submodule of a projective module is a direct summand.*

PROOF. $\Rightarrow$: If A is regular, every module is regular (Theorem 4), and the result follows from Corollary 1.

$\Leftarrow$: Every *fg* left ideal is a direct summand, hence pure, and A is regular by Proposition 1.

THEOREM 10. (Structure Theorem for Regular Projective Modules). *A left A-module P is regular projective iff* $P = \oplus J_i$ *where* J_i *is a regular projective principal left ideal, which is a direct summand of A.*

REMARK. This generalizes and simplifies the proof of a theorem of Kaplansky ([13], Thm. 4).

PROOF. $\Leftarrow$: If each J_i is regular so is P (Theorem 7). If each J_i is projective, so is P. Hence the result in one direction.

$\Rightarrow$: By Kaplansky's Theorem ([13], Thm. 1), every projective module is the direct sum of *cg* (= countably generated) projective modules; hence we can reduce our problem to this case, and assume that P is *cg*. Let $x_i, i = 1, 2, 3, \cdots$ generate P. We shall define inductively y_i in P, $i = 1, 2, 3, \cdots$ such that: For all n, the sum $P_n = \sum_1^n Ay_i$ is direct and $P = \oplus_1^\infty Ay_i$. Define $y_1 = x_1$ and assume y_i defined for $i \leqq n$, so that $P_n = \oplus_1^n Ay_i$. Since P_n is *fg* pure and P is projective, there exists Q so that $P = P_n \oplus Q$. Let $x_{n+1} = p_n + y_{n+1}$ (p_n in P_n, y_{n+1} in Q). Clearly the sum $P_{n+1} = \sum_1^{n+1} Ay_i$ is direct. Since for all n the sum $P_n = \oplus_1^n Ay_i$ is direct, so is the sum $P' = \oplus_1^\infty Ay_i$. Also for each n, x_{n+1} is in $P_n \oplus Ay_{n+1} = P_{n+1}$. Therefore P is contained in P'. The opposite inclusion holds to since each y_n is in $P = P_n \oplus Q$. Since P is regular projective, so is Ay_n for all n. Since Ay_n is projective, it is isomorphic to a left ideal J_n, which is a direct summand of A, and hence principal.

COROLLARY 1. *If P is a regular projective module, every cg submodule is projective.*

PROOF. Let x_i $i = 1, 2, \cdots$ generate the *cg* submodule M of P. We define y_i, $i = 1, 2, \cdots$ exactly as we did in the proof of the theorem. Then $M = \oplus_1^\infty Ay_i$ is projective since each Ay_i is projective.

COROLLARY 2. *If A is regular, every cg submodule of a projective module is projective.*

6. Socles

In this section, we define socles which are generalizations of the usual socle. We remark that both Maranda [16] and Dickson [8] have studied radicals, and in doing so, have introduced preradicals which correspond to our socles. However, there is little, if any, overlap with our work.

Let C be the category of all left A-modules. *A socle* is a function T which assigns to each module M of C a submodule $T(M)$ of M in such a way that $f: M \to N \Rightarrow f(T(M)) \leqq T(N)$, i.e. $f(T(M))$ in a submodule of $T(N)$ (or equivalently $f_T = f|T(M)$ is a map from $T(M)$ to $T(N)$). In categorical language, a socle is a subfunctor of the identify functor. Let T be a socle. We make the following definitions:

T is *torsion* iff $T(N) = N \cap T(M)$ for all submodules N of M.

T is *idempotent* iff $T^2 = T$, i.e. $T(T(M)) = T(M)$ for all M.

T has *radical property* iff $T(M/T(M)) = 0$ for all M.

A module M is *T-complete* iff $T(M) = M$.

If T and T' are socles, $T \leqq T'$ iff $T(M)$ is a submodule of $T'(M)$ for all modules M. It is easy to verify that if T is torsion, then it is idempotent and $T(M)$ is T-complete.

We now prove a theorem which establishes the basic properties of socles.

THEOREM 11. *Let T be any socle.*

(1) *If N is a submodule of M, then $T(N)$ is a submodule of $T(M)$ and $(T(M)+N)/N$ is a submodule of $T(M/N)$.*

(2) *$T(A)$ is a two-sided ideal of A.*

(3) *T commutes with direct sums, i.e. $T(\oplus M_i) = \oplus T(M_i)$.*

(4) *$T(P) = T(A)P$ for all projective modules P.*

(5) *$T(A)M$ is a submodule of $T(M)$ for all modules M.*

(6) *If M is T-complete, so is any image of M.*

(7) *If $T(M)$ is T-complete then it is the largest T-complete submodule of M.*

PROOF. (1) Let $k: N \to M$ and $f: M \to M/N$ be the canonical maps. Since T is a socle, $T(N) = k(T(N)) \leqq T(M)$ and $(T(M)+N)/N = f(T(M)+N) \leqq T(M/N)$.

(2) $T(A)$ is a left ideal by definition. For any a in A, define $f_a : A \to A$ (as left A-modules) by $f_a(x) = xa$ for all x in A. Since T is a socle, $(T(A))a = f_a(T(A)) \leqq T(A)$. Hence $T(A)$ is also a right ideal.

(3) Let $M = \oplus M_i$. By (1) for each i, $T(M_i)$ is a submodule of $T(M)$. Hence $\sum T(M_i) = \oplus T(M_i) \leqq T(M)$. The sum is direct since for each i, $T(M_i)$ is a submodule of M_i. Let $p_i : M \to M_i$ be the canonical projection. Then $p_i(T(M))$ is a submodule of $T(M_i)$. If x is in $T(M)$ then $x = \sum x_i$ with x_i in M_i. Then $x_i = p_i(x)$ which is in $T(M_i)$. Hence $T(M) = \oplus T(M_i)$.

(4) If F is free, $F = \oplus A$ and by (3), $T(F) = \oplus T(A) = \oplus (T(A)A) =$

$T(A)(\oplus A) = T(A)F$. If P is projective then $F = P \oplus Q$ with F free. Hence $T(P) \oplus T(Q) = T(F) = T(A)(P \oplus Q) = T(A)P \oplus T(A)Q$. Therefore $T(P) = T(A)P$.

(5) For any M, let $f: F \to M$ be epi with F free. Then $f(T(F)) = f(T(A)F) = T(A)f(F) = T(A)M$. But $f(T(F))$ is a submodule of $T(M)$. Hence the result.

(6) Let $f: M \to N$ be epi and $T(M) = M$. Then $N = f(M) = f(T(M)) \leqq T(N) \leqq N$.

(7) If $T(N) = N \leqq M$ then $N = T(N) \leqq T(M)$ by (1).

Corollary. *$T(P) = 0$ for all projective modules P iff $T(A) = 0$.*

Proof. Obvious since $T(P) = T(A)P$ by (4).

Proposition 2. *If the socle T has radical property, then $T(M)$ is the smallest of the submodules N of M such that $T(M/N) = 0$.*

Proof. By definition of radical property, $T(M/T(M)) = 0$. If for some N we have $T(M/N) = 0$, then

$$(T(M)+N)/N \leqq T(M/N) = 0$$

and $T(M)+N = N$. Hence $T(M) \leqq N$.

7. Semi-simple and regular socles

In this section, we shall define the regular socle of a module which is analogous to the semi-simple (= usual) socle of a module, with purity playing the role of direct summand.

A left A-module $M \neq 0$ is called *simple* iff O and M are its only submodules, and *semi-simple* iff it is the sum of simple modules. For basic facts on (semi-) modules and rings, see Lambek [14].

For any left A-module M, its *ss* (= semi-simple) *socle* $S(M)$ is defined to be the sum of all its simple submodules (= the sum of all its semi-simple submodules). In an analogous way, we will define the *regular socle* $R(M)$ of a module M to be the sum of all its regular submodules (i.e. submodules which are regular modules). Thus $R(M) = \sum Ax$ (x in M and Ax regular).

Theorem 12. *Both the ss socle S and the regular socle R are torsion socles, and hence have all the properties given in Theorem 11. A module is S-complete iff it is semi-simple, and R-complete iff it is regular, i.e. $S(M) = M$ iff M is semi-simple and $R(M) = M$ iff M is regular. Also $S \leqq R$.*

Remark. (i) Neither S nor R are radicals. See Proposition 3 for examples and discussion.

(ii) By taking A to be a ring which is one of

(a) regular,
(b) not regular,
(c) semi-simple,
(d) not semi-simple,
(e) regular but not semi-simple,

we easily have examples where

(a) $R(M) = M$,
(b) $R(M) \neq M$,
(c) $S(M) = M$,
(d) $S(M) \neq M$,
(e) $S(M) \neq R(M)$, etc.

PROOF. Let T be either S or R.

Socle: $T(M)$ was defined to be a submodule. $T(M)$ is the sum of simple (resp. regular) modules. Hence if $f: M \to N$ then $f(T(M))$ is the sum of simple (resp. regular) modules, since the image of a simple module is simple or zero (easy to verify), and the image of a regular module is regular (Theorem 6).

Torsion. If N is a submodule of M, we know that $T(N)$ is a submodule of $T(M)$, by Theorem 11. $T(M)$ is the sum of simple or regular modules and hence $T(M)$ is either semi-simple (well known) or regular (Theorem 7). Hence the submodule $N \cap T(M)$ of $T(M)$ is either semi-simple (well known) or regular (Theorem 6), and therefore contained in $T(N)$.

The properties concerning S-complete and R-complete are clear. Also $S \leqq R$ since every simple module is regular.

For any socle T, the ring A will be called *left T-faithful* iff for all left A-modules $M \neq 0$, we have $T(M) \neq 0$.

For example, Bass [1] has shown that a left perfect ring is *rt.* S-faithful, where $S = ss$ socle.

Clearly if $T \leqq T'$ then if A is left T-faithful, it is left T'-faithful. Hence A left perfect $\Rightarrow A$ is *rt.* S-faithful $\Rightarrow A$ is *rt.* R-faithful.

PROPOSITION 3. *If A is left T-faithful for some socle T, then the left A-module M is T-complete iff* $T(M/T(M)) = 0$.

PROOF. $\Rightarrow$: is clear since $T(M) = M$.

$\Leftarrow$: Since A is T-faithful, we have $M/T(M) = 0$ whence $M = T(M)$ and M is T-complete.

COROLLARY 1. *If A is left T-faithful, then the socle T has radical property iff all left A-modules are T-complete.*

PROOF. $\Rightarrow$: For any M, $T(M/T(M)) = 0$ whence $M/T(M) = 0$ and $M = T(M)$.

$\Leftarrow$: is clear since $T(M) = M$ for all M.

COROLLARY 2. *Neither S nor R have radical property.*

PROOF. Let A be left perfect, but not regular. Then A is *rt*. T-faithful for $T = R$ or S by Bass [1], but not all *rt*. modules are regular ($= R$-complete) and hence not all are semi-simple ($= S$-complete).

EXAMPLE 1. In [9] it is shown that the singular submodule functor of Johnson [12] is a torsion socle.

EXAMPLE 2. Brown and McCoy [5] define a regular radical $M(A)$ of the ring A, so that A is (von Neumann) regular iff $A = M(A)$. In [9] it is shown that $M(A)$ is contained in $R(A)$ if A is a commutative semi-principal ring (i.e. every *fg* ideal is principal). An example is given in [9] to show that this inclusion can be strict.

EXAMPLE 3. Let A be a Dedekind domain. For any element $x \neq 0$, in the A-module E let the order ideal $0(x) = \pi P^{n(P)}$, with the product ranging over the prime ideals P of A. Call x *square free* iff $n(P) \leqq 1$ for all P. In [9] we have shown that the regular socle of E consists of the square free elements of E.

8. Localization

In this section we let A be a commutative ring, S a multiplicative set of A and $M = M(A)$ be the collection of maximal ideals m of A. We let E_S, E_m, E_p, u_S etc. denote the localization of the A-module E at S, at m, at p (a prime ideal), and of the A-homomorphism u at S, etc. Also $\otimes_S$ and $\otimes_m$ denote $\otimes_{A_S}$ and $\otimes_{A_m}$ respectively.

THEOREM 13. *Let E be any A-module, D any submodule of E and S any multiplicative set of A.*

(1) *If D is an A-direct summand of E (resp. A-pure in) E, then D_S is an A_S-direct summand of (resp. A_S-pure in) E_S.*

(2) *If E is A-semi-simple (resp. A-regular), then E_S is A_S-semi-simple (resp. A_S-regular).*

(3) $(T(E))_S \leqq T_S(E_S)$ *where T (resp. T_S) is either the ss socle or the regular socle with respect to A (resp. A_S).*

(4) *If E is A-simple, then E_S is A_S-simple.*

PROOF. (1) The direct summand case is given in [2] and the pure case in [4]

(2) Any A_S-submodule of E_S has the form D_S where D is an A-submodule of E. If E is A-semi-simple (resp. A-regular) than D is A-direct summand of (resp. A-pure in) E, and the result follows from (1).

(3) $T(E)$ is A-semi-simple (resp. A-regular); therefore by (2) $(T(E))_S$ is an A_S-semi-simple (resp. A-regular) submodule of E_S and hence contained in $T_S(E_S)$.

(4) As in (2), any A_S-submodule of E_S has the form D_S where D is an A-submodule of E. If E is A-simple then $D = 0$ or $D = E$ and $D_S = O$ or $D_S = E_S$. Hence E_S is A_S-simple.

COROLLARY. *Let S be any multiplicative set of A.*

(1) *If A is semi-simple (resp. regular), so is A_S.*

(2) *If A is simple, so is A_S.*

PROOF. Apply Parts (2) and (4) of the theorem with $E = A$.

THEOREM 14. *Let E be any A-module and D any submodule of E.*

(1) *D is A-pure in E iff D_m is A_m-pure in E_m for all m in M.*

(2) *E is A-regular iff E_m is A_m-regular for all m in M.*

PROOF. By Theorem 13 we need only show $\Leftarrow$ in each case.

(1) $\Leftarrow$: Let $j : D \to E$ be the canonical injection. For any A-module F, let $f = l_F \otimes j$. Then $f_m = l_{F_m} \otimes_m j_m$. Since D_m is A_m-pure in E_m, f_m is a monomorphism, for all m in M, and therefore f is a monomorphism. Hence D is A-pure in E.

(2) $\Leftarrow$: If D is any submodule of E, D_m is A_m-pure in E_m, and therefore D is A-pure in E by (1), and E is A-regular.

In general, the property of being a direct summand is not local. However, we have:

THEOREM 15. *Let $O \to D \to E \to F \to O$ be an exact sequence of A-modules. If F is A-pure projective, (see [10]), then*

(1) *D is an A-direct summand of E iff D_m is an A_m-direct summand of E_m for all m in M.*

(2) *E is A-semi-simple iff E_m is A_m-semi-simple for all m in M.*

PROOF. By Theorem 13 we need only show $\Leftarrow$ in each case.

(1) $\Leftarrow$: If D_m is A_m-pure in E_m for all m in M, then D is A-pure in E by Theorem 14. Since F is pure projective, the sequence is split exact by Theorem 4 of [10].

(2) $\Leftarrow$: For any submodule D of E, D_m is an A_m-direct summand of E_m for all m in M. Therefore by (1), D is a direct summand of E.

COROLLARY. *If A is PDS, (see (10)) then any exact sequence $O \to D \to E \to F \to O$ is split exact iff $O \to D_m \to E_m \to F_m \to O$ is split exact for all m in M.*

PROOF. If A is PDS, then all A-modules are pure projective by Theorem 7 of [10]. Apply the above theorem.

THEOREM 16. *Let $O \to D \to E \to F \to O$ be an exact sequence of A-modules. If F is fg flat, then D_p is an A_p-direct summand of E_p for all prime ideals p of A.*

PROOF. For all p, F_p is a *fg* flat A_p-module and hence A_p-free since A_p is a local ring. Since any free module is pure projective, the exact sequence $O \to D_p \to E_p \to F_p \to O$ splits.

9. Commutative perfect rings

Bass [1] has conjectured that a ring A is left perfect iff every nonzero left A-module has a maximal submodule and A has no infinite sets of orthogonal idempotents. As he remarks, this is the natural dual to Part (7) of Theorem P of [1].

Hamsher [11] has given an affirmative solution for commutative noetherian rings. We shall extend his solution to arbitrary commutative rings.

For the rest of this section, let A be commutative. We quote without proof:

LEMMA A. (Hamsher [11]) *If every nonzero module has a maximal submodule, then every prime ideal of A is a maximal ideal and the obvious*:

COROLLARY. *In this case the Jacobson radical $J = J(A)$ of A coincides with the prime radical of A.*

PROOF. $J(A)$ is the intersection of all maximal ideals and the prime radical is the intersection of all prime (= maximal) ideals.

For our main theorem we prove:

LEMMA B. *If A has the property that every prime ideal is maximal, then*

(1) *Every quotient ring A/I has the same property.*

(2) *A_S has the same property for any multiplicative set S.*

PROOF. (1) is an immediate consequence of the one-one correspondence between the prime (resp. maximal) ideals of A and the prime (resp. maximal) ideals of A/I.

(2) Any prime ideal of A_S has the form p_S where p is a prime ideal of A disjoint from S. But p is maximal and disjoint from S, and therefore a maximal ideal among ideals disjoint from S. Hence P_S is a maximal ideal of A_S.

An ideal I of A is *T-nilpotent* (1) iff for any sequence $a_1, a_2, a_3, \cdots$ of elements of I there exists an integer $n > 0$ such that $a_1 a_2 \cdots a_n = 0$.

THEOREM 17. *The ring A is perfect iff every nonzero A-module has a maximal submodule and A has no infinite sets of orthogonal idempotents.*

PROOF. $\Rightarrow$: has been shown by Bass (1).

$\Leftarrow$: Bass has also shown that under these conditions the Jacobson radical J

of A is T-nilpotent. Therefore by Theorem P of [1] it only remains to show that $B = A/J$ is semi-simple.

Lambek ([14], p. 72) has shown that if J is a nil ideal of A, any countable orthogonal set of nonzero idempotents in $B = A/J$ can be lifted to an orthogonal set of nonzero idempotents of A. Since any T-nilpotent ideal is clearly a nil ideal, this implies that B has no infinite sets of orthogonal idempotents. Osofsky [17] has remarked that any regular ring with no infinite sets of orthogonal idempotents is a semi-simple ring. Therefore to complete the proof it suffices to show that B is a regular ring.

We will prove that B is a regular ring by showing that B_n is a field for all maximal ideals n of B.

By the corollary of Lemma A, $J =$ the prime radical of A and hence $B = A/J$ is a semi-prime ring (see Lambek [14], p. 56). Therefore B_n is semi-prime for all maximal ideals n of B by ([4], Prop 17, p. 97). Since in A every prime ideal is maximal, the same is true for B and B_n by Lemma B. Consequently for all n, B_n is a local semi-primitive ring i.e. a field, and B is a regular ring.

Remarks. (1) Hamsher has informed me that he has also obtained a solution to Bass' conjecture in the commutative case. However the solution presented above is different from his, and was obtained independently in [9].

(2) Parts of the above solution are also valid in the non-commutative case. Details will appear in a subsequent publication.

References

[1] H. Bass, 'Finitistic Dimension and a Homological Generalization of Semi-Primary Rings', *Trans. Amer. Math. Soc.* 95 (1960), 466—488.
[2] N. Bourbaki, Algebre, Chap. 2.
[3] N. Bourbaki, Algebre, Chap. 8
[4] N. Bourbaki, Algebre Commutative, Chap. 1 and 2.
[5] B. Brown and N. McCoy, 'The maximal regular ideal of ring', *Proc. Amer. Math. Soc.* 1 (1950) 165—171.
[6] S. Chase, 'Direct Products of Modules', *Trans. Amer. Math. Soc.* 97 (1960), 457—473.
[7] P. M. Cohn, 'On the Free Product Associative Rings, I', *Math. Z.* 71 (1959), 380—398.
[8] S. E. Dickson, 'A Torsion Theory for Abelian Categories', *Trans. Amer. Math. Soc.* 121 (1966), 223—235.
[9] D. Fieldhouse, *Purity and Flatness* (PhD Thesis, McGill University, Montreal, Canada, 1967).
[10] D. Fieldhouse, '*Purity and Flat Covers*', (Queen's Preprint Series, 1967).
[11] R. M. Hamsher, 'Commutative Noetherian Rings over which every Module has a maximal Submodule' *Proc. Amer. Math. Soc.* 17 (1966), 1471—1472.
[12] R. Johnson, 'The extended Centralizer of a ring over a module', *Proc. Amer. Math. Soc.* 2 (1951), 891—895.
[13] I. Kaplansky, 'Projective Modules', *Ann. Math.* 68 (1958), 372—377.
[14] J. Lambek, *Lectures on Rings and Modules* (Waltham, Mass. (Blaisdell) 1966).
[15] B. Maddox, 'Absolutely Pure Modules', *Proc. Amer. Math. Soc.* 18 (1967), 155—158.

[16] J. M. Maranda, 'Injective Structures', *Trans. Amer. Math. Soc.* 110 (1964), 98--135.
[17] B. Osofsky, 'Rings all of whose finitely generated modules are injective', *Pacific J. Math.* 14 (1964), 645—650.

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